Computer Handout 9: MA, AR and ARMA Models
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This Computer Handout 9 will cover the estimation of MA, AR and ARMA models and how to select between competing models using AIC, SIC, the autocorrelation function and the partial autocorrelation function. For this exercise we will start with the same file as computer handout 8.

Artificial Data: MA(1) Process

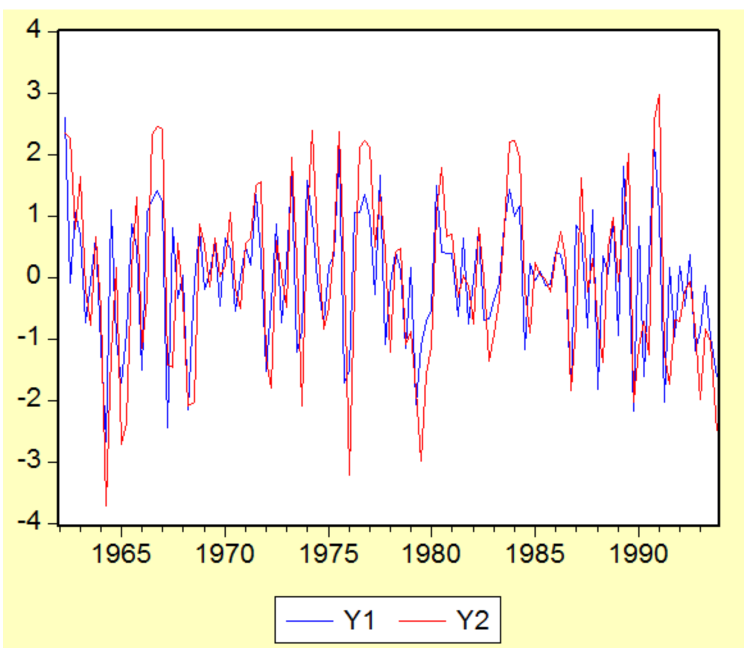
Lets generate an artificial MA(1) process.

Generate the random variable epsilon: genr epsilon=nrnd

Now, generate the following three MA(1) processes:

```
genr Y1=epsilon+0.08*epsilon(-1)
genr Y2=epsilon+0.98*epsilon(-1)
genr Y3=epsilon-0.98*epsilon(-1)
```

Graph Y1 and Y2. Notice that Y2 is more volatile than Y1. Recall the formula for the variance?



Now, let's look at the autocorrelation and the partial autocorrelation functions. Notice that even though the series have undistinguishable forms based on just looking at the graph, they have very characteristic autocorrelation and partial correlation functions.

For Y1, the coefficient on the lagged espilon is so small that based on the autocorrelation and the partial autocorrelation function, we conclude that is is white noise (all coefficients are within the bands). Moreover, the Ljung-Box Q-statistic reaches the same conclusion.

Correlogram of Y1

Date: 10/22/10 Time: 01:00
Sample: 1962Q1 1993Q4
Included observations: 127

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.059	-0.059	0.4460	0.504
		2	-0.104	-0.108	1.8631	0.394
		3	0.069	0.057	2.4893	0.477
		4	-0.014	-0.018	2.5157	0.642
		5	-0.040	-0.029	2.7299	0.742
		6	0.013	0.001	2.7513	0.839
		7	-0.087	-0.093	3.7816	0.805
		8	-0.140	-0.149	6.4704	0.595
		9	0.099	0.063	7.8320	0.551
		10	-0.132	-0.150	10.261	0.418
		11	-0.150	-0.147	13.451	0.265
		12	0.083	0.017	14.437	0.274

