$$k_n L = n\pi \quad n = 1, 2, 3$$

$$\frac{2\pi f_n}{c} L = n\pi$$

$$\boxed{f_n = \left(\frac{n}{2}\right)\left(c/L\right) \quad Eq\ (3.4.10)}$$

natural frequencies of a free-free bar (overtones are harmonics)

if $m \gg m_b$ (fixed end)

$$\tan(k_n L) \Rightarrow -\infty$$

$$\cos(k_n L) = 0$$

$$k_n L = \frac{(2n-1)}{2}\pi$$

$$\frac{2\pi f_n L}{c} = \frac{(2n-1)}{2}\pi$$

$$\boxed{f_n = \frac{(2n-1)}{4}\left(c/L\right) \quad Eq\ (3.4.13)}$$

nat. freq. of a free-fixed bar (overtones are harmonics) (note: no even harmonics of fund.)

otherwise

$$\tan(k_n L) = -\left(\frac{m}{m_b}\right)(k_n L)$$

f_n not nec. harmonics of fundamental! (overtones not harmonic)

---

finish Chap 3

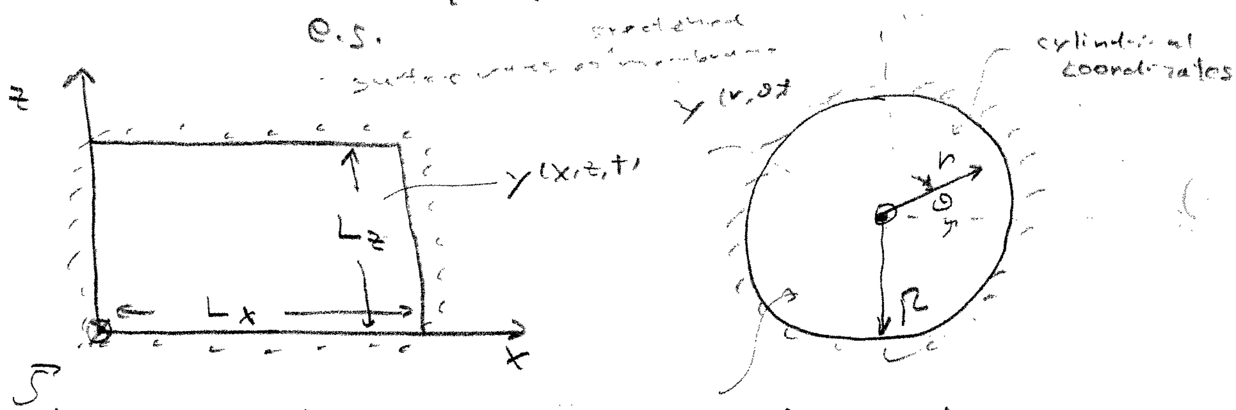
advanced topics → flextural waves
→ shear waves

Chap 4 The Two-Dimensional Wave Equation: Vibrations of Membranes

So far have considered one-dimensional waves (waves that can propagate only in one spatial direction) and the corresponding one-dimensional form of the wave equation

as we gradually progress towards the full three dimensional wave equation, in Chap 4 consider transverse vibrations of a stretched membrane
→ two-dim wave equation

e.g.



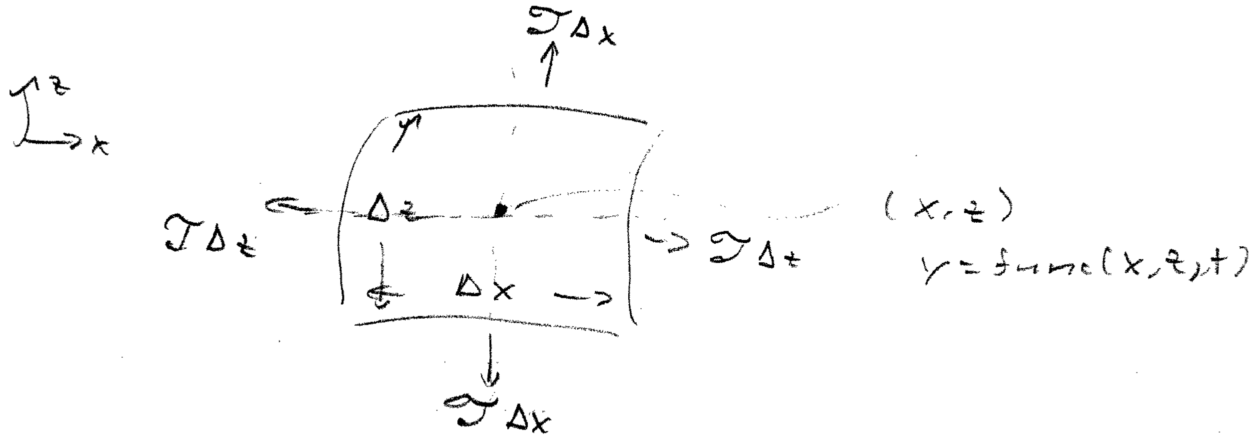
rectangular membrane, fixed on perimeter vibrates in y-direction

