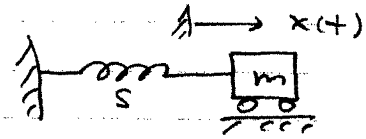
REVIEW NOTES

CHAPTER 1 FUNDAMENTALS OF VIBRATION

undamped spring-mass oscillator (initial condition)



$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

$$x(0) = x_0$$
$$u(0) = u_0$$

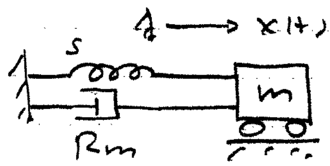
$\omega_0 = \sqrt{s/m}$

= undamped natural frequency

$$x(t) = x_0 \cos(\omega_0 t) + (u_0/\omega_0) \sin(\omega_0 t)$$
$$= A \cos(\omega_0 t + \phi)$$

$$A = \sqrt{x_0^2 + (u_0/\omega_0)^2} \quad , \quad \phi = -\tan^{-1}\left[\frac{(u_0/\omega_0)}{x_0}\right]$$

damped spring-mass oscillator (initial condition)



$$m\frac{d^2x}{dt^2} + R_m\frac{dx}{dt} + sx = 0$$

$$x(t) = A e^{-\beta t} \cos(\omega_d t + \phi) \qquad \left(\begin{array}{l}\beta < \omega_0 \\ \Rightarrow \text{underdamped}\end{array}\right)$$

$\beta = R_m/2m$ = temporal absorption coefficient

$\tau = 1/\beta = 2m/R_m$ = time constant

ω_d = damped natural frequency

$$= \sqrt{\omega_0^2 - \beta^2} < \omega_0$$

energy of vibration $E = \underbrace{\tfrac{1}{2} m u^2}_{\text{kinetic}} + \underbrace{\tfrac{1}{2} s x^2}_{\text{potential}}$

<u>damped spring-mass oscillator (sinusoidally forced)</u>

complex exponential method

$$m\frac{d^2\bar{x}}{dt^2} + R_m\frac{d\bar{x}}{dt} + s\bar{x} = \bar{F}e^{j\omega t}$$

$$\bar{x}(t) = \bar{A}e^{j\omega t}$$

physical part $x(t) = \text{Real}\{\bar{x}(t)\}$

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$\bar{A} = a + jb$$

$$\text{Real}\{\bar{A}e^{j\omega t}\} = a\cos(\omega t) - b\sin(\omega t)$$

$$= A\cos(\omega t + \phi)$$

$$A = \sqrt{a^2+b^2}\ ,\quad \phi = \tan^{-1} b/a$$

$$\bar{u}(t) = \frac{d\bar{x}}{dt} = (j\omega)\bar{x}(t) = \frac{\bar{F}e^{j\omega t}}{R_m + j(\omega m - s/\omega)}$$

$$= \bar{f}(t)/\bar{Z}_m(\omega)$$

$$\bar{Z}_m(\omega) = \frac{\bar{f}(t)}{\bar{u}(t)} = R_m + j(\omega m - s/\omega)$$

$$= R_m + j(m\omega_0)[(\omega/\omega_0) - (\omega_0/\omega)]$$

= mechanical impedance

$$= \underbrace{R_m}_{\text{mechanical resistance}} + j\underbrace{X_m(\omega)}_{\text{mechanical reactance}}$$

$$= |\bar{Z}_m|e^{j\Theta}$$

$$|\bar{Z}_m| = \sqrt{R_m^2 + X_m^2}\ , \quad \Theta = \tan^{-1}(X_m/R_m)$$

$$x(t) = \frac{F}{\omega|\bar{Z}_m|} \sin(\omega t - \Theta)$$

$$u(t) = \frac{F}{|\bar{Z}_m|} \cos(\omega t - \Theta)$$

- instantaneous power input

$$\Pi_i(t) = Re\{\bar{f}(t)\}\ Re\{\bar{u}(t)\}$$

- average power input

$$\Pi = \frac{1}{2} Re\{\bar{f}(t)\ \bar{u}^*(t)\} = \frac{|\bar{F}|^2}{2|\bar{Z}_m|} \cos(\Theta)$$

