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Self-adjoint differential equations for classical orthogonal polynomials

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Abstract

This paper deals with spectral type differential equations of the self-adjoint differential operator, $2r$ order:

$$L_{(2r)}[Y](x) = \frac{1}{\rho(x)}\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left(\rho(x)\beta^r(x)\frac{\mathrm{d}^r Y(x)}{\mathrm{d}x^r}\right) = \lambda_{rn} Y(x).$$

If $\rho(x)$ is the weight function and $\beta(x)$ is a second degree polynomial function, then the corresponding classical orthogonal polynomials, $\{Q_n(x)\}_{n=0}^{\infty}$, are shown to satisfy this differential equation when λ_{rn} is given by

$$\lambda_{rn} = \prod_{k=0}^{r-1}(n-k)[\alpha_1 + (n+k+1)\beta_2],$$

where α_1 and β_2 are the leading coefficients of the two polynomial functions associated with the classical orthogonal polynomials. Moreover, the singular eigenvalue problem associated with this differential equation is shown to have $Q_n(x)$ and λ_{rn} as eigenfunctions and eigenvalues, respectively. Any linear combination of such self-adjoint operators has $Q_n(x)$ as eigenfunctions and the corresponding linear combination of λ_{rn} as eigenvalues.

Keywords: Orthogonal polynomials; Self-adjoint differential equations; Singular eigenvalue problems

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1. Introduction

With important applications, orthogonal polynomials theory was developed during the 20th century. Two major publications marked this evolution, Szegö [15] and Chihara [5]. Recent considerations of orthogonal polynomials and their differential equations have been published. Krall [11] gave necessary and sufficient conditions for orthogonal polynomial systems satisfying an Nth order linear differential equation of spectral type with polynomial coefficients. He further showed that the differential equation must have even order as an orthogonal polynomial system exists as a solution set. Kwon and Yoon [14] found that if such a differential equation has an orthogonal polynomial system of solutions, then its differential operator must be symmetrizable (self-adjoint). Sufficient conditions for symmetrizability of differential operators with polynomial coefficients were previously presented by Krall and Littlejohn [12]. Caruntu [3] presented a fourth-order differential equation of classical orthogonal polynomials and its associated singular eigenvalue problem. Kwon et al. [13] showed that a classical orthogonal system satisfying a second-order differential equation also satisfies a differential equation of order N, where N is an even number and the Nth-order differential operator is a linear combination of iterations of the second-order operator. Moreover, they showed that orthogonal polynomials satisfying a spectral type differential equation of order N, where N is greater than 2, must be Hermite polynomials if and only if the leading coefficient is a nonzero constant. Koekoek and Koekoek [10] reported differential equations satisfied by generalized Jacobi polynomials. A survey of the latest results concerning the classification of linear differential equations off spectral type having a sequence of polynomial eigenfunctions that are orthogonal with respect to some real bilinear form can be found in Everitt et al. [6]. Shifted Jacobi operators were introduced by Hajmirzaahmad [9]. He showed that they are self-adjoint.

The computer revolution of the last few decades led to a development of approximation theory and numerical analysis, and consequently to an increased interest in orthogonal polynomials. Numerical methods and software packages have been developed for solving Sturm–Liouville problems. A review of numerical methods for self-adjoint and non-self-adjoint nonsingular boundary eigenvalue Sturm–Liouville problems can be found in Greenberg and Marletta [7]. Bayley et al. [2] reported a software package, SLEIGN, for the computation of eigenvalues and eigenfunctions of either regular or singular second-order Sturm–Liouville boundary value problems. This code is based on the Prufer transformation and the knowledge of the precise number of zeros of the eigenfunctions. In the singular case, SLEIGN "has no serious competitor" [1]. The only code available dealing with fourth-order Sturm–Liouville boundary value problems is SLEUTH, Greenberg and Marletta [8]. Even so, it is limited to regular problems. Solving singular problems is a future direction of their research. Chanane [4] extended his results on the computation of eigenvalues of second-order Sturm–Liouville problems to a class of fourth-order problems. This approach was based on iterated integrals and Fliess series.