circular membrane (e.g. kettle drum) fixed on circumference vibrates in y-direction

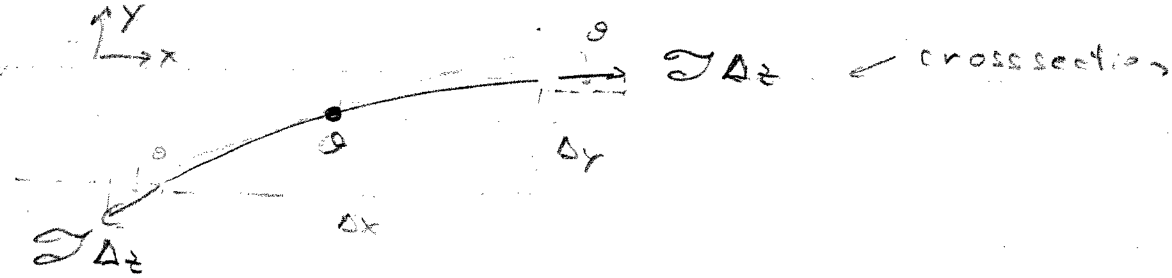
assume ⇒ (as with string)
- small displacements
- uniform surface tension ($\mathcal{T}$)
- uniform surface density (ρ_s = mass/area)
- neglect gravitational effects

consider free body diagram of incremental surface element of membrane

(rectangular coordinates)



$\mathcal{T}$ = surface tension of membrane $\left[\frac{\text{force}}{\text{length}}\right], \left[\frac{N}{m}\right]$



$$\Delta F_y\Big)_x = (\mathcal{T}\sin\theta\,\Delta z)_{x+\Delta x/2} - (\mathcal{T}\sin\theta\,\Delta z)_{x-\Delta x/2}$$

$$= \frac{\partial\,\mathcal{T}\sin\theta}{\partial x}(\Delta x\,\Delta z) = \frac{\partial}{\partial x}\left(\mathcal{T}\frac{\partial y}{\partial x}\right)\Delta x\,\Delta z$$

$$= \mathcal{T}\frac{\partial^2 y}{\partial x^2}\,\Delta x\,\Delta z$$

likewise

$$\Delta F_y\Big)_z = \mathcal{T}\frac{\partial^2 y}{\partial z^2}\,\Delta x\,\Delta z$$

Newton's 2nd law of motion

$$(\underbrace{\rho_s\,\cancel{\Delta x\,\Delta z}}_{\text{mass}})\frac{\partial^2 y}{\partial t^2} = \Delta F_y$$

$$= \mathcal{T}\left[\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial z^2}\right)\Delta x\,\Delta z$$

$$\frac{\partial^2 y}{\partial t^2} = \left(\frac{\mathcal{T}}{\rho_s}\right)\left(\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial z^2}\right)$$

define $\quad c = \sqrt{\dfrac{T}{\rho_s}}$ = phase speed $\quad (c\uparrow \text{ as } T\uparrow, \rho\downarrow)$

so

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \left(\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial z^2} \right) \qquad \text{Eq } (4.2.2)}$$

two-dimensional wave equation

or can introduce a vector calculus operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$ (two-dim) Laplacian operator

so equivalently

$$\boxed{\nabla^2 y = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \qquad \text{Eq } (4.2.4)}$$

even though derived in rectangular coordinates, in vector calculus form, valid for any orthogonal coordinate system, as long as use correct definition of ∇^2

e.g., if use cylindrical (polar) coordinates (see App A7)

