VISCOUS FLOW II
Review Material

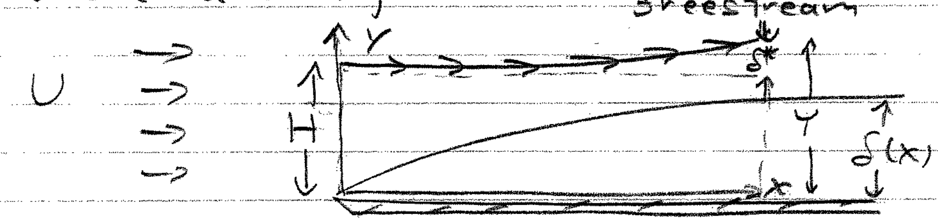
Chap 4 Laminar Boundary Layer Theory

for sufficiently large Reynolds number ($Re_L \gtrsim 1000$), appreciable viscous effects confined to a relatively thin layer near the surface => "viscous boundary layer"

patch together thin viscous boundary layer to inviscid freestream => "boundary layer method"

Sec 4-1 integral analysis of viscous boundary layer

flow over a flat plate



$\delta(x)$ = boundary layer thickness , $u)_{y=\delta} = 0.99U$

$\delta^*(x)$ = boundary layer displacement thickness

$$\delta^*(x) = \int_0^{y>\delta} (1 - u/U)\, dy$$

$\theta(x)$ = boundary layer momentum thickness

$$\theta(x) = \int_0^{y>\delta} (1 - u/U)(u/U)\, dy$$

H = boundary layer shape factor = δ^*/θ

for flat plate ($\partial P/\partial x = 0$, U = const.)

$$\tau_w = \rho U^2 \frac{d\theta}{dx}, \quad C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = 2\frac{d\theta}{dx}$$

$$C_D = \frac{D'}{\frac{1}{2}\rho U^2 L} = 2\,\frac{\theta(L)}{L}$$

if substitute $u \approx U\left[2(y/\delta)-(y/\delta)^2\right]$ find

$$\frac{\delta}{x} \approx \frac{5.5}{\sqrt{Re_x}}, \text{ --- etc.}$$

in the boundary layer ($Re_x >> 1$)

$y \sim \delta \sim x/\sqrt{Re_x} \Rightarrow y << x$

$u \sim U, \; v \sim \dfrac{U}{\sqrt{Re_x}} \Rightarrow v << u$

$\dfrac{\partial u}{\partial x} \sim \dfrac{U}{x}, \; \dfrac{\partial u}{\partial y} \sim \dfrac{U}{\delta} \sim \dfrac{U}{x}\sqrt{Re_x} \Rightarrow \dfrac{\partial u}{\partial x} << \dfrac{\partial u}{\partial y}$

$\dfrac{\partial v}{\partial x} \sim \dfrac{U}{x}\dfrac{1}{\sqrt{Re_x}}, \; \dfrac{\partial v}{\partial y} \sim \dfrac{v}{\delta} \sim \dfrac{U}{\sqrt{Re_x}}\dfrac{\sqrt{Re_x}}{x} \Rightarrow \dfrac{\partial v}{\partial x} << \partial v/\partial y$

$\boxed{\text{skip Secs. 4-1.6 → 4-1.7}}$

$\boxed{\text{Sec 4.2}}$ the laminar boundary layer equations

continuity $\quad \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

x-mom $\quad \rho\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y}\right) = -\dfrac{\partial p}{\partial x} + \mu\dfrac{\partial^2 u}{\partial y^2}$

y-mom $\quad \partial p/\partial y = 0$

energy $\quad \rho c_p\left(\dfrac{\partial T}{\partial t} + u\dfrac{\partial T}{\partial x} + v\dfrac{\partial T}{\partial y}\right) = k\dfrac{\partial^2 T}{\partial y^2} + \mu\left(\dfrac{\partial u}{\partial y}\right)^2$

observations

- applicable for curved surfaces if $R >> \delta$
- flow direction transport negligible $\left(\dfrac{\partial^2 u}{\partial x^2} << \dfrac{\partial^2 u}{\partial y^2}\right)$
- for constant fluid properties, energy equation is decoupled