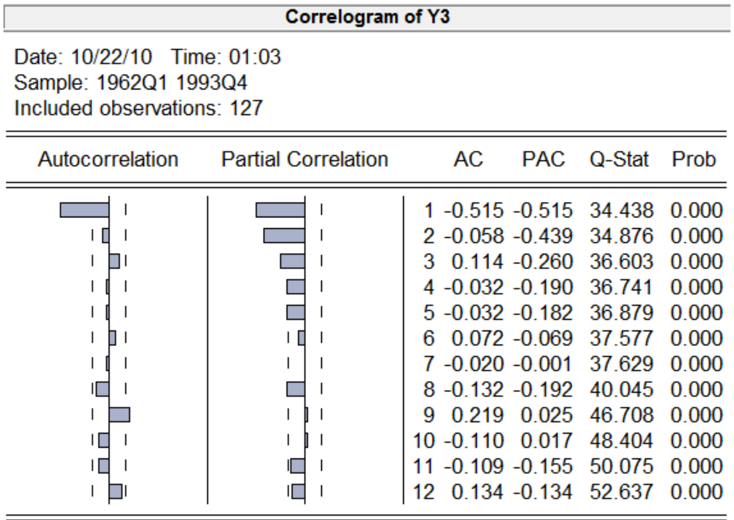
For Y2, theretically speaking, $\rho(\tau) = \theta/(1+\theta^2) = 0.98/(1+0.98^2) = 0.499$. The simulated series gives un 0.358. Because $\theta>0$, the coefficients of the partial autocorrelation function alternate signs. Ljung-Box Q-statistic rejects white noise.

Correlogram of Y2

Date: 10/22/10 Time: 01:02
Sample: 1962Q1 1993Q4
Included observations: 127

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.358	0.358	16.710	0.000
		2	-0.150	-0.319	19.648	0.000
		3	0.017	0.258	19.684	0.000
		4	-0.011	-0.245	19.701	0.001
		5	-0.050	0.164	20.034	0.001
		6	-0.044	-0.205	20.298	0.002
		7	-0.165	-0.056	24.010	0.001
		8	-0.129	-0.051	26.301	0.001
		9	-0.004	-0.007	26.304	0.002
		10	-0.164	-0.263	30.092	0.001
		11	-0.194	0.046	35.408	0.000
		12	0.045	-0.021	35.698	0.000

For Y3, we should have $\rho(\tau) = \theta/(1+\theta^2) = -0.98/(1+(-0.98)^2) = -0.499$. The simulated series gives un -0.515. Because $\theta<0$, the coefficients of the partial autocorrelation function are all negative. Ljung-Box Q-statistic rejects white noise.



Correlogram of Y3

Date: 10/22/10 Time: 01:03
Sample: 1962Q1 1993Q4
Included observations: 127

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.515	-0.515	34.438	0.000
		2	-0.058	-0.439	34.876	0.000
		3	0.114	-0.260	36.603	0.000
		4	-0.032	-0.190	36.741	0.000
		5	-0.032	-0.182	36.879	0.000
		6	0.072	-0.069	37.577	0.000
		7	-0.020	-0.001	37.629	0.000
		8	-0.132	-0.192	40.045	0.000
		9	0.219	0.025	46.708	0.000
		10	-0.110	0.017	48.404	0.000
		11	-0.109	-0.155	50.075	0.000
		12	0.134	-0.134	52.637	0.000

Artificial Data: AR(1) Process

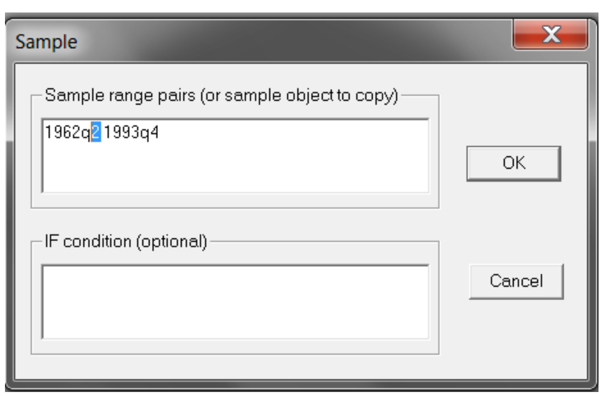
Lets generate an artificial AR(1) process.