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \qquad y(r, \theta, t)$$

(simple harmonic)

now for sinusoidal solutions, expect

$$\bar{y} = \bar{\Psi} e^{j\omega t} \quad \text{so} \quad \leftarrow \bar{\Psi} = \text{func}(x, z) \text{ or func}(r, \theta)$$

$$\nabla^2 \bar{\Psi} \cancel{e^{j\omega t}} = -\frac{\omega^2}{c^2} \bar{\Psi} \cancel{e^{j\omega t}}$$

$$k = \omega / c$$

$$\boxed{\nabla^2 \bar{\Psi} + k^2 \bar{\Psi} = 0 \qquad \text{Eq } (4.2.9)}$$

Lecture #14 Vibrating Membrane
H.W. #11 Prob 4.4.4, 4.5.1 (do not turn in)
EXAM #1, Wed., Feb. 16, Chap 1, 2, 3, 4

- will finish up, review for exam on Monday

- H.W. #11, do not turn in, do to help prepare for exam, will make solutions available

- last time => vibrating membrane

two-dim wave equation $\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial z^2}\right)$

$$= c^2 \nabla^2 y$$

$$c = \sqrt{\frac{\mathcal{T}}{\rho_s}}$$

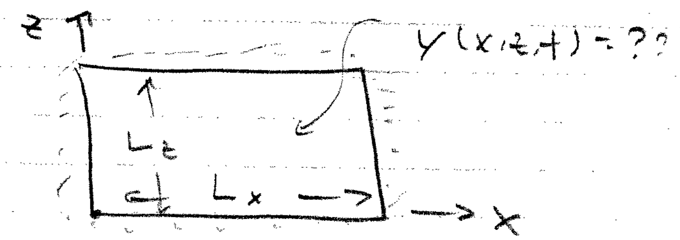
if $\bar{y} = \bar{\Psi} e^{j\omega t}$ $\quad \bar{\Psi} \to$ func(x, z), $\to$ func(r, θ)

then $$\boxed{\nabla^2 \bar{\Psi} + k^2 \bar{\Psi} = 0 \quad Eq\ (4.2.9)}$$

Helmholtz equation
(time independent wave equation)

Helmholtz equation
(time-independent wave equation)

consider now unforced vibration of a rectangular membrane with a fixed rim



B.C.'s $\quad y(0,z,t)=0 \qquad y(L_x,z,t)=0$

$\qquad y(x,0,t)=0 \qquad y(x,L_z,t)=0$

I.C.'s $\quad y(x,z,0) \Rightarrow$ known $\quad \partial y/\partial t\,(x,z,0) \Rightarrow$ known

assume solution of the form

$$\bar{y}(x,z,t) = \bar{\Psi} e^{j\omega t}$$

(can build up more general solutions by summing)

then

$$\frac{\partial^2 \bar{\Psi}}{\partial x^2} + \frac{\partial^2 \bar{\Psi}}{\partial z^2} + k^2 \bar{\Psi} = 0$$

try to solve by separation of variables, assume a solution of the form

$$\bar{\Psi}(x,z) = \bar{X}(x)\,\bar{Z}(z) \quad \text{then}$$

$$\bar{Z}\frac{d^2\bar{X}}{dx^2} + \bar{X}\frac{d^2\bar{Z}}{dz^2} + k^2\bar{X}\bar{Z} = 0$$

or

$$\underbrace{\frac{1}{\bar{Z}}\frac{d^2\bar{Z}}{dz^2}}_{\text{func}(z)\text{ only}} = -k^2 - \underbrace{\frac{1}{\bar{X}}\frac{d^2\bar{X}}{dx^2}}_{\text{func}(x)\text{ only}} = \text{constant} = -k_z^2$$

so $\qquad \dfrac{d^2\bar{Z}}{dz^2} + k_z^2\,\bar{Z} = 0 \qquad$ also

$$\frac{1}{\bar{X}}\frac{d^2\bar{X}}{dx^2} = -k^2 + k_z^2 = -k_x^2$$

where $\boxed{k_x^2 + k_z^2 = k^2}$ Eq (4.3.7)

so
$$\frac{d^2\bar{X}}{dx^2} + k_x^2\bar{X} = 0$$

several solutions

$$\bar{X}(x) = \bar{C}_1 \cos(k_x x) + \bar{C}_2 \sin(k_x x)$$
$$\bar{Z}(z) = \bar{C}_3 \cos(k_z z) + \bar{C}_4 \sin(k_z z)$$