- mechanical resonance

input power => maximum

input mechanical reactance $= X(\omega) \Rightarrow 0$

ω = resonance frequency $= \omega_o$

$$\Pi \Rightarrow \Pi_{max} = \frac{|\bar{F}|^2}{2R_m}$$

- resonance quality (Q)

$$Q = \frac{\omega_o}{(\omega_u - \omega_l)} = \omega_o\left(\frac{m}{R_m}\right) = \frac{\omega_o}{2\beta} = \frac{1}{2}\omega_o T$$

$$= \frac{\text{"energy stored"}}{\text{"energy input per cycle"}}$$

- periodic forcing function

if $f(t)$ is periodic with fundamental frequency ω

then Fourier series analysis

$$f(t) = \tfrac{1}{2} A_0 + \sum_{n=1}^{\infty} [A_n \cos(n\omega t) + B_n \sin(n\omega t)]$$

where $$A_n = \frac{2}{T} \int_0^T f(t) \cos(n\omega t)\, dt$$

$$B_n = \frac{2}{T} \int_0^T f(t) \sin(n\omega t)\, dt$$

$$f(t) = A_n \cos(n\omega t) \Rightarrow x(t) = \mathrm{Re}\{\bar{x}(t)\}$$
$$= \frac{A_n}{(n\omega)|\bar{Z}_m(n\omega)|} \sin(n\omega t - \Theta_n)$$
$$\Theta_n = \tan^{-1} X_m(n\omega)/R_m$$

$$f(t) = B_n \sin(n\omega t) \Rightarrow x(t) = \mathrm{Im}\{\bar{x}(t)\}$$
$$= \frac{-B_n}{(n\omega)|\bar{Z}_m(n\omega)|} \cos(n\omega t - \Theta_n)$$

so
$$x(t) = \frac{1}{2}\frac{A_0}{s} + \sum_{n=1}^{\infty} \frac{1}{(n\omega)|\bar{Z}_m(n\omega)|}\Big[A_n \sin(n\omega t - \Theta_n) - B_n \cos(n\omega t - \Theta_n)\Big]$$

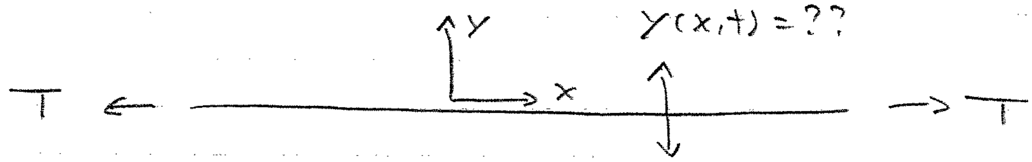
- if $f(t)$ is a transient pulse 

then can use Fourier transform

$$f(t) = \int_{-\infty}^{\infty} \bar{F}(\omega) e^{j\omega t} d\omega \quad \text{where } \bar{F}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

thus $$x(t) = \int_{-\infty}^{\infty} \frac{\bar{F}(\omega)}{(j\omega)\, \bar{Z}_m(\omega)} e^{j\omega t} d\omega$$

CHAPTER 2 TRANSVERSE MOTION: THE VIBRATING STRING



small displacements, uniform ρ_L and T

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$ one-dimensional wave equation

$$c = \sqrt{\frac{T}{\rho_L}}$$ = phase speed transverse force $= T \frac{\partial y}{\partial x}$