This paper reports self-adjoint differential equations for classical orthogonal polynomials and their associated eigenvalue singular problems. Besides contributing to the continuous effort of studying orthogonal polynomials' properties, this paper can be very useful as reference to researchers interested in seeking numerical solutions of singular two-point higher order Sturm–Liouville eigenvalue problems.

2. Self-adjoint differential equations of orthogonal polynomials

Orthogonal systems play an important role in analysis, mainly because functions belonging to very general classes can be expanded in series of orthogonal functions. Classical orthogonal polynomials

(Jacobi, Legendre, Hermite, Laguerre and Tchebycheff) are important classes of orthogonal systems. They are commonly encountered in many applications. In addition to the orthogonal property, they are the integrals of differential equations of a simple form, and can be defined as the coefficients in expansions of powers of t of suitable chosen functions $w(x,t)$, called generating functions. Classical orthogonal polynomials can be found using Rodrigues' formula [5,15]. A system of polynomials, $\{Q_n(x)\}_{n=0}^{\infty}$, is said to be orthogonal with weight $\rho(x)$ on the interval $[a,b]$ if their inner product is given by

$$(Q_n, Q_m)_\rho = \int_a^b \rho(x) Q_n(x) Q_m(x)\,\mathrm{d}x = 0, \quad n \neq m, \tag{1}$$

where $Q_n(x)$ is the orthogonal polynomial of nth degree of the considered system and m and n are any nonnegative integers. Let us consider that $Q_n(x)$ denotes any classical orthogonal polynomial of nth degree. Classical theory of orthogonal polynomials shows that the classical orthogonal polynomials satisfy the following second-order differential equation [5,15]

$$\beta \frac{\mathrm{d}^2 Q_n}{\mathrm{d}x^2} + \left[\alpha + \frac{\mathrm{d}\beta}{\mathrm{d}x}\right] \frac{\mathrm{d}Q_n}{\mathrm{d}x} - n[\alpha_1 + (n+1)\beta_2] Q_n = 0, \tag{2}$$

where $\alpha(x)$ and $\beta(x)$ are two polynomial functions as follows:

$$\alpha(x) = \alpha_1 x + \alpha_0, \quad \beta(x) = \beta_2 x^2 + \beta_1 x + \beta_0, \quad \alpha_1^2 + \beta_2^2 > 0, \tag{3}$$

$\rho(x)$ is the weight function of the inner product, and the following requirements are met:

$$\frac{1}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}x} = \frac{\alpha}{\beta}, \tag{4}$$

$$\lim_{\substack{x \to a \\ x \to b}} \rho\beta = 0. \tag{5}$$

We present here self-adjoint differential equations, $2r$ order, satisfied by classical orthogonal polynomials, where r is any natural number. Four lemmas and a consequence precede Proposition 1 in which the differential equation is presented.

Lemma 1. *If Eq.* (4) *is satisfied, then the jth order derivative of* $\rho\beta^r$ *is given by*

$$\frac{\mathrm{d}^j}{\mathrm{d}x^j}(\rho\beta^r) = \rho\beta^{r-j}\gamma_j, \tag{6}$$

where j is any nonnegative integer less than or equal to r and γ_r *is a polynomial of rth degree satisfying the following recurrence relation*:

$$\gamma_j = \left[\alpha + (r-j+1)\frac{\mathrm{d}\beta}{\mathrm{d}x}\right]\gamma_{j-1} + \beta \frac{\mathrm{d}\gamma_{j-1}}{\mathrm{d}x} \tag{7}$$

for any natural number j less than or equal to r, and $\gamma_0 = 1$.

Proof. This is proved by induction. If $j=0$, then Eq. (6) gives $\gamma_0 = 1$. If $j=1$, using Eq. (4), the first derivative of $\rho\beta^r$ becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho\beta^r) = \rho\beta^{r-1}\gamma_1 \quad \text{where} \quad \gamma_1 = \alpha + r\frac{\mathrm{d}\beta}{\mathrm{d}x}. \tag{8}$$

Assuming that Eqs. (6) and (7) are true for j, it is proven that they are also true for $j+1$. Since the $(j+1)$th order derivative of $\rho\beta^r$ is given by

$$\frac{\mathrm{d}^{j+1}}{\mathrm{d}x^{j+1}}(\rho\beta^r)=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}^j}{\mathrm{d}x^j}(\rho\beta^r)\right]=\frac{\mathrm{d}}{\mathrm{d}x}(\rho\beta^{r-j}\gamma_j)$$

using Eq. (4), Eqs. (6) and (7) are obtained for $j+1$.