B.C.'s $\bar{y}(0,z,t) = \bar{C}_1 \bar{Z}(z) e^{j\omega t} = 0$ → $\bar{y}(x,z,t) = \bar{X}(x)\bar{Z}(z)$

$$\underline{\bar{C}_1 = 0}$$

$$\bar{y}(x,0,t) = \bar{C}_3 \bar{X}(x) e^{j\omega t} = 0$$

$$\underline{\bar{C}_3 = 0}$$

$$\bar{y}(x,z,t) = \bar{A} \sin(k_x x) \sin(k_z z) e^{j\omega t}$$
$$\bar{y}(L_x,z,t) = \bar{A} \sin(k_x L_x) \sin(k_z z) e^{j\omega t} = 0$$

$$k_x L_x = n\pi \qquad n = 1, 2, 3 \ldots$$

$$\boxed{k_{x,n} = \frac{n\pi}{L_x}} \leftarrow \text{eigenvalue}$$

$$\bar{y}(x,L_z,t) = \bar{A} \sin(k_x x) \sin(k_z L_z) e^{j\omega t} = 0$$

$$\boxed{k_{z,m} = \frac{m\pi}{L_z}} \qquad m = 1, 2, 3 \ldots$$

thus

$$\bar{y}_{n,m}(x,z,t) = \bar{A}_{n,m} \sin\left(n\pi \frac{x}{L_x}\right) \sin\left(m\pi \frac{z}{L_z}\right) e^{j\omega_{nm} t}$$

eigenfunction / natural mode / normal mode where $$k^2_{n,m} = k^2_{x,n} + k^2_{z,m}$$

$$= \left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_z}\right)^2 = \frac{\omega^2_{m,n}}{c^2}$$

$$\omega_{n,m} = c\left\{\left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_z}\right)^2\right\}^{1/2} = 2\pi f_{n,m}$$

$$f_{n,m} = \left(\frac{c}{2}\right)\left[\left(\frac{n}{L_x}\right)^2 + \left(\frac{m}{L_z}\right)^2\right]^{1/2}$$

eigenfrequencies (natural/normal frequencies)

$m = 1, 2, 3 --$

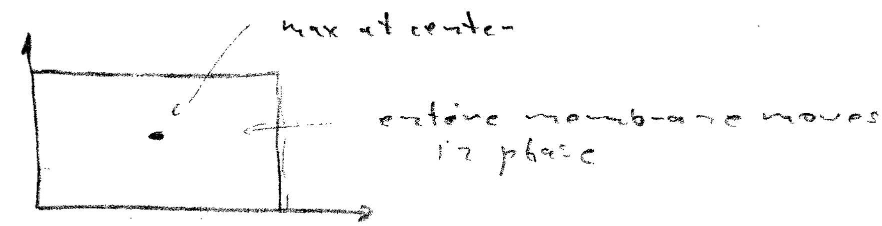
$n = 1, 2, 3 --$

fundamental (lowest) $f_{1,1} = \frac{c}{2}\left[\left(\frac{1}{L_x}\right)^2 + \left(\frac{1}{L_z}\right)^2\right]^{1/2}$

overtones not nec. harmonic!

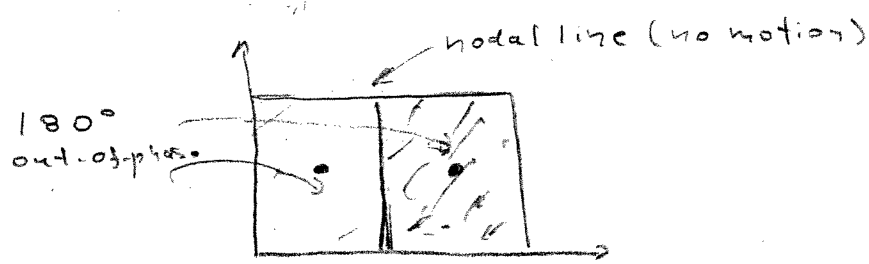
(1,1) mode

$$\bar{y}_{1,1} \sim \sin\left(\pi \frac{x}{L_x}\right) \sin\left(\pi \frac{z}{L_z}\right)$$