general traveling-wave solutions

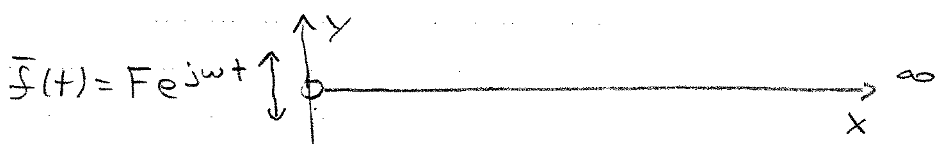
$$y(x,t) = y_1(ct - x) + y_2(ct + x)$$

- boundary conditions, reflection at boundaries

  fixed end $y(L,t) = 0$ ⇒ inverted reflection

  free end $\frac{\partial y}{\partial x}(L,t) = 0$ ⇒ noninverted reflection

- sinusoidally forced semi-infinite string



$$\bar{f}(t) = Fe^{j\omega t} = -T \frac{\partial \bar{y}}{\partial x}(0,t)$$

$$\bar{y}(x,t) = \bar{A} e^{j(\omega t - kx)}$$ ← rightward propagating harmonic traveling-wave

$$k = \omega/c = 2\pi/\lambda$$ = wavenumber

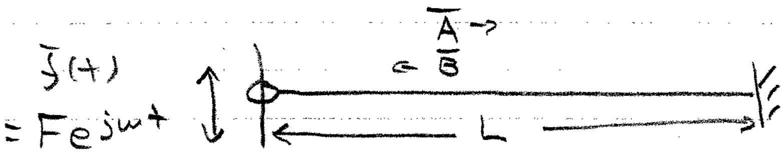
$$\lambda = 2\pi/k = \frac{2\pi c}{\omega} = c/f$$ = wavelength

$$\bar{A} = \frac{F}{jkT}$$

$$\bar{u}(x,t) = \frac{\partial \bar{x}}{\partial t} = j\omega\,\bar{x}(t) = \frac{F}{(\rho_L c)} e^{j(\omega t - kx)}$$

$$\bar{Z}_{mo} = \frac{\bar{f}(t)}{\bar{u}(0,t)} = (\rho_L c)$$ characteristic string impedance

<u>sinusoidally forced fixed end string</u>

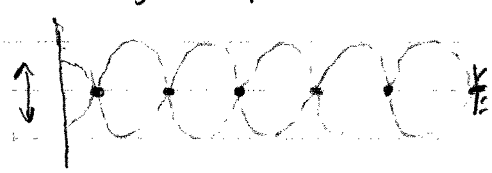


$$\bar{y}(x,t) = \bar{A}e^{j(\omega t - kx)} + \bar{B}e^{j(\omega t + kx)}$$

B.C.'s $\quad \bar{f}(t) = -T\frac{\partial \bar{y}}{\partial x}(0,t) \qquad \bar{y}(L,t) = 0$

find $\quad \bar{y}(x,t) = \dfrac{F}{kT}\,\dfrac{\sin[k(L-x)]}{\cos(kL)}\,e^{j\omega t}$

nodes, anti-nodes, resonance, antiresonance
standing waves, loops



<u>normal (natural) modes of a fixed-fixed string</u>
(initial condition driven)

assume $\bar{y}(x,t) = \bar{A}e^{j(\omega t - kx)} + \bar{B}e^{j(\omega t + kx)}$

eigenfunction (normal mode, natural mode)

$$y_n(x,t) = [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)]\sin(k_n x)$$

eigenvalues (natural frequencies)

$$\omega_n = n\pi(c/L) \Rightarrow f_n = \left(\tfrac{n}{2}\right)(c/L) \quad n = 1, 2, 3 \ldots$$

general solution

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t)$$

$$= \sum_{n=1}^{\infty} \left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \right] \sin(k_n x)$$

orthogonality of eigenfunctions, initial conditions

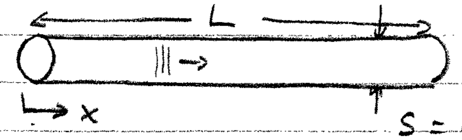
$$A_n = \frac{2}{L} \int_0^L y(x,0) \sin(n\pi x/L)\, dx$$

$$B_n = \frac{2}{\omega_n L} \int_0^L u(x,0) \sin(n\pi x/L)\, dx$$

- fundamental frequency, partials, overtones, harmonics
- energy density $\quad dE/dx = \frac{1}{2}\rho_L c^2 \Big[\underbrace{(\partial y/\partial x)^2}_{\text{potential}} + \underbrace{\left(\frac{1}{c}\frac{\partial y}{\partial t}\right)^2}_{\text{kinetic}} \Big]$