Lemma 2. *If Eq.* (4) *is satisfied, then the first derivative of the polynomials* γ_j *satisfy the following relation*:

$$\frac{\mathrm{d}\gamma_j}{\mathrm{d}x}=b_j\gamma_{j-1}, \tag{9}$$

where j is any natural number less than or equal to r, and the constants b_j *are given by*

$$b_j=\frac{j}{2}\left[2\frac{\mathrm{d}\alpha}{\mathrm{d}x}+(2r-j+1)\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2}\right]. \tag{10}$$

Proof. This is proved by induction. If $j=1$, using γ_1 from Eq. (8), it results

$$\frac{\mathrm{d}\gamma_1}{\mathrm{d}x}=b_1\gamma_0 \quad \text{where} \quad b_1=\frac{\mathrm{d}\alpha}{\mathrm{d}x}+2r\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2}.$$

Let us suppose that Eqs. (9) and (10) are true for j and then prove that they are also true for $j+1$. Using Eq. (7) for $j+1$, Eq. (4) and then Eq. (6) for j, it results

$$\frac{\mathrm{d}\gamma_{j+1}}{\mathrm{d}x}=b_{j+1}\gamma_j \quad \text{where} \quad b_{j+1}=b_j+\frac{\mathrm{d}\alpha}{\mathrm{d}x}+(r-j)\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2}.$$

Therefore, using (10) for j, Eq. (10) results for $j+1$.

Consequence 3. *The polynomials* γ_j *satisfy the following recurrence relation*:

$$\gamma_j=\left[\alpha+(r-j+1)\frac{\mathrm{d}\beta}{\mathrm{d}x}\right]\gamma_{j-1}+b_{j-1}\beta\gamma_{j-2}, \tag{11}$$

where j is any natural number greater than or equal to 2 *and less than or equal to r, and* b_j *are given by Eq.* (10). *Also,* $\gamma_0=1$ *and* $\gamma_1=\alpha+r(\mathrm{d}\beta/\mathrm{d}x)$.

Proof. Obviously, it results from Lemmas 1 and 2.

Lemma 4. *If Eqs.* (3) *and* (4) *are satisfied, then the leading coefficients* c_j *of the polynomials* γ_j *are given by*

$$c_j=\begin{cases}1 & \text{if } j=0,\\ \prod_{i=1}^{j}[\alpha_1+(2r-i+1)\beta_2] & \text{if } j\neq 0 \text{ and } j\leqslant r.\end{cases} \tag{12}$$

Proof. This is also proved by induction. If $j=0$, then $\gamma_0=1$ and consequently its leading coefficient is $c_0=1$. If $j=1$, then according to Eqs. (7) and (3), the leading coefficient of γ_1 is $c_1=\alpha_1+2r\beta_2$. Let us assume that Eq. (12) is true for j and prove that it is also true for $j+1$. Using Eq. (7) for $j+1$, the leading coefficient c_{j+1} of γ_{j+1} results as follows:

$$c_{j+1} = [\alpha_1 + (2r - i)\beta_2]c_j.$$

This gives Eq. (12) for $j+1$.

Lemma 5. *If S_k is a sum defined by*

$$S_k = \sum_{j=0}^{r-k} \frac{(n-r)!}{(n-2r+j+k)!} \binom{r-k}{j} \beta_2^{r-j-k} c_j, \tag{13}$$

where k is any nonnegative integer less than r, then the following recurrence relation occurs

$$S_k = [\alpha_1 + (n+k+1)\beta_2]S_{k+1}, \tag{14}$$

where k is any natural number less than $r-1$. Consequently, the sum S_0 is given by

$$S_0 = \prod_{k=0}^{r-1} [\alpha_1 + (n+k+1)\beta_2]. \tag{15}$$

Proof. Since

$$\binom{r-k}{j} = \binom{r-k-1}{j} + \binom{r-k-1}{j-1} \tag{16}$$

the Eq. (13) can be written as

$$S_k = \sum_{j=0}^{r-k-1} \frac{(n-r)!}{(n-2r+j+k)!} \binom{r-k-1}{j} \beta_2^{r-j-k} c_j + \sum_{j=1}^{r-k} \frac{(n-r)!}{(n-2r+j+k)!} \binom{r-k-1}{j-1} \beta_2^{r-j-k} c_j. \tag{17}$$