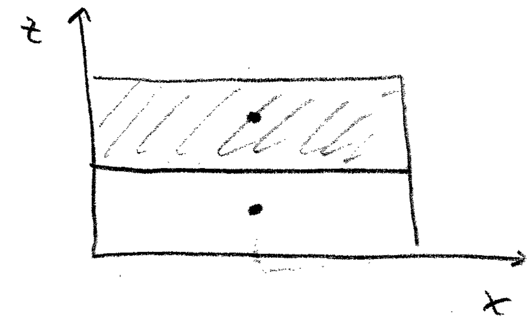


(2,1) mode

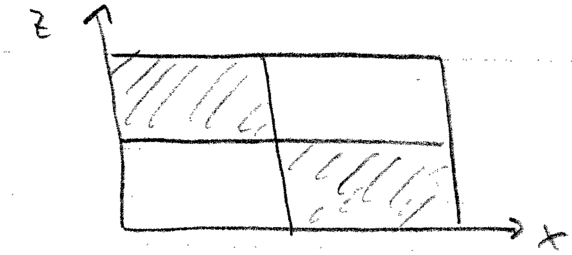
$$\bar{y}_{2,1} \sim \sin\left(2\pi \frac{x}{L_x}\right) \sin\left(\pi \frac{z}{L_z}\right)$$



(1,2)



(2,2)



modes are orthogonal

etc.

$\bar{y} = \sum_{n,m}^{\infty} \bar{y}_{n,m}$

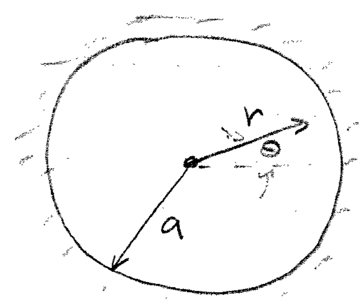
(I.C.) choose $\bar{A}_{n,m}$ to sat. initial con.

any general unforced motion can be expressed by summing over these natural modes

---

now consider general vibration of a circular membrane (e.g. a kettledrum) with fixed rim

cylindrical coord.



a fixed

a = radius

as before $$\nabla^2 y = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

but now $y = y(r,\theta,t)$

if assume $\bar{y}(r,\theta,t) = \bar{\Psi}(r,\theta)\, e^{j\omega t}$ then

$$\nabla^2 \bar{\Psi} + k^2 \bar{\Psi} = 0$$

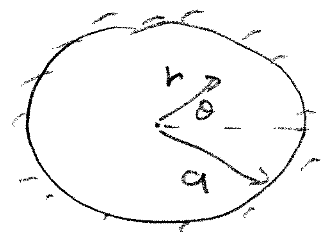
Lecture #15 Vibrating Membrane /
Review for EXAM #1

Read Secs. 5.1 → 5.5 (for Friday)
EXAM #1, Wed., Feb. 16, Chap 1, 2, 3, 4

- return H.W. #9, collect H.W. #10

- will discuss/review for exam in second half of period, first finish up Chap 4

last time ⇒ circular membrane, fixed rim



Cyl. coord

$$\bar{y}(r,\theta,t) = \bar{\Psi}(r,\theta)\, e^{j\omega t}$$

$$\nabla^2 \bar{\Psi} + k^2 \bar{\Psi} = 0$$

$$\frac{\partial^2 \bar{\Psi}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\Psi}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{\Psi}}{\partial \theta^2} + k^2 \bar{\Psi} = 0$$

in cylindrical coordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \qquad \text{so}$$

two-dim

$$\frac{\partial^2 \overline{\Psi}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{\Psi}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \overline{\Psi}}{\partial \theta^2} + k^2 \overline{\Psi} = 0$$

try again separation of variables solution where assume $\overline{\Psi}(r,\theta) = \overline{R}(r)\,\overline{\Theta}(\theta)$