CHAPTER 3 VIBRATION OF BARS

- longitudinal (compression) waves in a "thin" bar



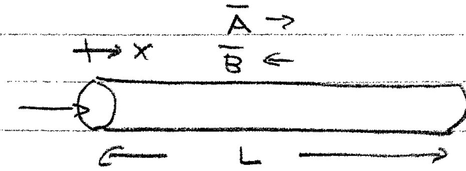
S = cross sectional area

$\xi(x,t)$ = displacement from equilibrium

$$f/S = -Y \frac{\partial \xi}{\partial x} \quad \text{(stress and strain relationship)}$$

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2} \qquad c = \sqrt{Y/\rho}$$

forced - free end bar



$\bar{f}(t) = Fe^{j\omega t}$ $\qquad \frac{\partial \bar{\xi}}{\partial x}(L,t) = 0$

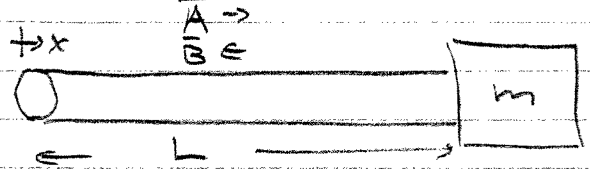
$$\bar{\xi}(x,t) = \bar{A}e^{j(\omega t - kx)} + \bar{B}e^{j(\omega t + kx)}$$

find $$\bar{\xi}(x,t) = \frac{-F}{SYk} \frac{\cos[k(L-x)]}{\sin(kL)} e^{j\omega t}$$

$$\bar{Z}_{mo} = \frac{\bar{f}(t)}{\bar{u}(0,t)} = j(\rho S)c\tan(kL)$$

nodes, antinodes, resonance, etc.

natural modes of a free-mass loaded bar



$\frac{\partial \bar{\xi}}{\partial x}(0,t) = 0 \qquad -SY\frac{\partial \bar{\xi}}{\partial x}(L,t) = m\frac{\partial^2 \bar{\xi}}{\partial t^2}(L,t)$

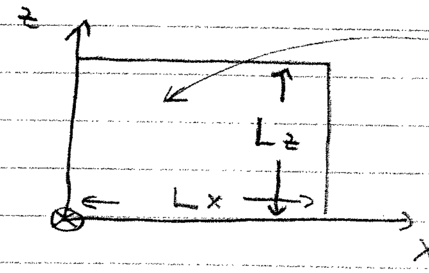
assume $\bar{\xi}(x,t) = \bar{A}\, e^{j(\omega t - kx)} + \bar{B}\, e^{j(\omega t + kx)}$

and determine eigenfrequencies (natural frequencies)

$$\tan(kL) = -\frac{m}{m_b}(kL) \qquad m_b = \rho S L$$

natural frequencies not necessarily harmonic

CHAPTER 4 THE TWO DIMENSIONAL WAVE EQUATION: VIBRATION OF MEMBRANES



$y(x,z,t) = ??$

τ = surface tension

transverse force $\sim \partial y / \partial x,\ \dfrac{\partial y}{\partial z}$

$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}\right) \qquad c = \sqrt{\frac{\tau}{\rho_S}}$$

two-dim wave equation

two-dim Laplacian $= \nabla^2_{2Dim}$

or in cylindrical coordinates $y(r,\theta,t)$

$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) y$$

natural modes of vibration

assume $\bar{y} = \bar{\Psi} e^{j\omega t}$ $\qquad \bar{\Psi} = \text{func}(x,z)$ only

then $\nabla^2_{2Dim}\bar{\Psi} + k^2\bar{\Psi} = 0$ Helmholtz equation natural modes for a rectangular membrane with fixed rim