Changing the summation index of the second sum of the right-hand side from j to $j+1$ and then factoring out, the sum S_k becomes

$$S_k = \sum_{j=0}^{r-k-1} \frac{(n-r)!}{(n-2r+j+k+1)!} \binom{r-k-1}{j} \times \beta_2^{r-j-k-1} c_j \left[\beta_2(n-2r+j+k+1) + \frac{c_{j+1}}{c_j}\right]. \tag{18}$$

According to Eq. (12), the ratio between the coefficients c_{j+1} and c_j is given by

$$c_{j+1}/c_j = [\alpha_1 + (2r - j)\beta_2]. \tag{19}$$

Consequently, Eq. (18) can be rewritten as

$$S_k = [\alpha_1 + (n+k+1)\beta_2] \sum_{j=0}^{r-k-1} \frac{(n-r)!}{(n-2r+j+k+1)!} \binom{r-k-1}{j} \beta_2^{r-j-k-1} c_j. \tag{20}$$

Since the sum of the right-hand side of Eq. (20) is S_{k+1}, the recurrence relation (14) has been proved. In addition, the sum S_{r-1} resulting from Eq. (13) is given by

$$S_{r-1} = [\alpha_1 + (n+r)\beta_2]. \tag{21}$$

Therefore, using the recurrence relations (14) and (21), Eq. (15) is obtained.

Proposition 1. *If conditions* (3)–(5) *are satisfied, then classical orthogonal polynomials* $Q_n(x)$ (*Jacobi, Legendre, Hermite, Laguerre and Tchebycheff*) *satisfy the following differential equation of* $2r$ *order*:

$$\sum_{j=0}^{r} \binom{r}{j} \beta^{r-j} \gamma_j \frac{\mathrm{d}^{2r-j} Q_n}{\mathrm{d}x^{2r-j}} - \lambda_{rn} Q_n = 0 \tag{22}$$

or in self-adjoint form

$$\frac{1}{\rho} \frac{\mathrm{d}^r}{\mathrm{d}x^r} \left(\rho \beta^r \frac{\mathrm{d}^r Q_n}{\mathrm{d}x^r} \right) - \lambda_{rn} Q_n = 0, \tag{23}$$

where r is any natural number and the eigenvalue parameter λ_{rn} *is given by*

$$\lambda_{rn} = \prod_{k=0}^{r-1} (n-k)[\alpha_1 + (n+k+1)\beta_2]. \tag{24}$$

The weight function $\rho(x)$, *the polynomial functions* $\alpha(x)$ *and* $\beta(x)$ *and the coefficients* α_1 *and* β_2 *are given by Eqs.* (3) *and* (4), *and the polynomials* γ_j *and the coefficients* c_j *are given by Eqs.* (11) *and* (12), *respectively. The eigenvalue parameter* λ_{rn} *has nonzero values for* $n \geqslant r$.

Proof. Next integral

$$I = \int_a^b \frac{\mathrm{d}^r}{\mathrm{d}x^r} \left(\rho \beta^2 \frac{\mathrm{d}^r Q_n}{\mathrm{d}x^r} \right) x^k \, \mathrm{d}x, \tag{25}$$

where k is any nonnegative integer less than n, will be calculated in two different ways. First, through integrating repeatedly by parts and using Eqs. (4) and (5), it is proved that the integral (25) is zero. Calculating in this way, the integral I becomes

$$I = (-1)^r \frac{k!}{(k-r)!} \int_a^b Q_n \frac{\mathrm{d}^r}{\mathrm{d}x^r} (\rho \beta^r x^{k-r}) \, \mathrm{d}x. \tag{26}$$