- p = func (x) only and is known from freestream conditions

- x-dir mom. equ. is parabolic for $u(x,y)$, numerical marching schemes can be used in $+x$ direction (likewise for $T(x,y)$)

- boundary conditions @ $y=0$ $u=0$, $v=0$, $T=T_w$
  @ $y \Rightarrow$ large $u=U$, $T=T_\infty$

- viscous stresses $\tau'_{xy} \approx \mu \frac{\partial u}{\partial y}$
  $\tau'_{xy} \gg \tau'_{xx}, \tau'_{yy}$

- $Re \gtrsim 1000$

- for $Re \gtrsim 10^6 \Rightarrow$ instability, turbulence

- if $\partial U/\partial x < 0$, $\partial p/\partial x > 0$, possibility of boundary layer separation ($\tau_w \Rightarrow 0$)

Blasius solution $\Rightarrow$ flow over a flat plate
$U = \text{const}$, $\partial p/\partial x = 0$

similarity variable $\eta = y\sqrt{\frac{U}{2\nu x}}$

similarity function $f(\eta)$

$u = U f'(\eta)$, $v = \sqrt{\frac{\nu U}{2x}}\,[\eta f'(\eta) - f(\eta)]$

Blasius equation $f''' + f f'' = 0$

$f(0)=0$, $f'(0)=0$, $f'(\eta \to \text{large}) = 1$

displaced B.C.'s, solve by trial and error using Runge-Kutta

$$f''(0) = 0.469600$$

expressions for $\delta, \delta^*, \theta, C_f, C_D, H$

Falkner-Skan wedge flows

$$U(x) = K x^m$$

$$\eta = y\sqrt{\frac{U(m+1)}{2\nu x}}, \quad \beta = \frac{2m}{m+1}$$

$$f''' + ff'' + \beta[1-(f')^2] = 0$$

$$f(0) = 0, \; f'(0) = 0, \; f'(\eta \to \text{large}) = 1$$

diff. $f''(0)$, $f(\eta)$ for each β

$u = U(x) f'(\eta)$, expressions for $\delta, \delta^*, \theta, C_f, H$

$m < 0 \Rightarrow$ decelerating freestream, thick, low τ_w, relatively unstable, possible separation

$m > 0 \Rightarrow$ accelerating freestream, thin, high τ_w, relatively stable

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flat plate with suction / blowing

similarity solution $\quad v_w^* = \frac{v_w}{U}\sqrt{Re_x}$

$$v_w \sim 1/\sqrt{x}$$

$v_w^* > 0 \Rightarrow$ injection/blowing, thick, low τ_w, relatively unstable, possible separation

$v_w^* < 0 \Rightarrow$ suction, thin, high τ_w, relatively stable nonsimilar solutions

linearly retarded flow of Howarth

$$U(x) = U_0\left(1 - x/L\right)$$

uniform suction

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approximate integral methods

for general $U(x)$

$$\frac{C_f}{2} = \frac{1}{U^2}\frac{\partial}{\partial t}(U\delta^*) + \frac{\partial \theta}{\partial x} + (2\theta + \delta^*)\frac{1}{U}\frac{\partial U}{\partial x} - \frac{v_w}{U}$$

for steady, $v_w = 0$

$$\frac{C_f}{2} = \frac{d\theta}{dx} + (2 + H)\frac{\theta}{U}\frac{dU}{dx}$$

method of Thwaites

$$\lambda = \frac{\theta^2}{\nu}\frac{dU}{dx} \quad \text{correlation parameter}$$

$$\frac{\tau_w \theta}{\mu U} = S(\lambda) \quad \text{shear correlation}$$

$$H = H(\lambda) \quad \text{shape factor correlation}$$

$$\lambda(x) = \frac{0.45\,(dU/dx)}{U^6}\int_0^x U^5\,dx'$$

separation at $\lambda = -0.09$

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numerical methods

finite difference explicit / implicit
marching schemes

MECE 6373
VISCOUS FLOW II
Review for EXAM #1

Chap 5 The Stability of Laminar Flows

Sec 5-1 Introduction: The Concept of Small-Disturbance Stability

if slightly disturb a system, will it return to its original state? $\Rightarrow$ yes, then stable
$\Rightarrow$ no, then unstable