Generate the random variable epsilon: genr epsilon=nrnd

Now, generate the following four AR(1) processes. First you need to create the series:

genr Z1 = 0
genr Z2 = 0
genr Z3 = 0
genr Z4 = 0

Then modify the sample to exclude the first observation. (this is just a trick to make sure we eliminate the first observation)

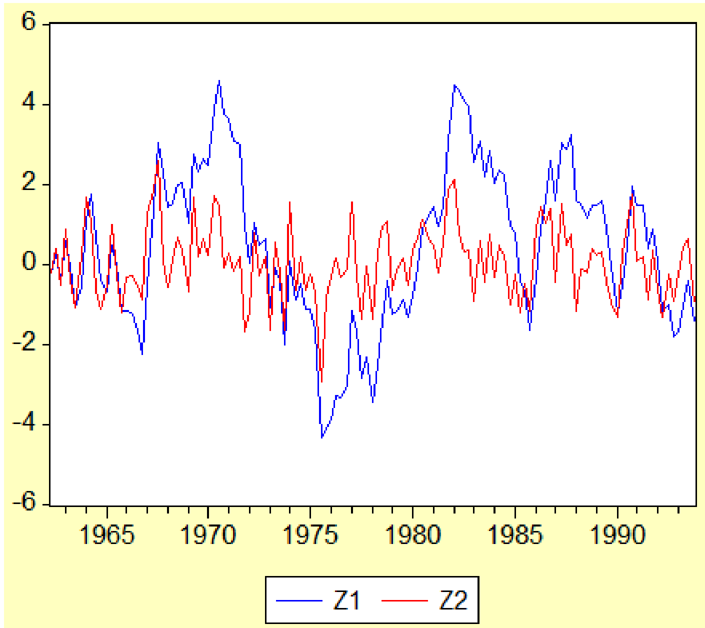


Now, proceed to create the series:

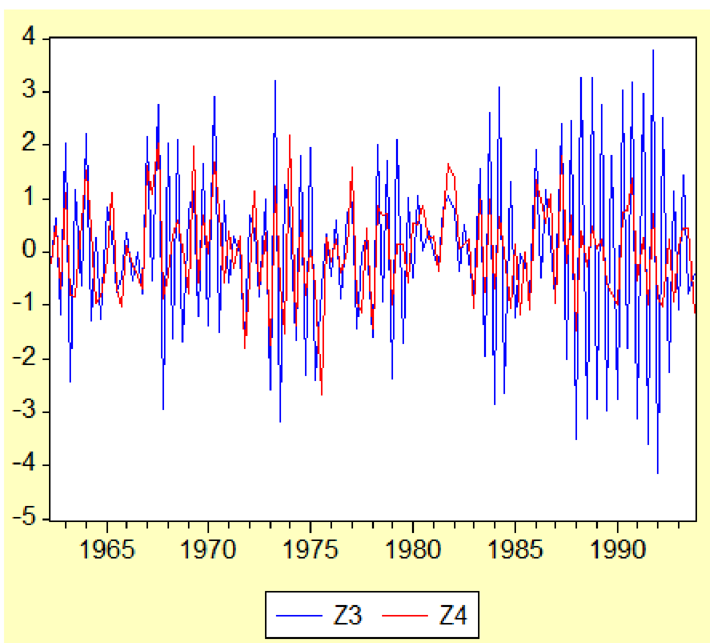
genr Z1 = +0.90*Z1(-1) + epsilon
genr Z2 = +0.20*Z2(-1) + epsilon
genr Z3 = -0.90*Z3(-1) + epsilon

genr Z4 = -0.20*Z4(-1) + epsilon

To see how a simple difference in the sign and the magnitude (size) of the autoregressive coefficient ϕ can have important differences in the series, let's graph Z1 and Z2:



And a graph of Z3 and Z4:



First, notice how different are the Z1 and Z2 series (positive ϕ) when compared with the Z3 and Z4 series (negative ϕ). Second, notice how a larger ϕ (either positive or negative) generates a series with higher dispersion (variance).

Now, let's see what the autocorrelation functions and the partial autocorrelation functions have to say:

