1.) zero equation => various eddy viscosity models can be tried

$$\mu_t = \text{func}(y, \bar{u}, \frac{\partial \bar{u}}{\partial y}, \text{etc.})$$

2.) one equation

add turbulent kinetic energy equation

$$\frac{\text{energy}}{\text{mass}} \rightarrow K = \tfrac{1}{2}(\overline{u'u'} + \overline{v'v'} + \overline{w'w'})$$

boundary layer form of equation presented earlier

$$\bar{u}\frac{\partial K}{\partial x} + \bar{v}\frac{\partial K}{\partial y} = -\frac{\partial}{\partial y}\left[\overline{v'(\tfrac{1}{2}u_i'u_i' + p'/\rho)}\right] + \frac{\tau}{\rho}\frac{\partial \bar{u}}{\partial y} - \epsilon \qquad E_q(6\text{-}26)$$

$$\epsilon = \nu\,\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} \quad \text{turbulent dissipation} \left(\frac{\text{power}}{\text{mass}}\right)$$

let L_e = turbulent length scale (effective eddy size)

char. eddy velocity $= K^{1/2}$

suggests then

$$\epsilon \sim \frac{\text{drag} \times \text{velocity}}{\text{mass}} \sim \frac{L^2(\rho K^{2/2}) \times K^{1/2}}{\rho L^3} \sim \frac{K^{3/2}}{L}$$

$$\sim \frac{\partial K}{\partial y} \qquad \tau = (\mu + \mu_t)\frac{\partial \bar{u}}{\partial y}$$

further modeling for $\mu_t = \text{func}(K, \ldots$ and $L_e = \text{func}(K, \ldots)$ for math closure then have three P.D.E.'s for $\bar{u}, \bar{v}, K$

author points out that one-equation model no longer popular

3. two equations $K - \epsilon$

author indicates that this is currently most widely used approach (as of ~1990)

non boundary layer form in text, use log-law to match near the wall

two-dimensional B.L. form

Eqs 6-109a and 6-109b

$$\bar{u}\frac{\partial K}{\partial x} + \bar{v}\frac{\partial K}{\partial y} \approx \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_K}\right)\frac{\partial K}{\partial y}\right] + \nu_t\left(\frac{\partial K}{\partial y}\right)^2 - \epsilon - 2\nu\left(\frac{\partial K^{1/2}}{\partial y}\right)^2 \quad (6\text{-}109a)$$

$$\bar{u}\frac{\partial \epsilon}{\partial x} + \bar{v}\frac{\partial \epsilon}{\partial y} \approx \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon}\right)\frac{\partial \epsilon}{\partial y}\right] + C_1\frac{\epsilon\nu_t}{K}\left(\frac{\partial \bar{u}}{\partial y}\right)^2 - C_2\frac{\epsilon^2}{K} + 2\nu\nu_t\left(\frac{\partial^2 \bar{u}}{\partial y^2}\right)^2 \quad (6\text{-}109b)$$

where, for example

$$\nu_t = C_\mu K^2/\epsilon$$

$$C_\mu = 0.09 \exp\left[-\frac{2.5}{1+R_T/50}\right]$$

$$R_T = K^2/\epsilon\nu$$ "turbulence Reynolds number"

then have four simultaneous P.D.E.'s for

$\bar{u}, \bar{v}, K, \epsilon$

4. Reynolds stress equation

next higher order analysis would be to eliminate $\nu_t \simeq \frac{C_\mu K^2}{\epsilon}$ and solve for Reynolds stresses directly

e.g. Eq (6-111)

$$\frac{D}{Dt}(\overline{u_i'u_j'}) = \underbrace{D_{ij}}_{\text{diffusion}} + \underbrace{P_{ij}}_{\text{production}} + \overbrace{\Pi_{ij}}^{\text{pressure strain}} - \underbrace{\epsilon_{ij}}_{\text{dissipation}} + \nu \nabla^2 (\overline{u_i'u_j'})$$

terms on right require further modeling as a function of K, ϵ, $\overline{u_i'u_j'}$, etc.

