Engineering Acoustics
EXAMPLE

response of a forced spring-damped-mass oscillator to a square wave input

in general, if $f(t)$ is periodic with $\omega = 2\pi/T$ then

$$f(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left[A_n \cos(n\omega t) + B \sin(n\omega t)\right]$$

where $$A_n = \frac{1}{T}\int_0^T f(t)\cos(n\omega t)\,dt$$

and $$B_n = \frac{1}{T}\int_0^T f(t)\sin(n\omega t)\,dt$$

and then

$$x(t) = \frac{1}{2}\frac{A_0}{s} + \sum_{n=1}^{\infty}\frac{1}{(n\omega)|Z_m(n\omega)|}\left[A_n \sin(n\omega t - \Theta_n) - B_n\cos(n\omega t - \Theta_n)\right]$$

where $$Z_m(n\omega) = R_m + j\left[m(n\omega) - \frac{s}{(n\omega)}\right]$$

and $$\Theta_n = \tan^{-1}\left\{\frac{m(n\omega) - s/(n\omega)}{R_m}\right\}$$

for a square wave where (odd function) $$f(t) = \begin{cases} +F_0 & 0 \le t < T/2 \\ -F_0 & T/2 < t < T \end{cases}$$

$A_n = 0$ for all n

$B_n = \frac{4F_0}{n\pi}$ for $n = 1, 3, 5 \cdots$ (odd)

$B_n = 0$ for $n = 0, 2, 4, \cdots$ (even) thus correspondingly

$$x(t) = -\frac{4F_o}{\pi}\sum_{n=1,3,5}^{\infty} \frac{1}{n^2\omega\,|\bar{Z}_m(n\omega)|}\cos(n\omega t - \Theta_n)$$

nondimensionalize before plotting

$$n\omega t = n\left(\frac{2\pi}{T}\right)t = 2\pi n\tilde{t} \qquad \boxed{\tilde{t} = t/T}$$

$$\omega\,|\bar{Z}_m(n\omega)| = \omega\left[R_m^2 + (\omega_o m)^2\left(\frac{n\omega}{\omega_o} - \frac{\omega_o}{n\omega}\right)^2\right]^{1/2}$$

$$= \omega\,\omega_o m\left[\left(\frac{R_m}{\omega_o m}\right)^2 + \left(\frac{n\omega}{\omega_o} - \frac{\omega_o}{n\omega}\right)^2\right]^{1/2}$$

$\beta = \frac{R_m}{2m}$

$$= \left(\frac{\omega}{\omega_o}\right)\omega_o^2 m\left[\left(\frac{2\beta}{\omega_o}\right)^2 + \left(n\left(\frac{\omega}{\omega_o}\right) - \frac{1}{n}\left(\frac{\omega_o}{\omega}\right)\right)^2\right]^{1/2}$$

($\omega_o^2 = s/m$)

$$= s\,\tilde{\omega}\left[\frac{1}{Q^2} + \left(n\tilde{\omega} - \frac{1}{n\tilde{\omega}}\right)^2\right]^{1/2}$$

where $\boxed{\tilde{\omega} = (\omega/\omega_o)}$

also

$$\Theta_n = \tan^{-1}\frac{m\omega_o\left(\frac{n\omega}{\omega_o} - \frac{\omega_o}{n\omega}\right)}{R_m}$$

$$\boxed{\Theta_n = \tan^{-1}\left[Q\left(n\tilde{\omega} - \frac{1}{n\tilde{\omega}}\right)\right]}$$