separation of variables solution of Helmholtz equation

- eigenfunctions (natural modes)

$$\bar{y}_{n,m}(x,y,t) = A_{n,m} \sin(k_{x,n} x) \sin(k_{z,m} z) e^{j\omega_{n,m} t}$$

eigenvalues $\quad k_{x,n} = \frac{n\pi}{L_x} \quad n = 1,2,3 \ldots$

$$k_{z,m} = \frac{m\pi}{L_z} \quad m = 1,2,3 \ldots$$

- eigenfrequencies (natural frequencies)

$$k^2 = k_x^2 + k_z^2 = \omega^2 / c^2$$

$$\omega_{n,m} = c \left[\left(\frac{n\pi}{L_x} \right)^2 + \left(\frac{m\pi}{L_z} \right)^2 \right]^{1/2}$$

overtones not necessarily harmonic!

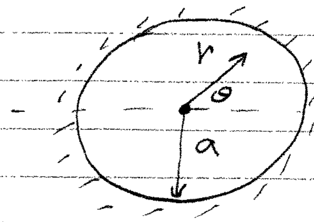
natural modes for a circular membrane with a fixed rim

separation of variables solution of Helmholtz equation in cylindrical coordinates

- eigenfunctions (natural modes)

$$\bar{y}_{m,n}(r,\theta,t) = \bar{A}_{m,n} J_m(k_{m,n} r) \cos(m\theta + \gamma_{m,n}) \times e^{j\omega_{m,n} t}$$

$J_m(\alpha) = m^{th}$ order Bessel's function of the first kind



radial nodal lines
circular nodal lines

- eigenvalues $k_{m,n} = \frac{j_{m,n}}{a}$

  $j_{m,n}$ = n^{th} zero of $J_m(\alpha)$

- eigenfrequencies (natural frequencies)

$$\omega_{m,n} = (c/a)\, j_{m,n}$$

overtones not harmonic!

angularly symmetric (purely radial) modes

$$y_n(r,t) = A_n J_0(k_n r) \cos(\omega_n t + \phi_n)$$

$$k_n = \frac{j_{0,n}}{a} = \omega_n / c$$

general solution

$$y(r,t) = \sum_{n=1}^{\infty} A_n J_0(k_n r) \cos(\omega_n t + \phi_n)$$

average surface displacement of n^{th} mode

$$\langle \Psi_n \rangle_s = \frac{2A_n}{j_{0,n}} J_1(j_{0,n})$$

decreases with increasing n!

CHAPTER 5 THE ACOUSTIC WAVE EQUATION AND SIMPLE SOLUTIONS

- basic concepts of fluid mechanics (what is a fluid, fluid particle)

- $\vec{\xi}(\vec{r},t)$ = acoustic particle displacement

  $\vec{u}(\vec{r},t) = \partial\vec{\xi}/\partial t$ = acoustic particle velocity

  $p(\vec{r},t) = \mathcal{P}(\vec{r},t) - \mathcal{P}_0$ = acoustic pressure

  $s(\vec{r},t) = (\rho - \rho_0)/\rho_0$ = acoustic condensation

  $(\rho - \rho_0) = \rho_0 s$ = acoustic mass density

- general conservation of mass equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

- linearized (acoustic) consv. of mass equation

$$\boxed{\frac{\partial s}{\partial t} + \vec{\nabla}\cdot\vec{u} = 0}$$

- general conservation of momentum equation (inviscid)

$$\rho\frac{\partial \vec{u}}{\partial t} + \rho\,\vec{u}\cdot\vec{\nabla}\vec{u} = -\vec{\nabla}\mathcal{P}$$

