GRADIENT DESCENT ALGORITHM

Intro to Deep Learning

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Optimization

Finding best `a` and `b` is an optimization problem.

From now on, we will use **w** instead of `a` to represent slope. Our linear model is `y = wx + b`,

Our goal is to estimate **w** (slope) and **b** that minimizes the Cost Function (MSE) given a particular data set (x and y)

Let's assume that it's linear model and we have a data set as follows.

x	y
1.0	2.0
2.0	3.0
3.0	4.0

What is optimal `w` and `b`? Can you guess?

Yes! Intuitively, we should be able to guess the answer.

w = 1 and **b = 1**

(if you could not guess that w and b should both be 1, try to picture the points on a graph in your mind. As x increases by 1, we can see that y increases by 1 as well so that tells us the slope of the graph. We then know can remember that in our high-school math class to find b, we simply put 0 into x and see what the result is, in this case when x = 0, y = 1 as expected)

But please let's assume we don't know the answer.

Example

x	y
1.0	2.0
2.0	3.0
3.0	4.0

- To find optimal `w` and `b`, the first step is to set `w` and `b` as 0 (zero).
- Then, estimate `w` first. (we don't change the value of `b`)
- We just calculate the cost (MSE) when w is 0, 1, 2. It is like a simple searching with different values.
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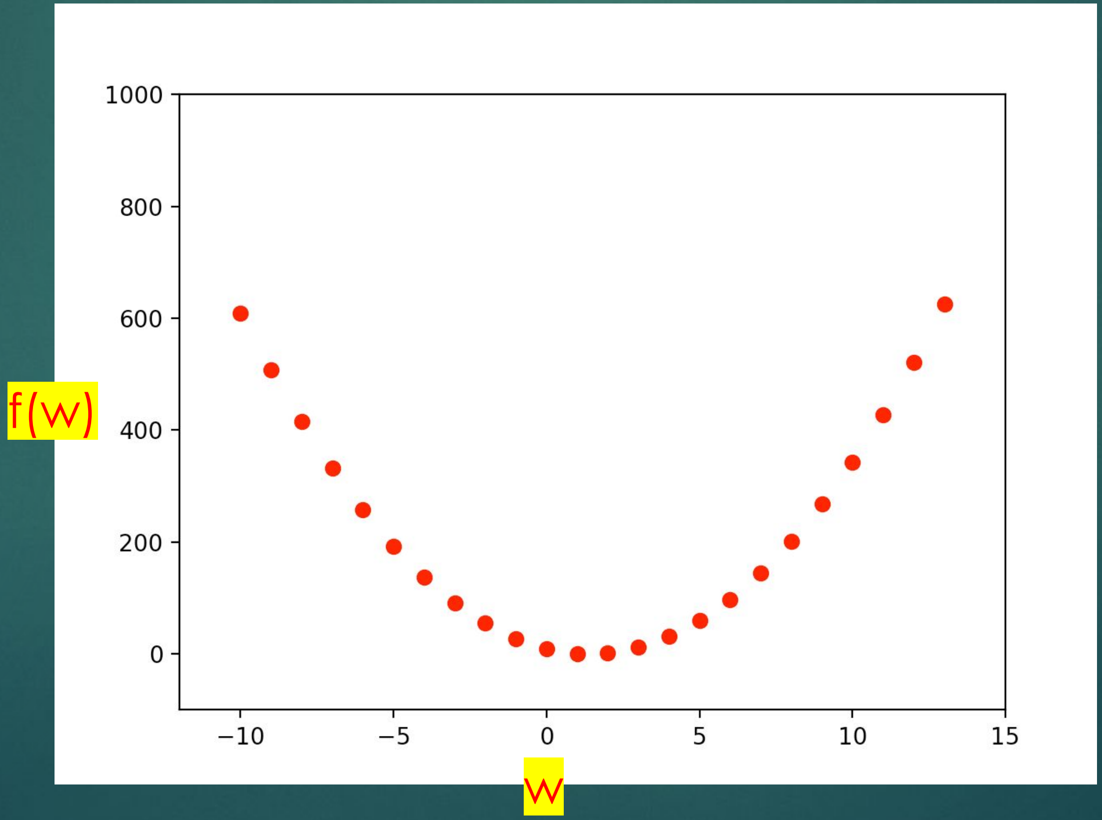
- w = 0, b = 0
  - Cost = ((1*0 + 0 - 2)^2 + (2*0 + 0 - 3)^2 + (3*0 + 0 - 4)^2) / 3 = 9.67
- w = 1, b = 0
  - Cost = ((1*1 + 0 - 2)^2 + (2*1 - 3)^2 + (3*1 - 4)^2) / 3 = 1
- w = 2, b = 0
  - Cost = ((1*2 + 0 - 2)^2 + (2*2 - 3)^2 + (3*2 - 4)^2) / 3 = 1.67