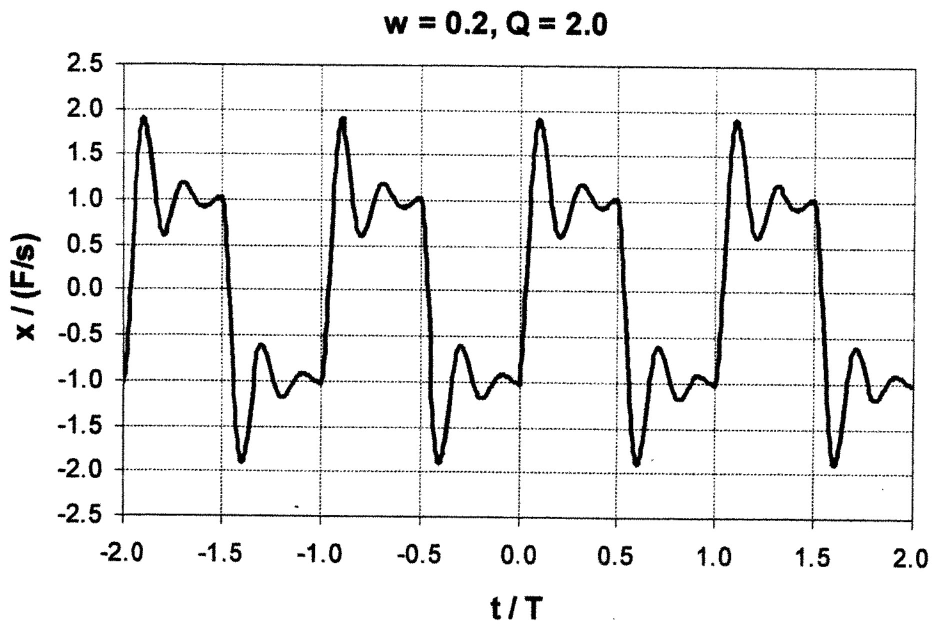
so

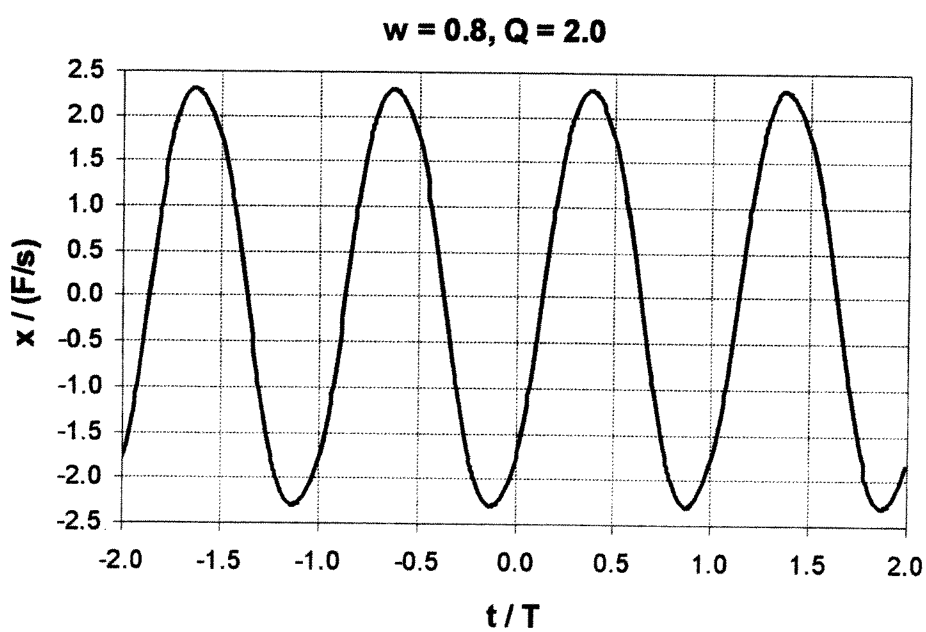
$$\boxed{\tilde{x}(t) = \frac{x(t)}{F_o/s} = -\frac{4}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2\tilde{\omega}\left[\frac{1}{Q^2} + \left(n\tilde{\omega} - \frac{1}{n\tilde{\omega}}\right)^2\right]^{1/2}} \times \cos(2\pi n\tilde{t} - \Theta_n)}$$

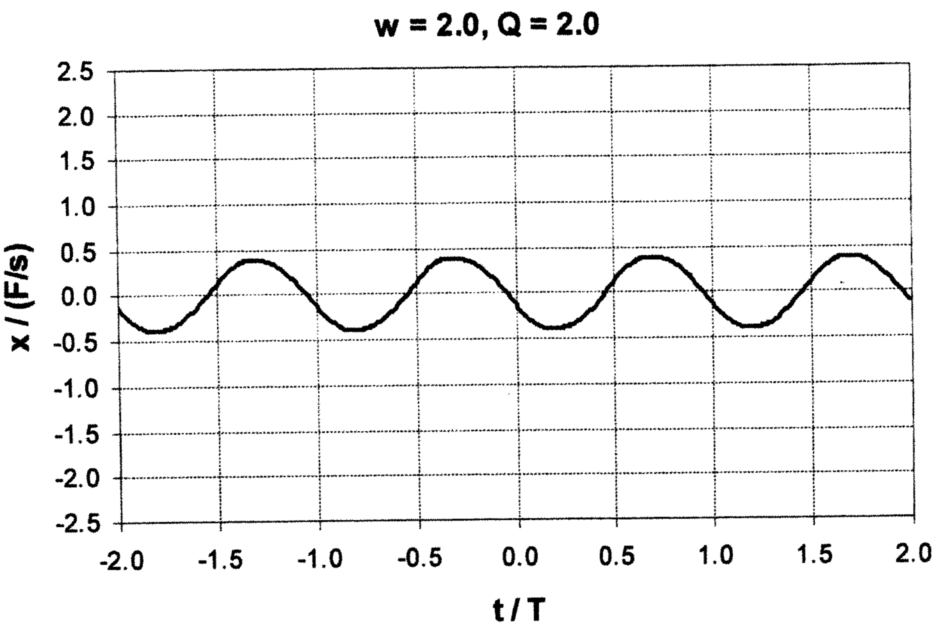
can determine $\tilde{x}(t)$ for any $\tilde{\omega}$, Q combination