Applying Leibniz' rule for the rth order derivative under the integral and then using Eq. (6), the integral becomes

$$I = (-1)^r \frac{k!}{(k-r)!} \int_a^b \rho Q_n p_{nk} \, \mathrm{d}x, \tag{27}$$

where p_{nk} is a polynomial of kth degree given by

$$p_{nk} = \sum_{j=0}^{n} \binom{r}{j} \beta^{r-j} \gamma_j \frac{\mathrm{d}^{r-j}}{\mathrm{d}x^{r-j}} (x^{k-r}). \tag{28}$$

As $Q_n(x)$ polynomials are orthogonal to any polynomial of a degree strictly smaller than n and $p_{nk}(x)$ are polynomials of degree less than or equal to $n-1$, the integral I is zero

$$I = 0. \tag{29}$$

Second, calculating the derivatives under integral (25) and then using Eq. (4), it will finally be proved that λ_{rn} is given by (24) due to (29). Calculating under the integral, the relation (25) becomes

$$I = \int_a^b \rho x^k \sum_{j=0}^{r} \binom{r}{j} \beta^{r-j} \gamma_j \frac{\mathrm{d}^{2r-j} Q_n}{\mathrm{d}x^{2r-j}} \,\mathrm{d}x. \tag{30}$$

Using the nomenclature of the inner product (1) into Eq. (30), Eq. (29) can be rewritten as

$$\left(x^k, \sum_{j=0}^{r} \binom{r}{j} \beta^{r-j} \gamma_j \frac{\mathrm{d}^{2r-j} Q_n}{\mathrm{d}x^{2r-j}} \right)_\rho = 0, \tag{31}$$

where k is any nonnegative integer less than n. According to the theorem of uniqueness of orthogonal polynomials, we can write

$$\sum_{j=0}^{r} \binom{r}{j} \beta^{r-j} \gamma_j \frac{\mathrm{d}^{2r-j} Q_n}{\mathrm{d}x^{2r-j}} = \lambda_{nr} Q_n, \tag{32}$$

where λ_{rn} is a constant to be determined. Therefore, writing the nth degree orthogonal polynomial as $Q_n(x) = q_n x^n + q_{n-1} x^{n-1} + \cdots + q_0$, where the leading coefficient is $q_n \neq 0$, and then equating the coefficients of x^n of both sides of Eq. (32), the parameters λ_{rn} result as follows:

$$\lambda_{rn} = \sum_{j=0}^{r} \frac{n!}{(n-2r+j)!} \binom{r}{j} \beta_2^{r-j} c_j.$$

Therefore, according to Lemma 5, the eigenvalue λ_{rn} is given by Eq. (24). Further, substituting λ_{rn} into Eq. (32), the differential equation (22), or in self-adjoint form (23), is obtained.

3. Eigenvalue singular problems

Eigenvalue problems, either regular or singular, associated with differential equations are frequently encountered in practice in connection with physical and engineering problems. The eigenvalue singular problem associated with the second-order differential equation Sturm–Liouville (2) is well known. The eigenvalue singular problem associated with the $2r$-order differential equation (22), or in self-adjoint form (23), over the interval $[a, b]$ is presented as follows:

Proposition 2. *We consider the* $2r$ *order differential equation with* $x=a$ *and* $x=b$ *singular points*

$$\sum_{j=0}^{r}\binom{r}{j}\beta^{r-j}\gamma_j\frac{\mathrm{d}^{2r-j}Y}{\mathrm{d}x^{2r-j}}-\lambda_r Y=0 \tag{33}$$

or in self-adjoint form

$$\frac{1}{\rho}\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left(\rho\beta^r\frac{\mathrm{d}^rY}{\mathrm{d}x^r}\right)-\lambda_r Y=0, \tag{33'}$$

where λ_r *is a real constant and* $[a,b]$ *is the interval of orthogonality. If relations* (3)–(5) *are satisfied and the following end conditions are met*:

$$Y(a),Y(b)\quad \textit{finite} \tag{34}$$

then the unique eigenvalues λ_{rn} *and eigenfunctions* $Y_n(x)$ *are*

$$\lambda_{rn}=\prod_{k=0}^{r-1}(n-k)[\alpha_1+(n+k+1)\beta_2] \tag{35}$$

and

$$Y_n(x)=Q_n(x). \tag{36}$$

$Q_n(x)$ *are the orthogonal polynomials of the considered system, and the polynomials* γ_j *are given by Eq.* (11). *The eigenvalue parameter* λ_{rn} *has nonzero values for* $n\geqslant r$.