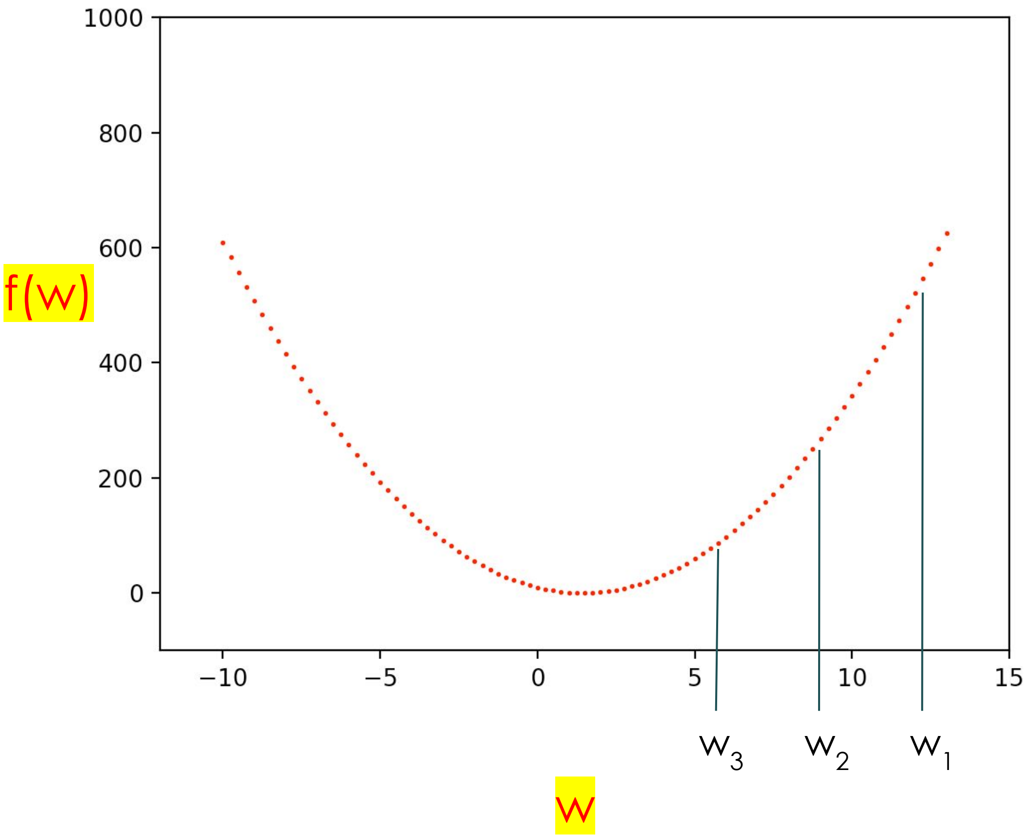
Example

- **Cost function** of w (given the data and b = 0)



The y-axis represents the error, and the x-axis represents `w`. The point with the smallest error is the lowermost convex part of the graph. That is, when w is 1, the error is the smallest. However, **since we assume we do not know the answer**, to find the optimal `w`, we need to calculate the error for a random point w_1 and move w to the side where the error decreases. In other words, the error is smaller for w_2 than for w_1. The error is smaller for w_3 than for w_2.

Gradient descent is a method to find `w` with the smallest error by comparing errors in this way.