Correlogram of Z1

Date: 10/25/10 Time: 02:09
Sample: 1962Q2 1993Q4
Included observations: 127

	AC	PAC	Q-Stat	Prob
1	0.885	0.885	101.89	0.000
2	0.779	-0.020	181.47	0.000
3	0.668	-0.081	240.47	0.000
4	0.589	0.080	286.71	0.000
5	0.504	-0.071	320.79	0.000
6	0.412	-0.095	343.79	0.000
7	0.385	0.258	363.99	0.000
8	0.326	-0.184	378.59	0.000
9	0.273	-0.044	388.93	0.000
10	0.188	-0.105	393.86	0.000
11	0.126	-0.015	396.08	0.000
12	0.095	0.110	397.36	0.000
13	0.035	-0.142	397.55	0.000
14	-0.011	-0.049	397.56	0.000
15	-0.079	-0.086	398.49	0.000
16	-0.131	-0.111	401.02	0.000
17	-0.183	0.040	406.00	0.000
18	-0.226	0.009	413.67	0.000
19	-0.265	-0.137	424.34	0.000
20	-0.299	0.021	438.03	0.000
21	-0.324	-0.107	454.30	0.000
22	-0.336	0.083	471.91	0.000
23	-0.360	-0.095	492.34	0.000
24	-0.382	-0.075	515.57	0.000
25	-0.412	-0.075	542.79	0.000
26	-0.431	-0.073	572.95	0.000
27	-0.432	0.058	603.48	0.000
28	-0.427	0.037	633.66	0.000
29	-0.410	-0.070	661.71	0.000
30	-0.387	0.031	687.04	0.000
31	-0.362	-0.080	709.36	0.000
32	-0.335	0.018	728.69	0.000
33	-0.305	0.095	744.87	0.000
34	-0.252	0.022	756.07	0.000
35	-0.204	0.010	763.47	0.000
36	-0.146	0.004	767.29	0.000

Correlogram of Z3

Date: 10/25/10 Time: 02:12
Sample: 1962Q2 1993Q4
Included observations: 127

	AC	PAC	Q-Stat	Prob
1	-0.868	-0.868	97.936	0.000
2	0.753	0.001	172.32	0.000
3	-0.635	0.079	225.51	0.000
4	0.493	-0.169	257.83	0.000
5	-0.350	0.110	274.28	0.000
6	0.225	-0.032	281.13	0.000
7	-0.161	-0.175	284.67	0.000
8	0.159	0.245	288.13	0.000
9	-0.168	-0.062	292.04	0.000
10	0.192	-0.011	297.19	0.000
11	-0.239	-0.020	305.29	0.000
12	0.246	-0.181	313.91	0.000
13	-0.218	0.157	320.74	0.000
14	0.179	-0.003	325.38	0.000
15	-0.151	-0.176	328.71	0.000
16	0.119	0.138	330.79	0.000
17	-0.123	-0.276	333.03	0.000
18	0.130	0.035	335.56	0.000
19	-0.140	0.214	338.51	0.000
20	0.137	-0.399	341.39	0.000
21	-0.131	0.310	344.05	0.000
22	0.090	-0.295	345.32	0.000

Correlogram of Z2

Date: 10/25/10 Time: 02:13
Sample: 1962Q2 1993Q4
Included observations: 127

	AC	PAC	Q-Stat	Prob
1	0.227	0.227	6.6903	0.010
2	0.073	0.023	7.3855	0.025
3	-0.080	-0.107	8.2371	0.041
4	0.020	0.063	8.2921	0.081
5	0.043	0.039	8.5369	0.129
6	-0.175	-0.223	12.678	0.048
7	0.144	0.265	15.499	0.030
8	0.020	-0.054	15.557	0.049
9	0.077	0.002	16.378	0.059
10	-0.061	-0.017	16.901	0.077
11	-0.105	-0.101	18.450	0.072
12	0.091	0.114	19.633	0.074

Correlogram of Z4

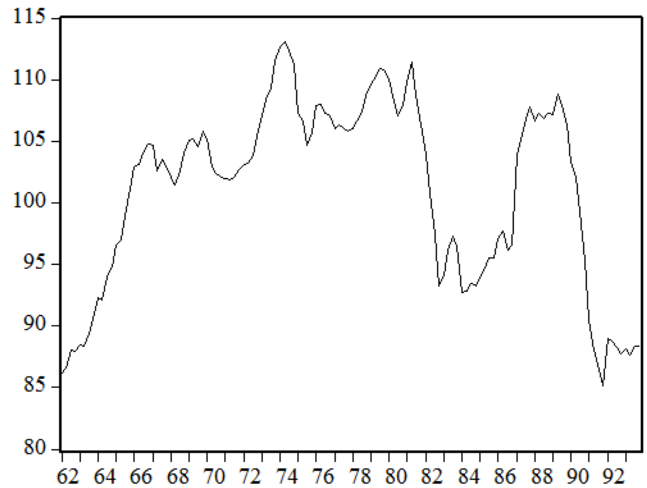
Date: 10/25/10 Time: 02:13
Sample: 1962Q2 1993Q4
Included observations: 127