would then have ~~ten~~ eleven equations for

$\bar{w}, \bar{u}, \bar{v}, K, \epsilon, \overline{u_i'u_j'}$ (3D)

five eqns. for $\bar{u}, \bar{v}, K, \epsilon, \overline{u'v'}$ (2-D)

knowing $\overline{u'v'}$ means eddy viscosity model no longer needed

expect more advanced/sophisticated than $K-\epsilon$ approach

simplification would be to use empirical algebraic equations for $\overline{u_i'u_j'} = \text{func}(K, \epsilon, \text{etc.})$ Eq (6-116)

then back to four P.D.E.'s for $\bar{u}, \bar{v}, K, \epsilon$ ala $K-\epsilon$ approach

beyond this have

5. large eddy simulation

6. direct (model free) numerical simulation

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solution of differential equations ordinarily must be done numerically using finite difference techniques, commercial codes are available, may require large memory and CPU time

as an alternative, simpler integral methods, may suffice for many engineering applications

consider steady, two-dim flow over nonporous surface

$$dU_e/dx \neq 0$$

consv. of mass $\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0\right) (U_e - \bar{u})$

apparent!

consv. of x-mom. $\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = U_e \frac{dU_e}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}$

multiply continuity by $(U_e - \bar{u})$ and subtract from momentum

$\partial \bar{u}^2/\partial x$

$$-U_e \frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial \bar{u}}{\partial x} - U_e \frac{\partial \bar{v}}{\partial y} + \bar{u}\frac{\partial \bar{v}}{\partial y} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}$$

$$-\bar{u}\frac{\partial U_e}{\partial x} + \bar{u}\frac{\partial U_e}{\partial x} = U_e \frac{dU_e}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

↑ add/subt.

$\frac{\partial \bar{u}\bar{v}}{\partial y}$

$-\frac{\partial \bar{u} U_e}{\partial x}$ ✓

regroup

$$-\frac{1}{\rho}\frac{\partial \tau}{\partial y} = \frac{\partial}{\partial x}\left(\bar{u}U_e - \bar{u}^2\right) + (U_e - \bar{u})\frac{\partial U_e}{\partial x}$$
$$+ \frac{\partial}{\partial y}\left(U_e \bar{v} - \bar{u}\bar{v}\right)$$

now integrate $\int_0^{y>\delta} - - - \, dy$

@ $y = 0$ $\quad \tau = \tau_w, \ \bar{u} = 0, \ \bar{v} = 0$

@ $y > \delta$ $\quad \tau \Rightarrow 0, \ \bar{u} \Rightarrow U_e$

$$+\frac{1}{\rho}\tau_w = \frac{\partial}{\partial x}\left[U_e^2 \int_0^{y>\delta} \left(1 - \frac{\bar{u}}{U_e}\right)\frac{\bar{u}}{U_e}\, dy\right]$$

$$+ U_e \frac{dU_e}{dx}\int_0^{y>\delta}\left(1 - \frac{\bar{u}}{U_e}\right) dy + \cancel{\left(\bar{v}\, U_e - \bar{u}\,\bar{v}\right)}^{\,0} \Big|_0^{y>\delta}$$

$$\frac{1}{\rho}\tau_w = \frac{d}{dx}\left[U_e^2 \theta\right] + U_e \delta^* \frac{dU_e}{dx}$$

$$\frac{\tau_w}{\rho U_e^2} = \frac{\theta\, 2 U_e}{U_e^2}\frac{dU_e}{dx} + \frac{\cancel{U_e^2}}{\cancel{U_e^2}}\frac{d\theta}{dx} + \frac{U_e \delta^*}{U_e^2}\frac{dU_e}{dx}$$

or ① $$\boxed{\frac{C_f}{2} = \frac{d\theta}{dx} + (2 + H)\frac{\theta}{U_e}\frac{dU_e}{dx} \quad \text{Eq } (6\text{-}28)}$$

van Karman integral equation for time-ave. turbulent flow (see pg: 265-266, for identical laminar result)

for known $U_e(x)$, need two more equations to relate

$$C_f, \theta, H$$

from Coles wall-wake law

$$u^+ = \frac{1}{\kappa}\ln y^+ + B + \frac{2\Pi}{\kappa} f(\eta)$$

find relationships between C_f, H, θ, Π

(integrate for δ^*, θ) $\quad H = \delta^*/\theta = \text{func}(C_f, \Pi)$

$\theta = \text{func}(C_f, \Pi)$

Eq (6-119)

(2) $$\boxed{\lambda = \frac{U_e}{v^*} = \left(\frac{2}{C_f}\right)^{1/2} = \frac{2 + 3.179\Pi + 1.5\Pi^2}{\kappa(1+\Pi)} \frac{H}{H-1}}$$

$\sqrt{\frac{\tau_w}{\rho}}$

also $$Re_\theta = \frac{1+\Pi}{\kappa H} \exp(\kappa\lambda - \kappa B - 2\Pi)$$

if eliminate Π, then can create curvefits among C_f, Re_θ, H

Eq (6-117) $$C_f \cong 0.246 (Re_\theta)^{-0.268} 10^{-0.678 H}$$

note typo!