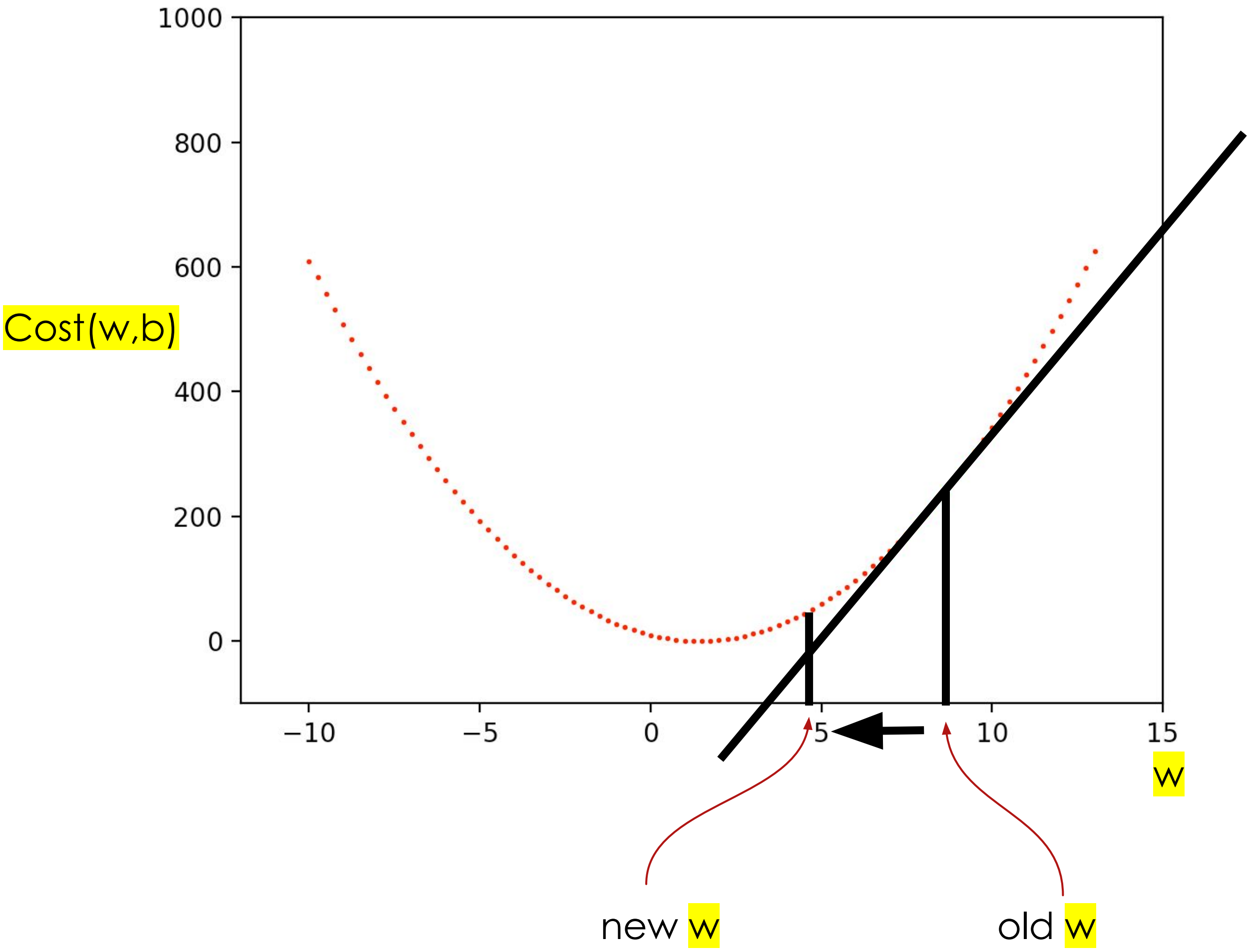


Gradient Descent Algorithm

- Step1: Initialize to 0
- Step2: update w and b that reduce cost function
- Use a gradient of cost function
- Step3: if w and b converge, stop step2. Otherwise, repeat step2
- New w