	AC	PAC	Q-Stat	Prob
1	-0.187	-0.187	4.5359	0.033
2	0.104	0.071	5.9438	0.051
3	-0.127	-0.099	8.0651	0.045
4	0.028	-0.018	8.1679	0.086
5	0.129	0.156	10.403	0.065
6	-0.289	-0.275	21.704	0.001
7	0.250	0.175	30.246	0.000
8	-0.087	0.043	31.284	0.000
9	0.107	-0.015	32.860	0.000
10	-0.050	0.033	33.207	0.000
11	-0.127	-0.117	35.500	0.000
12	0.160	0.046	39.153	0.000

Now, once you see the autocorrelation function and the partial autocorrelation function, can you guess whether the series is MA or AR?

Example: Canadian Employment

Consider the Canadian Employment series from the previous computer handout:



Let's estimate an MA(1) model: ls caemp c ma(1)

Dependent Variable: CAEMP
Method: Least Squares
Date: 10/29/10 Time: 09:12
Sample: 1962:1 1993:4
Included observations: 128
Convergence achieved after 11 iterations
Backcast: 1961:4

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	100.9893	0.710258	142.1867	0.0000
MA(1)	0.937751	0.030321	30.92718	0.0000

R-squared	0.696454	Mean dependent var	101.0176
Adjusted R-squared	0.694045	S.D. dependent var	7.499163
S.E. of regression	4.148027	Akaike info criterion	5.698644
Sum squared resid	2167.972	Schwarz criterion	5.743207
Log likelihood	-362.7132	F-statistic	289.0938
Durbin-Watson stat	0.439477	Prob(F-statistic)	0.000000

Inverted MA Roots	-.94

As soon as you estimated this model, you have to open the series: resid, to test if the regression residuals are White Noise.

Correlogram of RESID

Date: 10/29/10 Time: 09:15
Sample: 1962:1 1993:4
Included observations: 128

	AC	PAC	Q-Stat	Prob
1	0.741	0.741	71.908	0.000
2	0.865	0.702	170.80	0.000
3	0.624	-0.379	222.68	0.000
4	0.697	-0.032	287.79	0.000
5	0.485	-0.068	319.66	0.000
6	0.520	-0.073	356.54	0.000
7	0.343	-0.019	372.76	0.000
8	0.353	-0.052	390.02	0.000
9	0.210	-0.016	396.21	0.000
10	0.173	-0.199	400.43	0.000
11	0.056	-0.063	400.87	0.000
12	0.011	0.015	400.89	0.000

Based on these results you reject the null hypothesis of White Noise error terms. This is because the autocorrelation and the partial autocorrelation for various values of the displacement fall outside the two-standard deviation bands. Moreover, the Q-statistic (Ljung-Box Q-statistic) which is the weighted sum of squared autocorrelations has large values when compared to the χ^2 distribution with the corresponding degrees of freedom (the p-values are below 0.05). Hence, you may want to try other specifications and also keep track of the AIC and SIC.

You should consider MA(q) of different order q.

For an AR(1) we have to type: ls caemp c ar(1)

Dependent Variable: CAEMP
Method: Least Squares
Date: 10/29/10 Time: 09:26
Sample: 1962:1 1993:4
Included observations: 128
Convergence achieved after 3 iterations

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	101.7767	4.898049	20.77903	0.0000
AR(1)	0.970201	0.019320	50.21620	0.0000

R-squared	0.952411	Mean dependent var	101.0176
Adjusted R-squared	0.952033	S.D. dependent var	7.499163
S.E. of regression	1.642415	Akaike info criterion	3.845715
Sum squared resid	339.8886	Schwarz criterion	3.890278
Log likelihood	-244.1257	F-statistic	2521.666
Durbin-Watson stat	1.061546	Prob(F-statistic)	0.000000