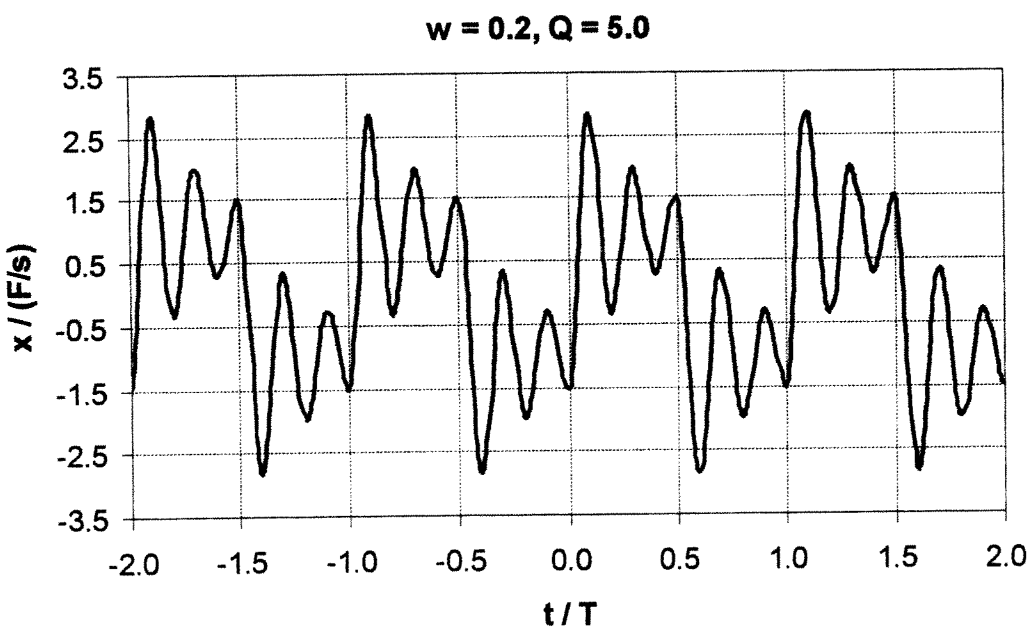
see attached FORTRAN program listing and sample plots