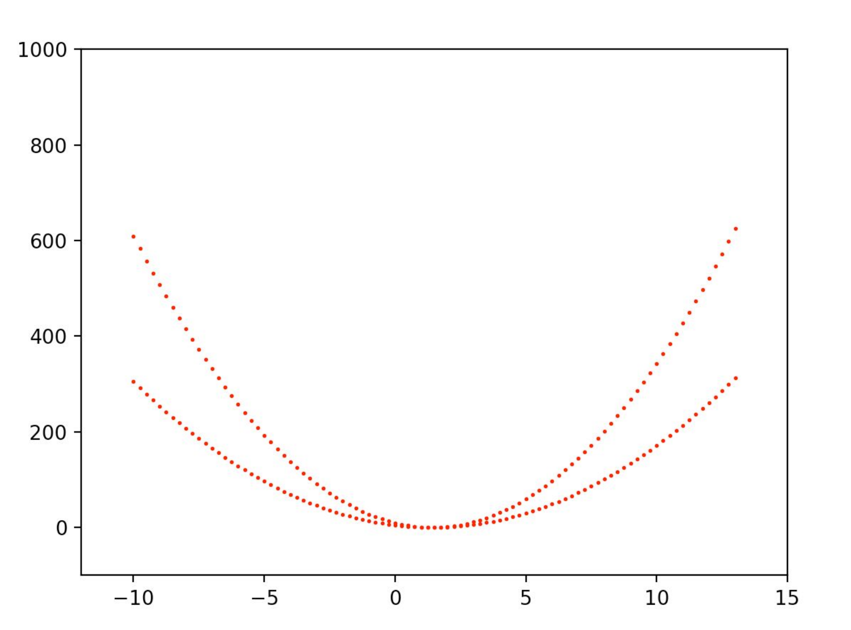
$$w' := w - \alpha \cdot \frac{\partial}{\partial w} Cost(w, b)$$

- α is "Learning Rate"



Little modification

- For optimization, these two cost functions will have same w and b to minimize each cost function
- $\min_{w} \frac{1}{n} \sum_{i=1}^{n}(wx_i + b - y_i)^2$
- $\min_{w} \frac{1}{2n} \sum_{i=1}^{n}(wx_i + b - y_i)^2$



Partial Differential

- $w' := w - \alpha \cdot \frac{\partial}{\partial w} Cost(w, b)$
- $w' := w - \alpha \cdot \frac{\partial}{\partial w} \frac{1}{2n} \sum_{i=1}^{n} (wx_i + b - y_i)^2$
- $w' := w - \alpha \cdot \frac{1}{2n} \sum_{i=1}^{n} (wx_i + b - y_i) 2x_i$
- $w' := w - \alpha \cdot \frac{1}{n} \sum_{i=1}^{n} (wx_i + b - y_i) x_i$ ← **Update Rule**

Partial Differential

$$b' = b + \alpha \cdot \frac{b}{\partial b} Cost(w, b)$$

$$b' = b + \alpha \cdot \frac{b}{\partial b} \frac{1}{2n} \sum_{i=1}^{n} (wx_i + b - y_i)^2$$

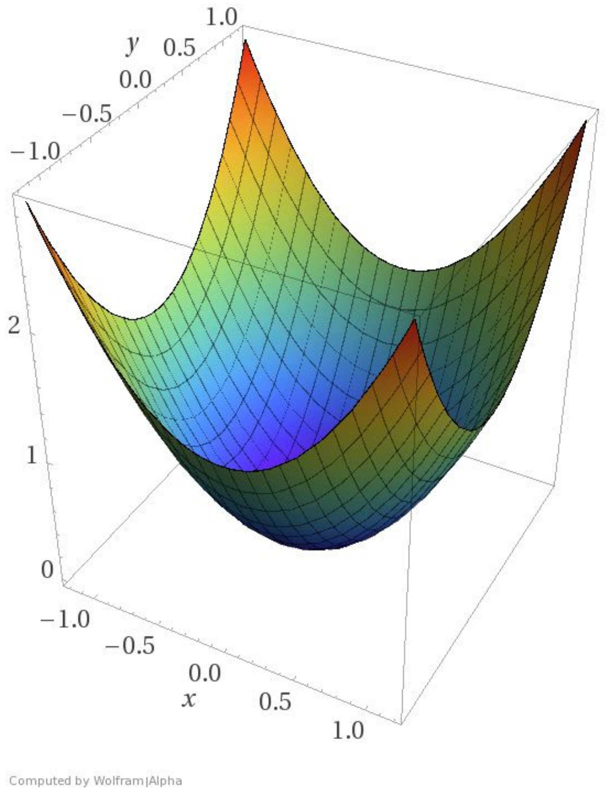
$$b' = b + \alpha \cdot \frac{1}{n} \sum_{i=1}^{n} (wx_i + b - y_i)$$

