MECE 6372/6373

Vector Calculus Operators

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let $f = f(\vec{r})$ = arbitrary scalar function

let $\vec{A} = \vec{A}(\vec{r})$ = arbitrary vector function

let $\vec{\vec{g}} = \vec{\vec{g}}(\vec{r})$ = arbitrary (symmetric) tensor of second rank

subscripts $i, j \Rightarrow x, y, z$

<u>gradient operator</u>

$$\text{grad} f = \vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

$$= \frac{\partial f}{\partial x_j}\hat{x}_j$$

<u>divergence operator</u>

$$\text{div}\vec{A} = \vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \frac{\partial A_j}{\partial x_j}$$

<u>curl operator</u>

$$\text{curl}\,\vec{A} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y}$$

$$+ \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$$

$$= \epsilon_{ijk} \frac{\partial A_j}{\partial x_i} \hat{x}_k$$

where ϵ_{ijk} = permutation

→ $= 0$ if any two indexes same

→ $= +1$ if even permutation of xyz

→ $= -1$ if odd permutation of xyz

<u>Laplacian operator</u>

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial x_i \partial x_i}$$

<u>divergence of second rank tensor (symmetric)</u>

$$\overline{\nabla} \cdot \overline{g} = \frac{\partial}{\partial x_j} g_{ij} \hat{x}_i = \frac{\partial}{\partial x_i} g_{ij} \hat{x}_j$$

$\boxed{\text{Operator Identities (for any coordinate system)}}$

(1) $\overline{\nabla} \times (\overline{\nabla} f) = 0$ (2) $\overline{\nabla} \cdot (\overline{\nabla} \times \overline{A}) = 0$

(3) $\overline{\nabla} \cdot (f \overline{A}) = \overline{A} \cdot \overline{\nabla} f + f \overline{\nabla} \cdot \overline{A}$

(4) $\overline{\nabla}(fg) = f \overline{\nabla} g + g \overline{\nabla} f$

(5) $\overline{\nabla} \cdot (\overline{A} \times \overline{B}) = \overline{B} \cdot (\overline{\nabla} \times \overline{A}) - \overline{A} \cdot (\overline{\nabla} \times \overline{B})$

(6) $\overline{\nabla} \times (f \overline{A}) = \overline{\nabla} f \times \overline{A} + f (\overline{\nabla} \times \overline{A})$

(7) $\overline{\nabla} \times (\overline{A} \times \overline{B}) = (\overline{B} \cdot \overline{\nabla}) \overline{A} - (\overline{A} \cdot \overline{\nabla}) \overline{B} + (\overline{\nabla} \cdot \overline{B}) \overline{A} - (\overline{\nabla} \cdot \overline{A}) \overline{B}$

(8) $\nabla^2 \overline{A} = \overline{\nabla} (\overline{\nabla} \cdot \overline{A}) - \overline{\nabla} \times (\overline{\nabla} \times \overline{A})$

(9) $(\overline{A} \cdot \overline{\nabla}) \overline{A} = \overline{\nabla} \frac{A^2}{2} - \overline{A} \times (\overline{\nabla} \times \overline{A})$

(a) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = \frac{\partial}{\partial x_k}\left(\epsilon_{ijk} \frac{\partial A_j}{\partial x_i}\right) = \epsilon_{ijk} \frac{\partial^2 A_j}{\partial x_i \partial x_k}$

(b) $\vec{\nabla} \times (\vec{\nabla} f) = \epsilon_{ijk} \frac{\partial\left(\frac{\partial f}{\partial x_j}\right)}{\partial x_i} \hat{x}_k$