- linearized (acoustic) momentum equation

$$\boxed{\rho_0 \frac{\partial \vec{u}}{\partial t} = -\vec{\nabla} p}$$

assume adiabatic process then

$$d\rho = \left.\frac{\partial \rho}{\partial P}\right)_{adiab} dP \Rightarrow \boxed{s = p/\mathcal{B}}$$

$$\mathcal{B} = \text{adiabatic bulk modulus} = \rho \left.\frac{\partial P}{\partial \rho}\right)_{adiab}$$

or isothermal process then

$$d\rho = \left.\frac{\partial \rho}{\partial P}\right)_T dP \Rightarrow s = p/\mathcal{B}_T \leftarrow \text{isothermal bulk modulus}$$

eliminate s, $\vec{u}$ in terms of p

$$\Rightarrow \boxed{\frac{\partial^2 p}{\partial t^2} = c^2 \nabla^2 p}$$

three dimensional wave equation

$$c = \sqrt{\mathcal{B}/\rho_0} = \sqrt{\left.\partial P/\partial \rho\right)} \qquad \boxed{s = p/\rho_0 c^2}$$

- acoustic velocity potential

$$\vec{\nabla} \times \vec{u} = 0 \quad \text{(irrotational)}$$

so $\boxed{\vec{u} = \vec{\nabla} \Phi}$ Φ = acoustic velocity potential

- p, s, Φ, and $\vec{u}$ all satisfy wave equation

<u>perfect gas</u>

$$P = \rho r T_K$$

$$r = \text{gas constant} = \frac{\mathcal{R} \leftarrow \text{universal gas constant } 8314 \frac{J}{kmole\,K}}{M \leftarrow \text{molecular weight}}$$

ideal gas undergoing an adiabatic, reversible process

$$(\mathcal{P}/\mathcal{P}_0) = (\rho/\rho_0)^{\gamma} \qquad \gamma = c_p/c_v$$

$$\mathcal{B} = \rho\, \partial\mathcal{P}/\partial\rho)_{adiab} = \gamma\mathcal{P} \Rightarrow c = \sqrt{\gamma\mathcal{P}/\rho} = \sqrt{\gamma r T_K}$$

$$\mathcal{B}_T = \rho\, \partial\mathcal{P}/\partial\rho)_T = \mathcal{P} \Rightarrow c_T = \sqrt{\mathcal{P}/\rho} = \sqrt{r T_K}$$

harmonic plane wave (x-direction)

$$\bar{p}(x,t) = \bar{A}e^{j(\omega t - kx)} + \bar{B}e^{j(\omega t + kx)}$$

$$\bar{u}_{\pm} = \pm\frac{\bar{p}_{\pm}}{\rho_0 c} \qquad s_{\pm} = \frac{\bar{p}_{\pm}}{\rho_0 c^2}$$

$$\bar{\xi}_{\pm} = \frac{\mp j\bar{p}_{\pm}}{\omega\rho_0 c} \qquad \bar{\Phi}_{\pm} = \frac{-\bar{p}_{\pm}}{j\rho_0\omega} = \frac{j\bar{p}_{\pm}}{\rho_0\omega}$$

harmonic planewave (arbitrary direction)

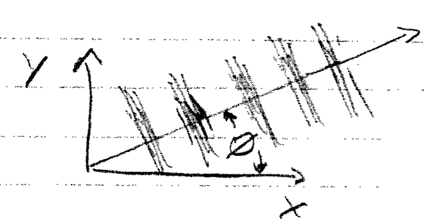
$$\bar{p}(\vec{r},t) = \bar{A}e^{j(\omega t - \vec{k}\cdot\vec{r})}$$

$$\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} = k\hat{k}$$

$$k = \sqrt{k_x^2 + k_y^2 + k_z^2} = \omega/c$$

if parallel to x-y plane then

$$\bar{p}(\vec{r},t) = \bar{A}e^{j(\omega t - kx\cos\phi - ky\sin\phi)}$$



acoustic energy density

instantaneous $\mathcal{E}_i(t) = \frac{1}{2}\rho_0\left[\underbrace{u^2}_{\text{kinetic}} + \underbrace{\left(p/\rho_0 c\right)^2}_{\text{potential}}\right]$

for a plane wave $\quad \mathcal{E}_i(t) = \rho_0 u^2 = \dfrac{p^2}{\rho_0 c^2}$

time-averge (plane wave)