typical stability analysis

- establish controlling equations, formulate basic solution, Q_0
- add a disturbance to basic solution and substitute into the controlling equations
  $$Q = Q_0 + Q_1, \quad Q_1 << Q_0$$
- subtract out basic solution
- linearize $(Q_1^2 << Q_1)$
- solve resultant linear, homogeneous equations and assess growth, maintainance, or decay of Q_1 with time or space

example: one dim, compressible, stationary fluid (supplementary)

example: buckling of simple beam under compression (text)

example: forced motion of a damped mass with a cubic spring (text)

Sec 5-2 Linearized Stability of Parallel Viscous Flows

Navier-Stokes equations with ρ, μ constant

mass $\qquad \vec{\nabla} \cdot \vec{V} = 0$

mom $\qquad \frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\vec{\nabla}p + \nu \nabla^2 \vec{V} \qquad$ Eq (5-10)

basic solution $\Rightarrow U, V, W, p_o$

disturbance $\Rightarrow \hat{u}, \hat{v}, \hat{w}, \hat{p}$

two-dim,
parallel (boundary layer) flows $\qquad U \simeq U(y)$
$V << U$
$W = 0$

Squire's theorem: for two-dim, parallel flow
$\Rightarrow$ minimum value of Re_{crit} corresponds to disturbances propagating in the flow direction

since linear, homogeneous equations with coefficients dependent only on y, expect traveling-wave solutions of the form

$$\hat{\xi}(x,y,t) = \xi(y) \exp[i\alpha(x-ct)]$$

Tollmein-Schlicting waves

α = wavenumber ($\alpha = 2\pi/\lambda$)

c = propagation speed

ω = angular frequency ($\omega = \alpha c$)

$$i\alpha u + v' = 0 \qquad \text{Eqs (5-15 a,b,c)}$$

$$i\alpha u(U-c) + U'v = -\frac{i}{\rho}\alpha p + \nu(u'' - \alpha^2 u)$$

$$i\alpha v(U-c) = -\frac{1}{\rho}p' + \nu(v'' - \alpha^2 v)$$

$$(\xi' = d\xi/dy)$$

combine Eqs (5-15 a, b, c) => Orr-Sommerfeld equ.

$$(U-c)(v''-\alpha^2 v) - U''v + \frac{i\nu}{\alpha}(v''''-2\alpha^2 v''+\alpha^4 v) = 0$$

Eq (5-18)

if inviscid ($\nu=0$, $Re \Rightarrow \infty$) flow then

$$v'' - \left(\frac{U''}{U-c} + \alpha^2\right) v = 0 \quad \text{Eq (5-21)}$$

for instability necessary that velocity profile have a point of inflection (PI), $|U'_{PI}|$ be a maximum, and $U''(U-U_{PI}) < 0$ somewhere on the profile

if viscous ($\nu \neq 0$, $Re \Rightarrow$ finite), nondim Orr-Sommerfeld equ.

$$(U^*-c^*)(\phi''-\alpha_\delta^2\phi) - U^{*''}\phi + \frac{i}{\alpha_\delta Re_\delta}(\phi''''-2\alpha_\delta^2\phi''+\alpha^4\phi) = 0$$

Eq (5-23)

for given $U^*(x,y)$ and Re_δ => eigenvalue problem
eigenvalue equation

$$f(Re_\delta, \alpha, c) = 0$$

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temporal stability $\qquad \alpha = 2\pi/\lambda$ => real and "known"

$C = C_r + i C_i$ => complex and to be determined

if $C_i > 0$ => growth, unstable

$C_i < 0$ => decay, stable

$C_i = 0$ => nuetral

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spatial stability $\qquad \omega = \alpha c$ => real and "known"