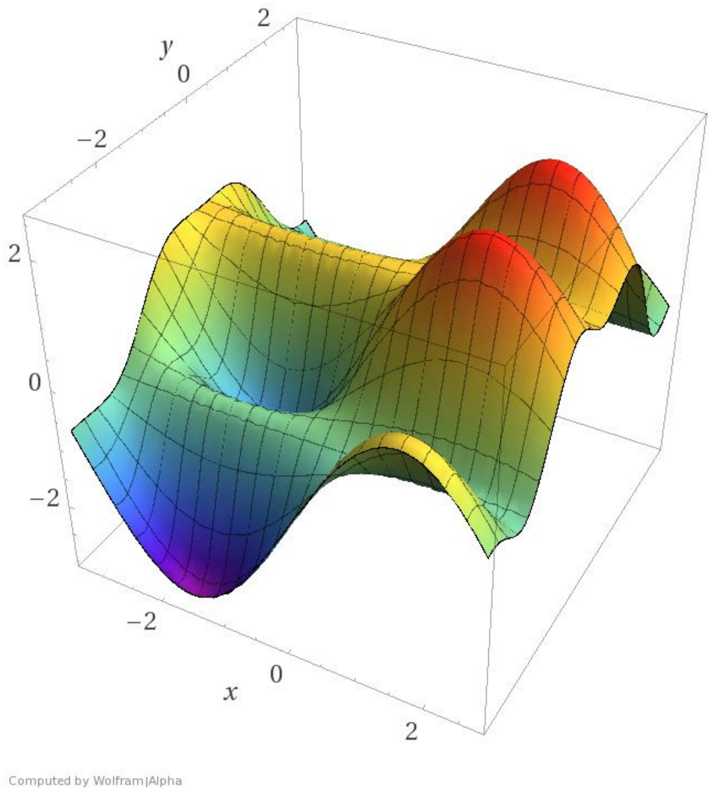
Gradient Descent Algorithm

$$w' := w - \alpha \cdot \frac{1}{n}\sum_{i=1}^{n}(wx_i + b - y_i)x_i$$

$$b' = b + \alpha \cdot \frac{1}{n}\sum_{i=1}^{n}(wx_i + b - y_i)$$

Convex Function



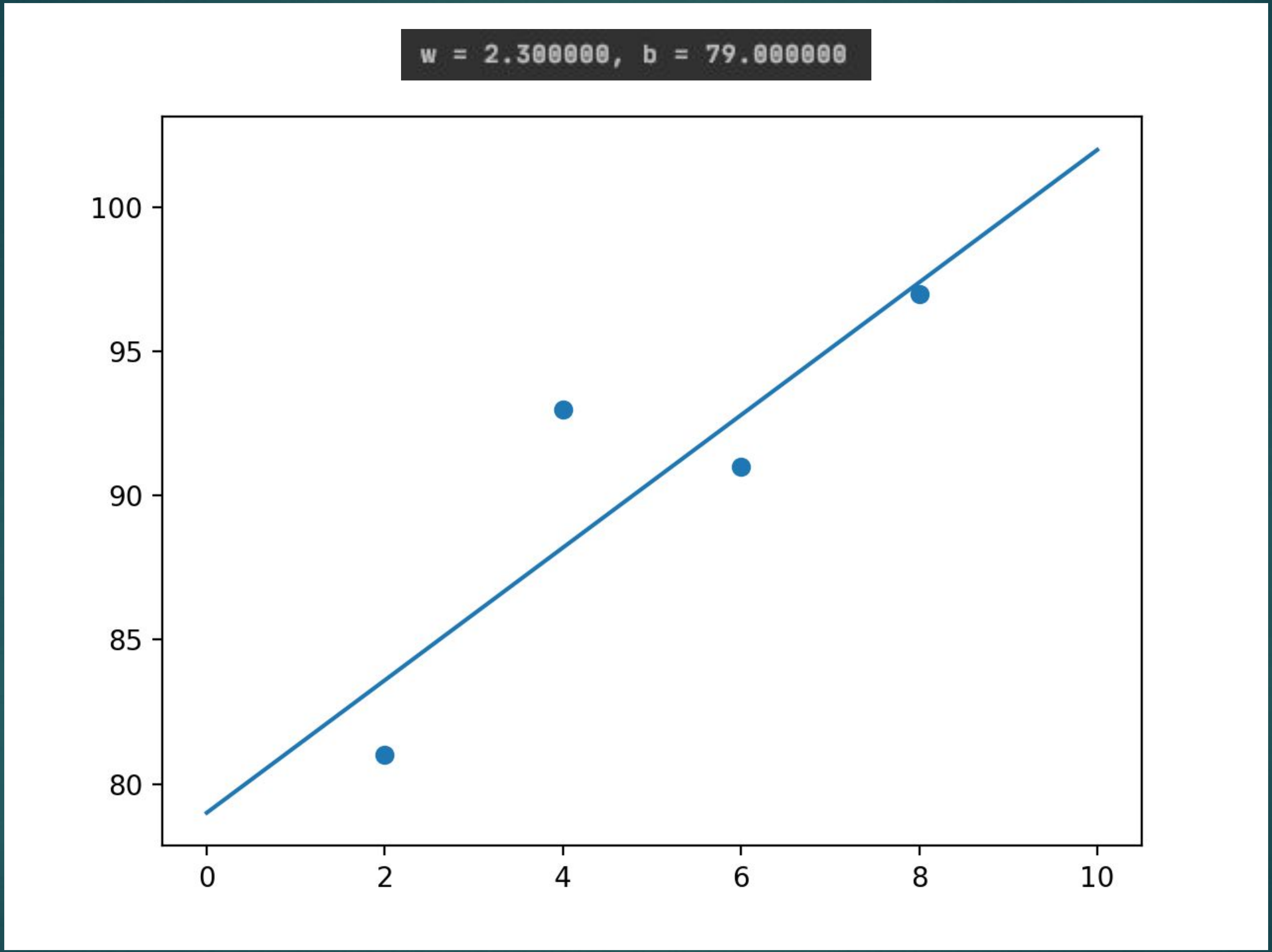


```python
import numpy as np
import matplotlib.pyplot as plt

data = np.array([[2, 81], [4, 93], [6, 91], [8, 97]])
x = data[:, 0]
y = data[:, 1]

# initialization
w, b = 0, 0
# learning rate
alpha = 0.05

plt.scatter(x, y)
xl = np.linspace(0, 10, 100)
# GD
for i in range(2000):
    w = w - alpha * (1/len(data)) * sum((w * x + b - y) * x)
    b = b - alpha * (1/len(data)) * sum((w * x + b - y))
print("w = %f, b = %f" % (w, b))
plt.plot(xl, w * xl + b)
plt.show()
```



w = 2.300000, b = 79.000000
100
95
90
85
80
0
2
4
6
8
10

Lab 9 - Gradient Descent Algorithm

- Write a program that estimates **optimal** w and b by implementing GD algorithm given a data below. (Hint: GD algorithm)
- Answer is about w = 2.x and b = 1.x.

x	y
2.3	6.13
1.2	4.71
4.3	11.13
5.7	14.29
3.5	9.54
8.9	22.43

Next time

Today, we have talked about how to optimize parameters (w and b) given a linear model that has only a single independent variable (x).

What if there are multiple variables?

For example, $y = w_1x_1 + w_2x_2 + w_3x_3 + b$

Next time, we are going to talk about how to estimate the parameters given a multivariable linear model.