$$\mathcal{E} = \langle \mathcal{E}_i(t) \rangle = \frac{1}{2}\rho_0 U^2 = \frac{1}{2}\frac{P^2}{\rho_0 c^2}$$

$$= \rho_0 U_e^2 = \frac{P_e^2}{\rho_0 c^2}$$

effective (root-mean-square) acoustic pressure

$$P_e = \left\{\frac{1}{T}\int_0^T p^2\,dt\right\}^{1/2} = \frac{P}{\sqrt{2}} \quad \text{if sinusoidal!}$$

acoustic intensity (rate of energy per unit area crossing a surface)

instantaneous $\quad I_i(t) = p\,u$

for a plane wave $\quad I_i(t) = \pm\dfrac{p^2}{\rho_0 c}$

time average (plane wave)

$$I = \langle I_i(t) \rangle = \pm\frac{P^2}{2\rho_0 c} = \pm\frac{P_e^2}{\rho_0 c}$$

specific acoustic impedance (per unit area)

$$\bar{z} = \bar{p}/\bar{u}$$

for a plane wave in "free" space $\quad \bar{z} = \pm\rho_0 c$

$\rho_o c$ = characteristic specific impedance of medium

$$\{\rho_o c\} = \left\{\frac{Pa \cdot s}{m}\right\} = \{rayl\}$$

in general $\bar{Z} = \underbrace{r}_{\text{resistance}} + j \underbrace{x}_{\text{reactance}}$

decibel scales

human audible range $\simeq 10^{-12}\ W/m^2 \rightarrow \simeq 100\ W/m^2$

($\simeq 20\ \mu Pa$) threshold of hearing — ($\simeq 200\ Pa$) threshold of hearing pain

- intensity level

$$IL = 10 \log_{10}\left(\frac{I}{I_{ref}}\right)$$

or since $I \sim P_e^2$

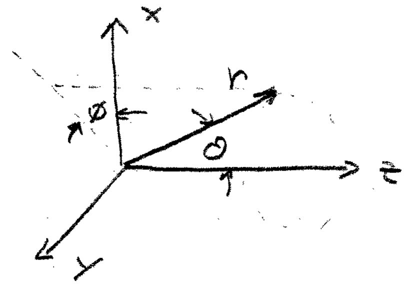
- soundpressure level

$$SPL = 20 \log_{10}\left(\frac{P_e}{P_{ref}}\right)$$

in air, typically $P_{ref} = 20\ \mu Pa$

in water, typically $P_{ref} = 1\ \mu Pa$

Sec 5-11 Spherical Waves



spherical coordinates

in general

$p(r,\theta,\phi,t)$

if spherically symmetric then $p(r,t)$ only and

$$p(r,t) = \frac{1}{r} f_1(ct-r) + \frac{1}{r} f_2(ct+r)$$

outgoing incoming

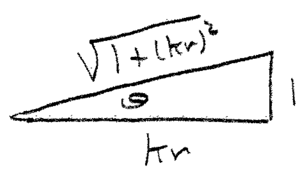
if outgoing harmonic spherical wave

$$\bar{p}(r,t) = \frac{\bar{A}}{r} e^{j(\omega t - kr)}$$

note $|\bar{p}(r,t)| = P = |\bar{A}|/r \sim 1/r$

$$\bar{u}(r,t) = \frac{1}{\rho_o c}\{1 - j/kr\}\,\bar{p}(r,t)$$

$$\bar{z} = \bar{P}/\bar{u} = \rho_o c \cos\theta\, e^{j\theta}$$



$\cos\theta = \frac{kr}{\sqrt{1+(kr)^2}}$, $\theta = \tan^{-1}\left(\frac{1}{kr}\right)$

$= \cot^{-1}(kr)$

intensity $I(r) = \frac{|\bar{A}|^2/r^2}{2\rho_o c} = \frac{P^2}{2\rho_o c} \sim 1/r^2$

power over spherical surface

$$\Pi = (4\pi r^2)\, I(r) = \text{constant!}$$