$\alpha = \alpha_r + i\alpha_i \Rightarrow$ complex and to be determined

if $\alpha_i < 0 \Rightarrow$ growth, unstable

$\alpha_i > 0 \Rightarrow$ decay, stable

$\alpha_i = 0$ nuetral

stability analysis for Blasius ($U_\infty =$ const.) and Falkner-Skan ($U_\infty = Kx^m$) flows

similarity solutions, $U(x,y) \Rightarrow$ known

nuetral stability curves $\Rightarrow$ Figs. 5-6 and 5-7

$x_{crit} \Rightarrow$ location where T-S waves first propagate with increasing amplitude ($c_i > 0$, $\alpha_i < 0$), first indication of instability

for flat plate, $Re_{x,crit} \simeq 91{,}000$

accelerating flows $\Rightarrow$ stabilizing ($dU/dx > 0$, $dP/dx < 0$)

deaccelerating flows $\Rightarrow$ destabilizing ($dU/dx < 0$, $dP/dx > 0$)

comparison of stability theory with experiment

Sec 5-3 Parametric Effects in the Linear Stability Theory

classical laminar flow profiles (internal flows)

suction $\Rightarrow$ stabilizing

shape factor correlation of Wazzan (1979)

$$Re_{\delta^*,crit} = \text{func}(H) \Rightarrow \text{Fig } 5\text{-}12$$

to find x_{crit} → if Falkner-Skan ⇒ Table 5-1
→ otherwise Fig 5-12 and method of Thwaite

wakes, jets, parallel fluids ⇒ very unstable
(PI in profiles)

wall temperature → gases, cold wall stabilizing
→ liquids, hot wall stabilizing

compressible effects ($\rho = \rho_0 + \rho_1$, $T = T_0 + T_1$, etc.)
⇒ stabilizing
⇒ higher modes
⇒ nonparallel wave growth

compliant boundaries ⇒ can be stabilizing

free convection flows ⇒ relatively unstable
(PI in profile)

centrifugal stability ⇒ concentric rotating cylinders, cylindrical coordinate anal., Taylor number, Taylor vorticies

Sec 5-4 Transition to Turbulence

transition from first amplification of T-S waves to full turbulence not well understood theoretically
rely great deal on visualization

flow over a flat plate, sequence of events

- at x_{crit}, small natural disturbances excite amplifying two-dim. T-S waves
- when T-S waves grow to about 1% of background properties, three dimensional (spanwise) waves occur (Λ vorties)
- Λ vorties begin to deteriate

- sporadic, finite regions of full turbulence spontaneously occur => "turbulent spots"

- coalescence of turbulent spots into fully turbulent flow

Sec 5-5 Engineering Prediction of Transition

x_{tr} = location of transition to full turbulence

empirical correlations

two-step method of Granville (1953)
one-step method of Michel (1952)

Orr-Sommerfeld method of Jaffe

transition when T-S waves amplify

$$\frac{A}{A_0} \approx 10^4 \approx e^9$$

purely theoretical prediction

Cebeci and Smith correlation (1974)

one-step method of Wazzan (1979, 1981)

$$Re_{x,tr} = \text{func}(H)$$

author prefers method of Wazzan

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effect of freestream turbulence
=> hastens onset of turbulence

turbulence level $\quad T = \frac{1}{U}\left[\frac{1}{3}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)\right]^{1/2}$

for Falkner-Skan => van Driest and Blumer (1963) correlation otherwise ⇒ correlation of Dunham with method of Thwaites

effect of surface roughness ⇒ can hasten transition to turbulence

"trip" wire criterion

$$Uk/\nu \gtrsim 850$$

$$k/\delta^* \gtrsim 0.3$$

oscillating boundary layers $U_\infty = U_0\,(1 + N_A \sin \omega t)$

unsteady Re number, $Re_{ns} = \dfrac{N_A U_0^2}{\omega \nu}$

hastens transition if $Re_{ns} \gtrsim 27{,}000$

accelerating flow in a pipe ⇒ can be very stabilizing

chaos ⇒ applications in turbulent modeling