Proof. First, it is proved that (35) and (36) are eigenvalues and eigenfunctions of problem (33)–(34). As shown in Proposition 1, (35) and (36) satisfy Eq. (33). As $Q_n(x)$ are polynomials, they are finite at $x=a$, $x=b$, and consequently they meet the requirements (34). Therefore, λ_{rn} and $Q_n(x)$ are eigenvalues and eigenfunctions of problem (33)–(34).

Second, it is proven by contradiction that the eigenvalue singular problem (33)–(34) does not admit eigenvalues and eigenfunctions other than (35) and (36), respectively. Multiplying Eq. (33) by $\rho(x)$, this equation can be rewritten in a self-adjoint form as

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left(\rho\beta^r\frac{\mathrm{d}^rY}{\mathrm{d}x^r}\right)-\lambda_r\rho Y=0. \tag{37}$$

Suppose that λ_{rn} and $Q_n(x)$ are not the unique eigenvalues and eigenfunctions of the eigenvalue singular problem (33)–(34). Let λ^* and $Y^*(x)$ be an eigenvalue and an eigenfunction, respectively, other than any λ_{rn} and $Q_n(x)$. Two eigenvalue singular problems are satisfied by λ_{rn} and $Q_n(x)$, and λ^* and $Y^*(x)$, respectively, as follows:

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left(\rho\beta^r\frac{\mathrm{d}^rQ_n}{\mathrm{d}x^r}\right)-\lambda_{rn}\rho Q_n=0\quad\text{and}\quad Q_n(a),\,Q_n(b)\ \text{finite} \tag{38}$$

and

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left(\rho\beta^r\frac{\mathrm{d}^rY^*}{\mathrm{d}x^r}\right)-\lambda^*\rho Y^*=0\quad\text{and}\quad Y^*(a),\,Y^*(b)\ \text{finite}. \tag{39}$$

Multiplying Eqs. (38) and (39) by $Y^*(x)$ and $Q_n(x)$, respectively, then subtracting the second equation from the first and integrating the resulting equation from *a* to *b*, the following equation is obtained:

$$(\lambda^* - \lambda_{rn}) \int_a^b (\rho Q_n Y^*)\,\mathrm{d}x = \int_a^b Q_n \frac{\mathrm{d}^r}{\mathrm{d}x^r}\left(\rho\beta^r \frac{\mathrm{d}^r Y^*}{\mathrm{d}x^r}\right)\mathrm{d}x - \int_a^b Y^* \frac{\mathrm{d}^r}{\mathrm{d}x^r}\left(\rho\beta^r \frac{\mathrm{d}^r Q_n}{\mathrm{d}x^r}\right)\mathrm{d}x. \quad (40)$$

Using relations (4)–(6), and the endpoint conditions of (38) and (39), the right-hand side of Eq. (40) is zero. As λ^* has been assumed not to be equal to any λ_{rn}, the difference $\lambda^* - \lambda_{rn}$ cannot be zero. Therefore, the next integral, which is an inner product between any $Q_n(x)$ and $Y^*(x)$, has to be zero

$$(Q_n, Y^*)_\rho = \int_a^b \rho Q_n Y^*\,\mathrm{d}x = 0, \quad (41)$$

where *n* is any nonnegative integer. As the function $Y^*(x)$ satisfies Eq. (39), it is continuous on the interval (a, b) along with its first $2r - 1$ derivatives. Moreover, $Y^*(a)$ and $Y^*(b)$ are assumed to be finite (30), so the function $Y^*(x)$ is continuous on the closed interval $[a, b]$. Thus, $Y^*(x)$ can be expanded in terms of orthogonal polynomials as follows:

$$Y^*(x) = c_0 Q_0(x) + c_1 Q_1(x) + \cdots + c_n Q_n(x) + \cdots, \quad (42)$$

where c_n are real coefficients. Multiplying Eq. (42) by ρQ_n, where *n* is any nonnegative integer, and then integrating over the interval (a, b), due to Eq. (41), results in