Inverted AR Roots	.97

With the corresponding correlogram:

Correlogram of RESID

Date: 10/29/10 Time: 09:29
Sample: 1962:1 1993:4
Included observations: 128

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.469	0.469	28.794	0.000
		2	0.253	0.043	37.269	0.000
		3	0.127	-0.009	39.415	0.000
		4	0.083	0.023	40.336	0.000
		5	0.031	-0.023	40.467	0.000
		6	0.023	0.013	40.538	0.000
		7	0.021	0.010	40.598	0.000
		8	0.087	0.092	41.648	0.000
		9	0.089	0.019	42.743	0.000
		10	0.028	-0.054	42.850	0.000
		11	-0.087	-0.125	43.928	0.000
		12	-0.193	-0.148	49.286	0.000

We still reject the null of White Noise errors.

For an AR(2): ls caemp c ar(1) ar(2)

Dependent Variable: CAEMP
Method: Least Squares
Date: 10/29/10 Time: 09:31
Sample: 1962:1 1993:4
Included observations: 128
Convergence achieved after 3 iterations

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	101.2413	3.399620	29.78017	0.0000
AR(1)	1.438810	0.078487	18.33188	0.0000
AR(2)	-0.476451	0.077902	-6.116042	0.0000

R-squared	0.963372	Mean dependent var	101.0176
Adjusted R-squared	0.962786	S.D. dependent var	7.499163
S.E. of regression	1.446663	Akaike info criterion	3.599554
Sum squared resid	261.6041	Schwarz criterion	3.666399
Log likelihood	-227.3715	F-statistic	1643.837
Durbin-Watson stat	2.067024	Prob(F-statistic)	0.000000
Inverted AR Roots	.92	.52	

Correlogram of RESID

Date: 10/29/10 Time: 09:31
Sample: 1962:1 1993:4
Included observations: 128

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.035	-0.035	0.1606	0.689
		2	0.044	0.042	0.4115	0.814
		3	0.011	0.014	0.4291	0.934
		4	0.051	0.050	0.7786	0.941
		5	0.002	0.004	0.7790	0.978
		6	0.019	0.015	0.8272	0.991
		7	-0.024	-0.024	0.9036	0.996
		8	0.078	0.072	1.7382	0.988
		9	0.080	0.087	2.6236	0.977
		10	0.050	0.050	2.9727	0.982
		11	-0.023	-0.027	3.0504	0.990
		12	-0.129	-0.148	5.4385	0.942

We finally have White Noise errors.

Now, you have to pick the most appropriate ARMA(p,q) for different orders of p and q based on the AIC and SIC. The result (not shown for all models) show that the ARMA(3,1) is the most appropriate:

Dependent Variable: CAEMP
Method: Least Squares
Date: 10/29/10 Time: 09:35
Sample: 1962:1 1993:4
Included observations: 128
Convergence achieved after 73 iterations
Backcast: 1961:4

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	101.1376	3.539424	28.57458	0.0000
AR(1)	0.500389	0.087408	5.724751	0.0000
AR(2)	0.872158	0.066839	13.04857	0.0000
AR(3)	-0.443209	0.080958	-5.474587	0.0000
MA(1)	0.971171	0.034669	28.01281	0.0000
R-squared	0.964535	Mean dependent var	101.0176	
Adjusted R-squared	0.963381	S.D. dependent var	7.499163	
S.E. of regression	1.435042	Akaike info criterion	3.598545	
Sum squared resid	253.2996	Schwarz criterion	3.709952	
Log likelihood	-225.3069	F-statistic	836.2915	
Durbin-Watson stat	2.057495	Prob(F-statistic)	0.000000	
Inverted AR Roots	.93	.51	-.94	
Inverted MA Roots	-.97			