Eq (6-118) $$C_f \cong 0.058 Re_\theta^{-0.268} (0.93 - 1.95 \log H)^{1.70}$$

(3) Eq (6-120) $$\boxed{C_f \simeq \frac{0.3 \exp(-1.33H)}{(\log Re_\theta)^{1.74 + 0.31H}}}$$

still one equation short (θ, H, C_f, Π)

from correlation of data

(4) $$\beta = -0.4 + 0.76\Pi + 0.42\Pi^2 \quad Eq (6-121)$$

Clauser's parameter $$\beta = \frac{\delta^*}{\tau_w}\frac{dp_e}{dx} = -\frac{\delta^*}{\underbrace{\tau_w}_{\rho v^{*2}}} \rho U_e \frac{dU_e}{dx} \times \frac{U_e}{U_e}$$

$$\beta = -\frac{\lambda^2 H \theta}{U_e}\frac{dU_e}{dx} \quad Eq (6-122)$$

note: 1 to 1 correlation for equilibrium turbulent flows basically have four equations for four unknowns

one ord. diff. equation and three algebraic for

$$\theta, H, C_f, \cancel{H}\ \Pi$$

begin with known $\theta(0)$

then numerically "march" downstream for any $U_e(x)$ to determine

$$\theta(x), C_f(x), H(x), \Pi(x)$$

$$\Rightarrow \delta^*(x),\ \tau_w(x),\ \bar{u}(x,y),\ \delta(x)$$

(somewhat analogous to Thwaites method for laminar boundary layer)

could use Runge Kutta, I used less sophisticated predictor-corrector

$$\frac{d\theta}{dx} = \frac{1}{\lambda^2} - (2+H)\frac{\theta}{U_e}\frac{dU_e}{dx} \qquad \text{Eq (6-28)}$$

$$\theta^p_{m+1} = \theta^c_m + \left.\frac{d\theta}{dx}\right)_m \Delta x$$

$$\theta^c_{m+1} = \theta^c_m + \left.\frac{d\theta}{dx}\right)_{m+1/2} \Delta x$$

where $\theta_{m+1/2} = (\theta^p_{m+1} + \theta^c_m)/2$

etc.

also

$$H = \frac{1}{1 - a/\lambda} \quad \text{where } a = \frac{2 + 3.179\Pi + 1.5\Pi^2}{\kappa(1+\Pi)}$$

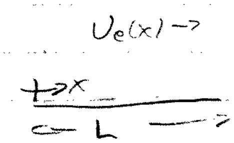
$$C_f = \frac{0.3\exp(-1.33H)}{\log(Re_\theta)^{1.74 + 0.31H}}$$

$$\Pi = -(0.905) + \sqrt{(0.905)^2 + (0.952 + 2.381\beta)}$$

$$\text{where } \beta = -\lambda^2 H \frac{\theta}{U_e}\frac{dU_e}{dx}$$

$$\lambda = \left(\frac{2}{C_f}\right)^{1/2}$$

→ iterate for given θ



handout for KRMNTBL.FOR

on fileserver H:\ME812

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note: three algebraic (nonlinear) equations relating θ/L, C_f, H, Π

only one independent

to find C_f for given (θ/L) (for example)

set $M = 0$ and guess (θ/L) until obtain proper C_f

questions?