$$c_n (Q_n, Q_n)_\rho = (Q_n, Y^*)_\rho = 0. \quad (43)$$

The nomenclature presented in (1) has been used here. Since $Q_n(x)$ are orthogonal polynomials, the inner product $(Q_n, Q_n)_\rho$ cannot be zero. So, $c_n = 0$ for any nonnegative integer *n*. Therefore, the function Y^* has to be identically zero. This is a contradiction, since $Y^*(x)$ has been considered an eigenfunction, which cannot be identically zero. Therefore, the assumption that the eigenvalues and eigenfunctions, λ_{rn} and $Q_n(x)$, respectively, are not unique is false. Consequently, the proposition has been proved.

4. General self-adjoint equation and eigenvalue problem

Since classical orthogonal polynomials satisfy any self-adjoint differential Eq. (23), they will satisfy any linear combination of these equations, as can be proved by another two propositions similar to Propositions 1 and 2. So, a general self-adjoint differential equation of classical orthogonal polynomials $Q_n(x)$, can be written as follows:

$$\sum_{i=1}^{r} c_i \frac{\mathrm{d}^i}{\mathrm{d}x^i}\left(\rho\beta^i \frac{\mathrm{d}^i Q_n}{\mathrm{d}x^i}\right) - \mu_{rn}\rho Q_n = 0, \quad (44)$$

where c_i are any real constants, *n* is any nonnegative integer, and the eigenvalue parameter μ_{rn} is given by

$$\mu_{nr} = \sum_{i=1}^{r} c_i \lambda_{ni}, \quad (45)$$

where λ_{in} are given by Eq. (24). In addition, the singular eigenvalue problem associated with this general self-adjoint differential equation is written as

$$\sum_{i=1}^{r} c_i \frac{\mathrm{d}^i}{\mathrm{d}x^i}\left(\rho\beta^i \frac{\mathrm{d}^i Y}{\mathrm{d}x^i}\right) - \mu_r \rho Y = 0, \quad Y(a), Y(b) \text{ finite} \tag{46}$$

and it gives the unique eigenvalues and eigenfunctions $\mu_{rn} = \sum_{i=1}^{r} c_i \lambda_{in}$ and $Y_n(x) = Q_n(x)$, respectively.

5. Second-, fourth- and sixth-order differential equations

If the order of differential equation (22), or in self-adjoint form (23), is two, then $r = 1$ and the second-order differential equation is Eq. (2), [5,15], where $\lambda_{1n} = n[\alpha_1 + (n+1)\beta_2]$, $n \geqslant 1$.

If the order of Eqs. (22) and (23) is four, then $r = 2$ and the fourth-order differential equation is

$$\beta^2 \frac{\mathrm{d}^4 Q_n}{\mathrm{d}x^4} + 2\beta\gamma \frac{\mathrm{d}^3 Q_n}{\mathrm{d}x^3} + \left[\frac{\mathrm{d}}{\mathrm{d}x}(\beta\gamma) + \alpha\gamma\right] \frac{\mathrm{d}^2 Q_n}{\mathrm{d}x^2} - \lambda_{2n} Q_n = 0 \tag{47}$$

or in self-adjoint form

$$\frac{1}{\rho} \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\rho\beta^2 \frac{\mathrm{d}^2 Q_n}{\mathrm{d}x^2}\right) - \lambda_{2n} Q_n = 0, \tag{48}$$

where λ_{2n} and the polynomial function $\gamma(x)$ are respectively given by

$$\lambda_{2n} = (n-1)n[\alpha_1 + (n+1)\beta_2][\alpha_1 + (n+2)\beta_2], \tag{49}$$

$$\gamma = \alpha + 2\frac{\mathrm{d}\beta}{\mathrm{d}x} \tag{50}$$

and $n \geqslant 2$ for $\lambda_{2n} \neq 0$. This fourth-order differential equation can be found in [3].