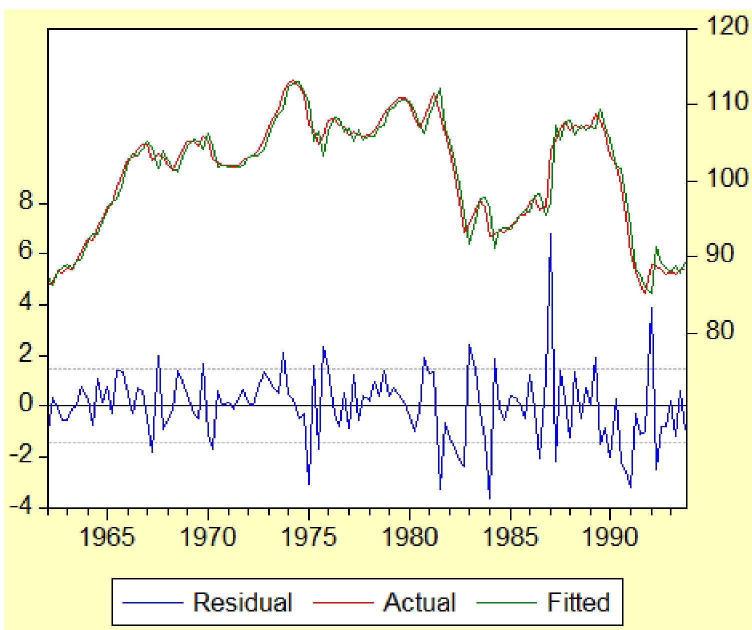
Correlogram of RESID

Date: 10/29/10 Time: 09:35
Sample: 1962:1 1993:4
Included observations: 128

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.032	-0.032	0.1384	0.710
		2	0.042	0.041	0.3683	0.832
		3	0.014	0.016	0.3937	0.942
		4	0.048	0.047	0.7030	0.951
		5	0.005	0.007	0.7070	0.983
		6	0.013	0.010	0.7311	0.994
		7	-0.017	-0.019	0.7723	0.998
		8	0.064	0.060	1.3485	0.995
		9	0.092	0.097	2.5236	0.980
		10	0.039	0.041	2.7345	0.987
		11	-0.017	-0.022	2.7736	0.993
		12	-0.136	-0.153	5.4430	0.942

And we have White Noise errors.

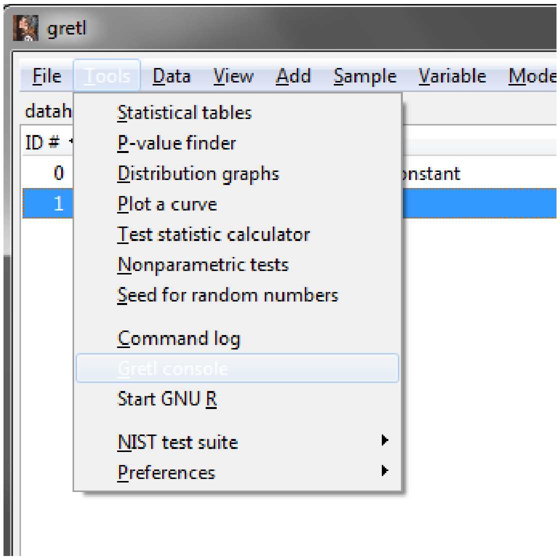
The actual, fitted (in-sample forecast), and residual (in-sample forecast errors) graph is:



In gretl

The command for the ARMA model is: arma 3 1 ; CAEMP 0

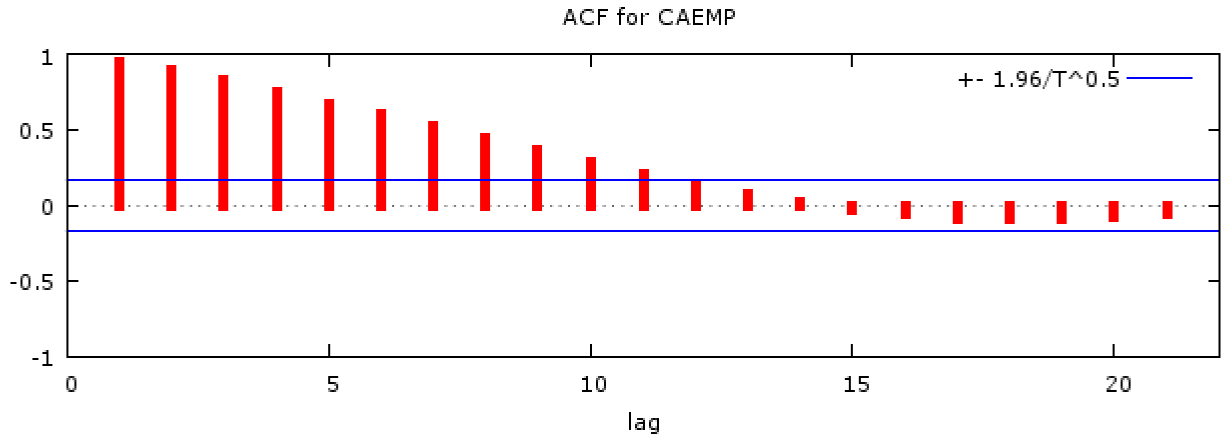
Go to "Tools" and then "gretl console"