$$\frac{\overline{\Theta}}{\overline{R}\,\overline{\Theta}}\frac{d^2\overline{R}}{dr^2} + \frac{\overline{\Theta}}{\overline{R}\,\overline{\Theta}}\frac{1}{r}\frac{d\overline{R}}{dr} + \frac{\overline{R}}{\overline{R}\,\overline{\Theta}}\frac{1}{r^2}\frac{d^2\overline{\Theta}}{d\theta^2} + k^2\overline{R}\,\overline{\Theta} = 0$$

function r only function θ only

$$r^2\left\{\frac{1}{\overline{R}}\frac{d^2\overline{R}}{dr^2} + \frac{1}{\overline{R}}\frac{1}{r}\frac{d\overline{R}}{dr} + k^2\right\} = -\frac{1}{\overline{\Theta}}\frac{d^2\overline{\Theta}}{d\theta^2} = \text{constant} = +m^2$$

thus
$$\frac{d^2\overline{\Theta}}{d\theta^2} + m^2\overline{\Theta} = 0$$

$$\overline{\Theta}(\theta) = \overline{C}_1 \cos(m\theta + \gamma)$$

but by symmetry $\overline{\Theta}(\theta \pm 2\pi) = \overline{\Theta}(\theta)$

thus m = integer = 0, 1, 2, ---

then
$$\frac{d^2\overline{R}}{dr^2} + \frac{1}{r}\frac{d\overline{R}}{dr} + \left(k^2 - \frac{m^2}{r^2}\right)\overline{R} = 0 \qquad \text{Eq (4.4.8)}$$

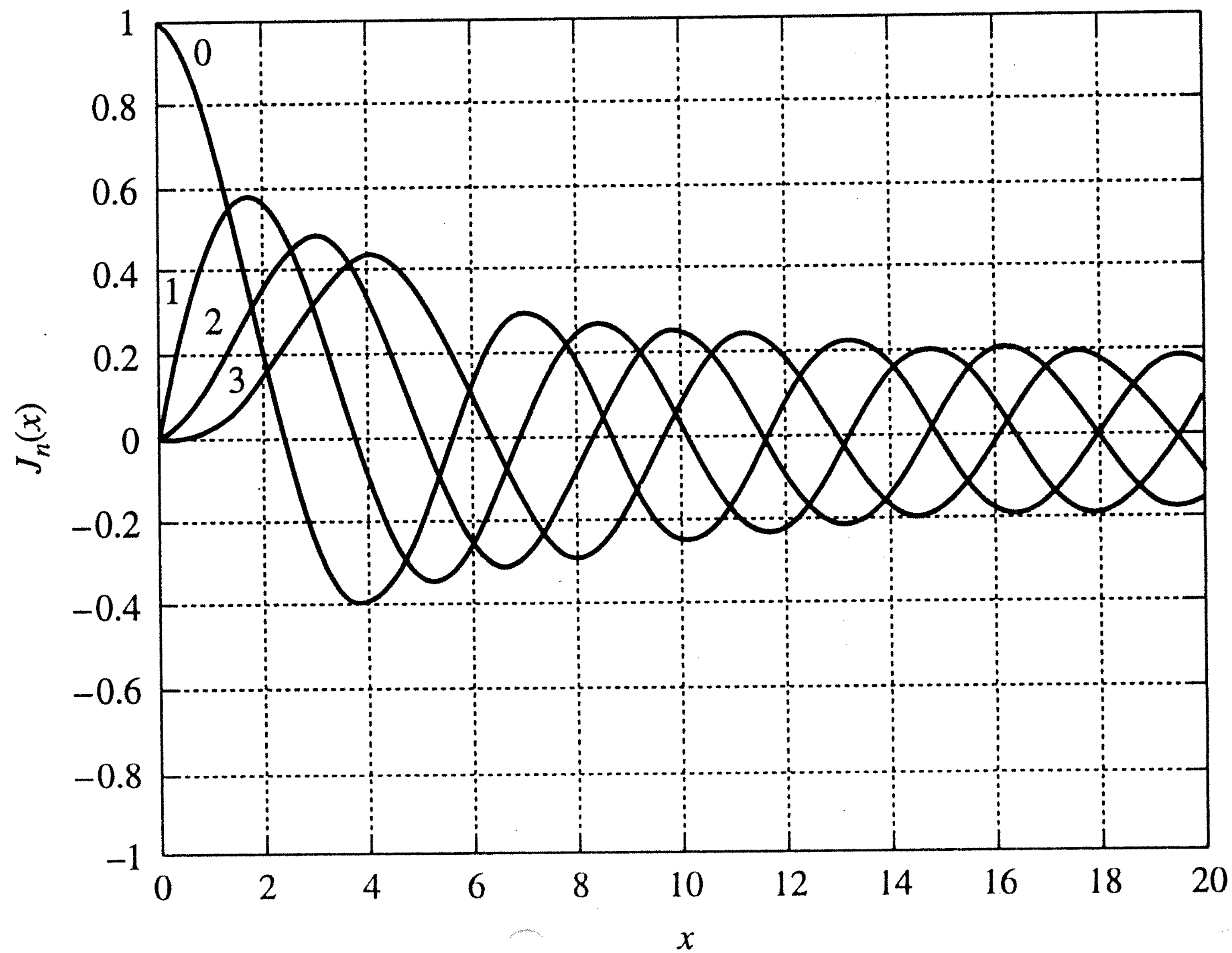
well-known ord. diff. eqn., Bessel's eqn. (m^{th}-order)