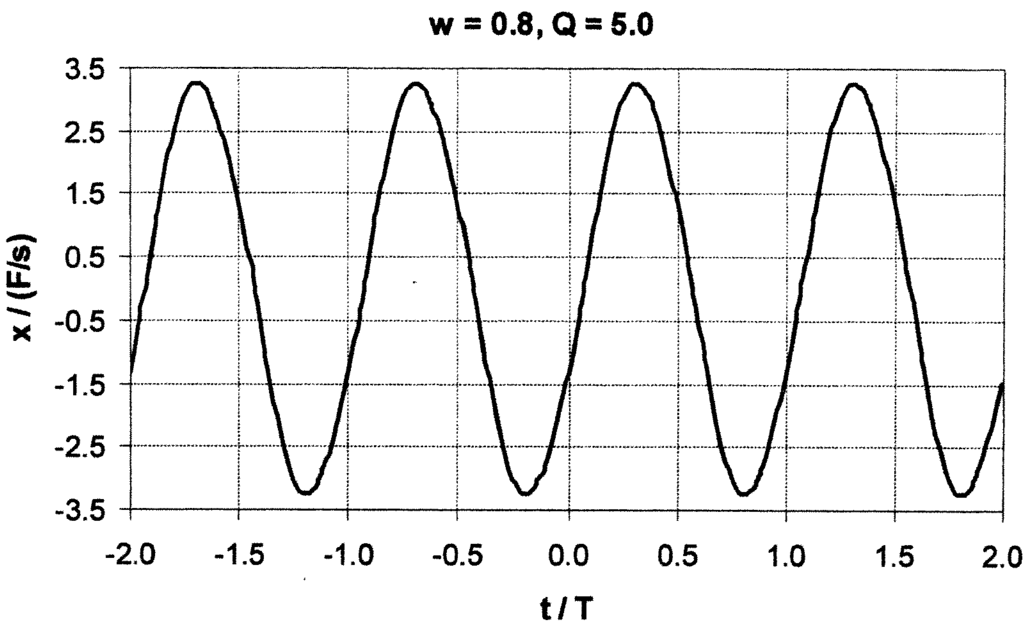
```
c
c       program to calculate the time response of a forced spring-damped-
c       mass oscillator to a square wave input (odd function)
c
        dimension x(0:100),th(51),den(51)
        real nf
        data pi/3.14159265/
        write(6,*) ' input omega_tilde,Q '
        read(5,*) w,q
        do 10 n=1,51,2
        nf=dfloat(n)
        den(n)=nf**2*w*sqrt(1./q**2+(nf*w-1./(nf*w))**2)
   10   th(n)=atan(q*(nf*w-1./(nf*w)))
        do 100 it=0,100
        t=dfloat(it)/100.
        x(it)=0.0
        do 20 n=1,51,2
        nf=float(n)
   20   x(it)=x(it)-4.*cos(2.*pi*nf*t-th(n))/(pi*den(n))
  100   continue
        open(unit=1,file='ome416x01.dat')
        do 200 j=1,4
        do 200 it=0,100
  200   write(1,*) x(it)
        close(unit=1)
        end
```

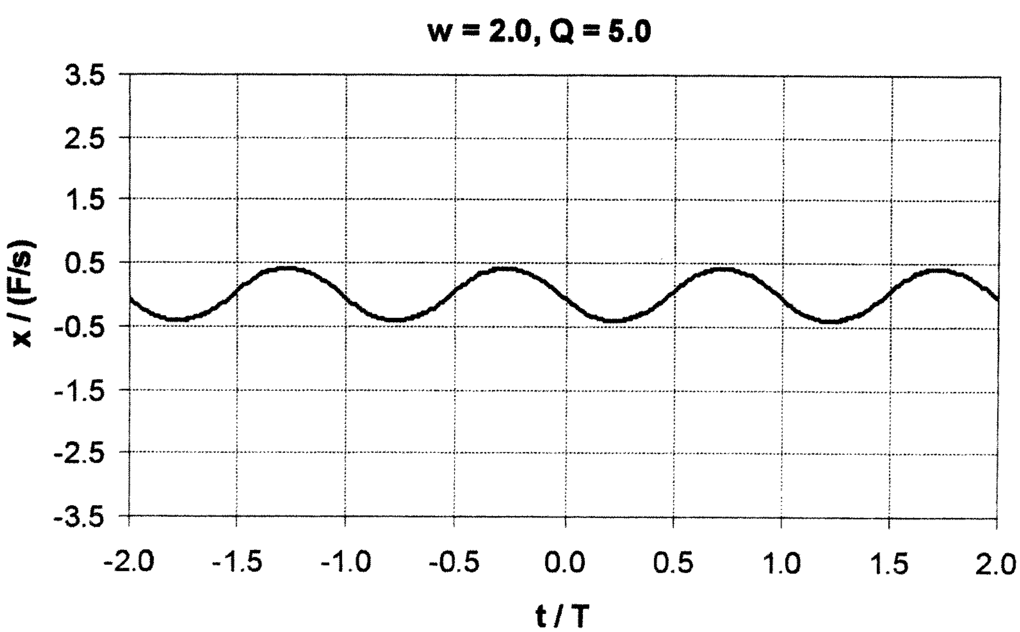

w = 0.2, Q = 2.0
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
-2.5
x / (F/s)
-2.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
t / T


w = 0.8, Q = 2.0
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
-2.5
x / (F/s)
-2.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
t / T


w = 2.0, Q = 2.0
x / (F/s)
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
-2.5
-2.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
t / T


w = 0.2, Q = 5.0
x / (F/s)
3.5
2.5
1.5
0.5
-0.5
-1.5
-2.5
-3.5
-2.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
t / T


w = 0.8, Q = 5.0
x / (F/s)
3.5
2.5
1.5
0.5
-0.5
-1.5
-2.5
-3.5
-2.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
t / T


w = 2.0, Q = 5.0
x / (F/s)
3.5
2.5
1.5
0.5
-0.5
-1.5
-2.5
-3.5
-2.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
t / T