other integral method approaches have been tried

Sec 6-8.1.2 entrainment integral methods

use van Karman integral equation

① $$\boxed{\frac{C_f}{2} = \frac{d\theta}{dx} + (2+H)\frac{\theta}{U_e}\frac{dU_e}{dx}} \quad Eq\,(6\text{-}28)$$

but now introduce entrainment equation which was derived from continuity earlier

$$E = \frac{d\delta}{dx} - \frac{v_e}{U_e} = \frac{1}{U_e}\frac{d}{dx}\left[U_e\,(\delta-\delta^*)\right] \quad Eq\,(6\text{-}31)$$

work of Head (1958), introduced new shape factor

$$H_1 = (\delta-\delta^*)/\theta$$

then ② $$\boxed{\frac{d\,(U_e\,\theta H_1)}{dx} = U_e E \simeq U_e \underbrace{F(H_1)}_{\text{curve fit}}}$$

where $F(H_1) = 0.0306\,(H_1-3.0)^{-0.6169}$

Eq (6-123)

then relate ③ $\boxed{H_1 = \text{func}(H)}$ Eq (6-124)

with ④ $\boxed{C_f = \text{func}(\theta, H)}$ Eq (6-120)

four equs. for C_f, H, $\underbrace{\theta}_{d\theta/dx}$, $\underbrace{H_1}_{dH_1/dx}$

two parameter system, good except for near-separating flows

note: does not use empirical $\beta = \text{func}(\pi)$

variation of this is approach of Ferziger, better in general, but not as good for favorable flows

Sec 6-8.2 inner variable integral method

Das (1988)

same as inner variable analysis for flow over flat plate but add wake

$$\frac{\bar{u}}{u^*} = \frac{1}{\kappa}\ln y^+ + B + \frac{2\pi}{\kappa}(3\eta^2 - 2\eta^3)$$

$$\eta = y/\delta$$

from continuity $\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$

$$\bar{v} = -\int_0^y \partial\bar{u}/\partial x \, dy'$$

use to eliminate $\bar{u}$ and $\bar{v}$ in x-mom equation, integrate from $y^+ = 0$ to $y^+ = \delta^+$

so $\tau = \tau_w \Rightarrow 0$

results in ordinary differential equation in x-direction

in non-dim form $\quad \varsigma = \frac{u^*}{U_e} = 1/\lambda = \left(\frac{C_f}{2}\right)^{1/2}$

$$V = U_e/U_0$$

$$x^* = x/L, \quad Re_L = \frac{U_0 L}{\nu}$$

Eq (6-134)

① $$\frac{d\varsigma}{dx^*} = -\frac{\varsigma}{V}\frac{dV}{dx^*} + \frac{1}{V}\,\frac{T_1 - T_2 + R_L T_3 (T_4 - R_L T_5)}{R_L T_3 (T_6 + T_7) + T_8 + T_9}$$

$T_1, T_2, \ldots T_9 \Rightarrow$ Table 6-2

$$= \text{func}\left(V, \frac{dV}{dx^*}, \frac{d^2V}{dx^{*2}}, \delta^+, \pi\right)$$

need relations for δ^+, π

evaluate Coles wall-wake law @ $y^+ = \delta^+$

$$\frac{1}{\kappa}\ln\delta^+ + B + \frac{2\pi}{\kappa} = \frac{1}{\varsigma} \quad ②$$

other is correlation between β and π

$$\beta = -0.4 + 0.76\pi + 0.42\pi^2$$

where

$$\beta = \frac{\delta^*}{\tau_w}\frac{dP_e}{dx} = -\frac{\delta^*}{\tau_w}\rho U_e\frac{dU_e}{dx}$$

Eq 6-98

$$= -\frac{\delta}{v^{*2}} U_e \frac{dU_e}{dx^*}\left(\cancel{\frac{\delta^*}{\delta}}\right) \times \frac{(1+\pi)v^*}{\kappa U_e}\Big/\cancel{(\delta^*/\delta)} \times \frac{\nu}{\nu} \times \frac{U_e}{U_e}$$

$\delta \to \delta^+$, $v^{*2} \to \varsigma^2$, $U_e \to V$

$$\beta = -\frac{\delta^+}{\varsigma^2}\frac{(1+\pi)}{\kappa Re_L}\frac{1}{V^2}\frac{dV}{dx^*} \quad \text{Eq (6-135)} \times \frac{U_o}{U_o} \times \frac{L}{L}$$

error in text??