If the order of Eqs. (22) and (23) is six, then $r = 3$ and the sixth-order differential equation in self-adjoint form is

$$\frac{1}{\rho} \frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(\rho\beta^3 \frac{\mathrm{d}^3 Q_n}{\mathrm{d}x^3}\right) - \lambda_{3n} Q_n = 0, \tag{51}$$

where

$$\lambda_{3n} = (n-2)(n-1)n[\alpha_1 + (n+1)\beta_2][\alpha_1 + (n+2)\beta_2][\alpha_1 + (n+3)\beta_2] \tag{52}$$

and $n \geqslant 3$ for $\lambda_{3n} \neq 0$.

6. Application

The above concepts can be used to study bending vibrations of nonuniform beams. Let us find eigenfrequencies and mode shapes for free bending vibrations of a beam of length L with free-free boundary conditions and a circular cross-section whose radius is as follows:

$$R(x) = R_0\left(1 - \frac{4x^2}{L^2}\right), \quad x \in \left(-\frac{L}{2}, \frac{L}{2}\right). \tag{53}$$

The Euler–Bernoulli differential equation of bending vibration of beams is

$$\frac{d^2}{dx^2}\left(EI_1(x)\frac{d^2Y(x)}{dx^2}\right) - \rho_0\omega^2 A_1(x)Y(x) = 0, \tag{54}$$

where $Y(x)$ is the transversal displacement, $A_1(x)$ and $I_1(x)$ are the area and the moment of inertia of the current cross-section, respectively; E, ρ_0 and ω are Young modulus, mass density and eigenfrequency, respectively, and x is the current longitudinal coordinate of the beam. According to (53) this is a beam with sharp ends. So, the boundary conditions are reduced to

$$Y\left(-\frac{L}{2}\right), \quad Y\left(\frac{L}{2}\right) \quad \text{finite.} \tag{55}$$

Using the change of variable $x = L\xi/2$, the Euler–Bernoulli differential equation (54) along with boundary conditions is

$$\frac{d^2}{d\xi^2}\left[(1-\xi^2)^4\frac{d^2Y}{d\xi^2}\right] - \lambda(1-\xi^2)^2 Y = 0, \quad Y(-1), Y(1) \quad \text{finite,} \tag{56}$$

where λ is given by

$$\lambda = \frac{\rho_0\omega^2 A_0 L^4}{16EI_0} \tag{57}$$

and where A_0 and I_0 are the cross-sectional area and moment of inertia at the reference longitudinal coordinate $\xi = 0$ and are given by

$$A_0 = \pi R_0^2, \quad I_0 = \frac{\pi R_0^4}{4}. \tag{58}$$

Problem (56) represents an eigenvalue singular problem of orthogonal polynomials. The functions $\rho(\xi)$ and $\beta(\xi)$, and $\alpha(\xi)$, resulting from Eqs. (48) and (56), and Eq. (4), respectively, are given by

$$\rho(z) = (1-\xi^2)^2, \quad \beta(z) = (1-\xi^2), \quad \alpha(\xi) = -4\xi. \tag{59}$$

The interval of orthogonality is $(-1, 1)$. The weight function $\rho(\xi)$ given by Eq. (59) is the weight function of Jacobi orthogonal polynomials $J_n^{p,q}(\xi)$, specifically, the case $p = q = 2$. Since the eigenfunctions and the eigenvalues of problem (56) are $J_n^{p,q}(\xi)$ and λ_{n2} (see (49)), respectively, the mode shapes $Y_n(x)$ and the eigenfrequencies ω_n are as follows:

$$Y_n(\xi) = J_{n+1}^{2,2}(\xi), \tag{60}$$

$$\omega_n = 4\sqrt{n(n+1)(n+6)(n+7)}\frac{1}{L^2}\sqrt{\frac{EI_0}{\rho_0 A_0}}, \tag{61}$$

where n is any natural number. The coefficients $\beta_2 = -1$ and $\alpha_1 = -4$, resulted from (59) and they were used in (49) to obtain (61). The lowest dimensionless eigenfrequency and its corresponding mode shape are as follows:

$$\omega_1 = 16\sqrt{7}\,\frac{1}{L^2}\sqrt{\frac{EI_0}{\rho_0 A_0}}, \quad Y_1(\xi) = c_1(1 - 7\xi^2), \tag{62}$$

where c_1 is a constant of proportionality.

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