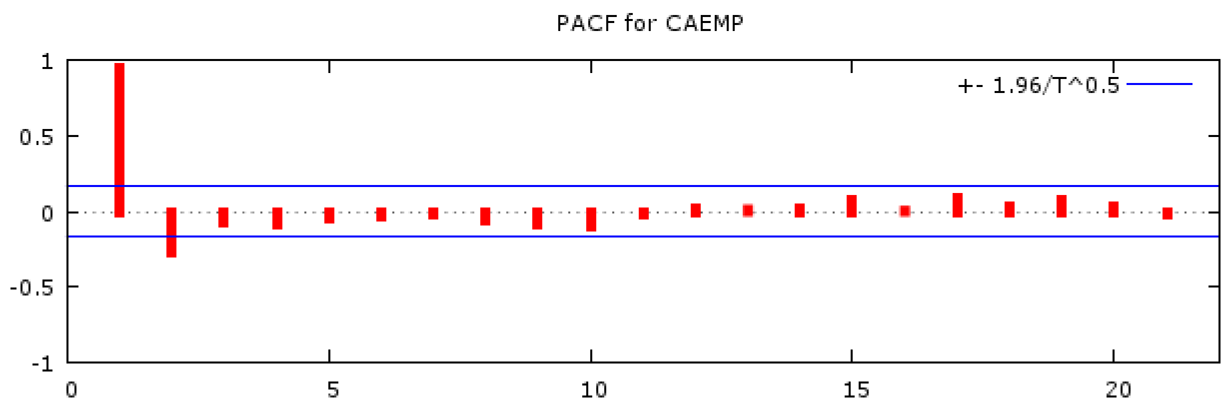


To obtain:

```
gretl console: type 'help' for a list of commands
? arma 3 1 ; CAEMP 0
Function evaluations: 96
Evaluations of gradient: 32

Model 1: ARMA, using observations 1961:1-1994:4 (T = 136)
Estimated using Kalman filter (exact ML)
Dependent variable: CAEMP
Standard errors based on Hessian

             coefficient   std. error      z       p-value
  ---------------------------------------------------------
  const       99.0506       3.22221      30.74    1.66e-207 ***
  phi_1        2.19046      0.224518      9.756   1.73e-022 ***
  phi_2       -1.49762      0.360611     -4.153   3.28e-05  ***
  phi_3        0.298463     0.145749      2.048   0.0406    **
  theta_1     -0.781041     0.199694     -3.911   9.18e-05  ***

Mean dependent var   100.2198   S.D. dependent var   7.997169
Mean of innovations -0.034017   S.D. of innovations  1.413822
Log-likelihood      -242.0093   Akaike criterion     496.0187
Schwarz criterion    513.4946   Hannan-Quinn         503.1205

                        Real  Imaginary    Modulus  Frequency
  -----------------------------------------------------------
  AR
    Root  1           1.0773    -0.0984     1.0817    -0.0145
    Root  2           1.0773     0.0984     1.0817     0.0145
    Root  3           2.8633     0.0000     2.8633     0.0000
  MA
    Root  1           1.2803     0.0000     1.2803     0.0000
  -----------------------------------------------------------
```

For the correlogram just go to “variable” and then “correlogram”:


ACF for CAEMP
+- 1.96/T^0.5
1
0.5
0
-0.5
-1
0
5
10
15
20
lag


PACF for CAEMP
+- 1.96/T^0.5
1
0.5
0
-0.5
-1
0
5
10
15
20