three equations (one diff.) to march along in ς, π, δ^+

I wasn't successful in getting this method to work, looked up original paper, might be some typos in Table 6-2

note: doesn't use $C_f = \text{func}(\theta, H)$ correlations

integral methods work best for equilibrium or near equilibrium flows

for rapidly changing freestream,

$$\beta = \beta_1 \text{ for } x < x_0$$

$$\beta = \beta_2 \text{ for } x > x_0$$

use "lag" equation where

$$\frac{dQ}{dx} = \frac{C}{\delta}(\underbrace{Q_{eq}}_{\text{asymptotic value}} - Q)$$

$$C = 0.025$$

Sec 6-8.3 finite difference methods

solve partial differential equations

Sec 6-8.3.1 zero equations

continuity, x-momentum for $\bar{u}, \bar{v}$

add eddy viscosity $\mu_t = \text{func}(y, \frac{\partial \bar{u}}{\partial y}, \text{etc.})$

GENBL.FOR ⇒ implicit marching method

more sophisticated codes than GENBL.FOR

e.g. variable gridding, coordinate transformations, wall functions to patch through sublayer and buffer layer

example Cebbi and Smith (1974)

Sec 6-8.3.2 two equations

$K-\epsilon$ Jones and Launder (1972)

Wilcox (1988) suggests $K-\omega$ instead of $K-\epsilon$

Sec 6-8.3.3 Reynolds stress equations
various approaches
Launder (1975)

Stanford Conferences provided experimental test cases for comparison

Karman-based
Head (1958)
Das (1988)
Ferzinger (1982)
} integral methods

Cebbi and Smith (1974) } zero equations

Sec 6-8.4 flat plate

C_f, θ vs. x → reasonably good agreement, all methods

for detail, $\bar{u}(y)$, $\bar{v}(y)$, $K(y)$ etc. need finite diff. methods

Sec 6-8.4.1 visualization and simulation turbulent flow over a flat plate

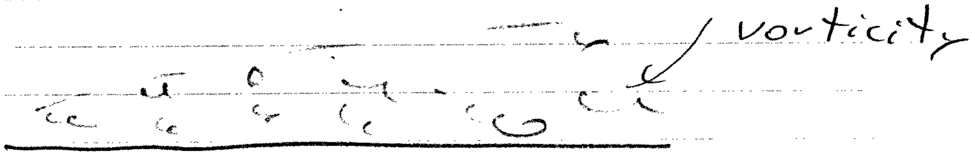
hydrogen bubble visualization at

$y^+ = 2, 4, 11.7, 40.7, 162$

sublayer buffer log layer

photos reversed ??

direct numerical simulation



good agreement with experiment

even at "low" Reynolds number, 10 million mesh points !!

Sec 6-8.5 adverse pressure gradient

$$U_e \approx 56.6\, x^{-0.28}$$

Karman-based integral method gave better agreement than more advanced approaches

perhaps not surprising since equilibrium turbulent flow

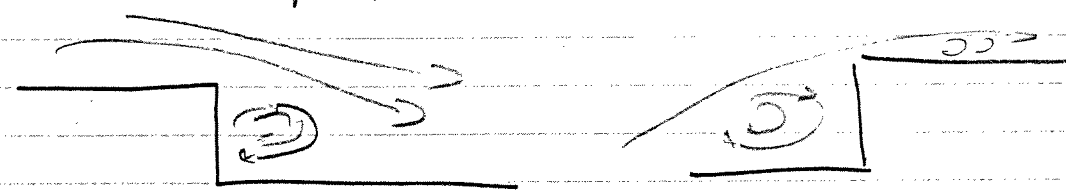
Sec 6-8.6 nonequilibrium separating flows

Cebbi and Smith provided best fit

inverse method for handling separating flows, assume $\delta^*(x)$ known, solve for $U_e(x)$, integral methods

Sec 6-8.7 "complex" turbulent flows

flow over steps, recirculation



roughness, suction/blowing ; periodic free stream velocity, etc.

"no method universally good"

"no method universally bad"

"no correlation between complexity of method and predictive capability"

often simplest integral method the best

might need to use "expert system" approach

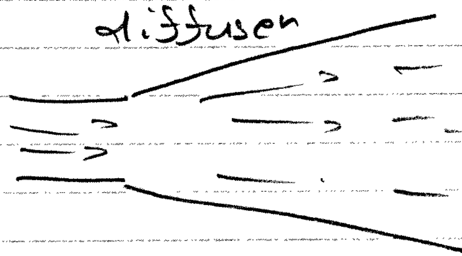
obviously, progress continues, much research left to be done

relaminarization

nozzle

strong acceleration stabilizing
% K decreases

diffuser



$Re_\theta < Re_{\theta cr}$
production of turbulence ceases

K-ε methods work best for this problem