see App A4 and A5

general solution

$$\overline{R}(r) = \overline{C}_2 J_m(kr) + \overline{C}_3 Y_m(kr)$$

(b) Graphs: Bessel Functions of the First Kind of Orders 0, 1, 2, and 3



(c) *Zeros: Bessel Functions of the First Kind,* $J_m(j_{mn}) = 0$

	j_{mn}					
$m \backslash n$	0	1	2	3	4	5
0	—	2.40	5.52	8.65	11.79	14.93
1	0	3.83	7.02	10.17	13.32	16.47
2	0	5.14	8.42	11.62	14.80	17.96
3	0	6.38	9.76	13.02	16.22	19.41
4	0	7.59	11.06	14.37	17.62	20.83
5	0	8.77	12.34	15.70	18.98	22.22

$J_m(kr) \Rightarrow m^{th}$ order Bessel function of 1^{st} kind

$Y_m(kr) \Rightarrow m^{th}$ order Bessel function of 2^{nd} kind

B.C.'s

note $Y_m(kr) \Rightarrow \pm\infty$ as $kr \Rightarrow 0$

for bounded solutions, require then $\bar{C}_3 = 0$

so

$$\bar{y}(r,\theta,t) = \bar{A}\, J_m(kr)\cos(m\theta+\gamma)\, e^{j\omega t}$$

@ $r=a$ $\quad \bar{y}(a,\theta,t) = \bar{A}\, J_m(ka)\cos(m\theta+\gamma)\, e^{j\omega t} = 0$

thus $J_m(ka) = 0$

$$k_{m,n}a = j_{m,n} \Rightarrow \underline{k_{m,n}} = \frac{j_{m,n}}{a} = \frac{\omega_{m,n}}{c} = \frac{2\pi f}{c}$$

where $J_m(j_{m,n}) = 0$

$j_{m,n}$ = zeroes of J_m

such that $j_{m,1} < j_{m,2} < j_{m,3} < j_{m,4} <$ etc.

see Table A5 c.) page 516

thus $\omega_{m,n} = (c/a)\, j_{m,n}$

$$f_{m,n} = \frac{1}{2\pi}(c/a)\, j_{m,n}$$

fundamental frequency $f_{0,1} = \frac{1}{2\pi}(c/a)\, j_{0,1}$

$$= \left(\frac{1}{2\pi}\right)(c/a)(2.40)$$

$$= 0.7639\left(\frac{c}{2a}\right)$$

in general overtones $\underline{\text{not}}$ harmonic!

e.g. $\dfrac{f_{1,1}}{f_{0,1}} = \dfrac{3.83}{2.40} \neq$ integer, etc.

so natural mode

$$\bar{Y}_{mn}(r,\theta,t) = \bar{A}_{mn} J_m(k_{mn} r) \cos(m\theta + \gamma_{mn}) e^{j\omega_{mn} t}$$

m = number of radial nodal lines
(number of θ cycles)

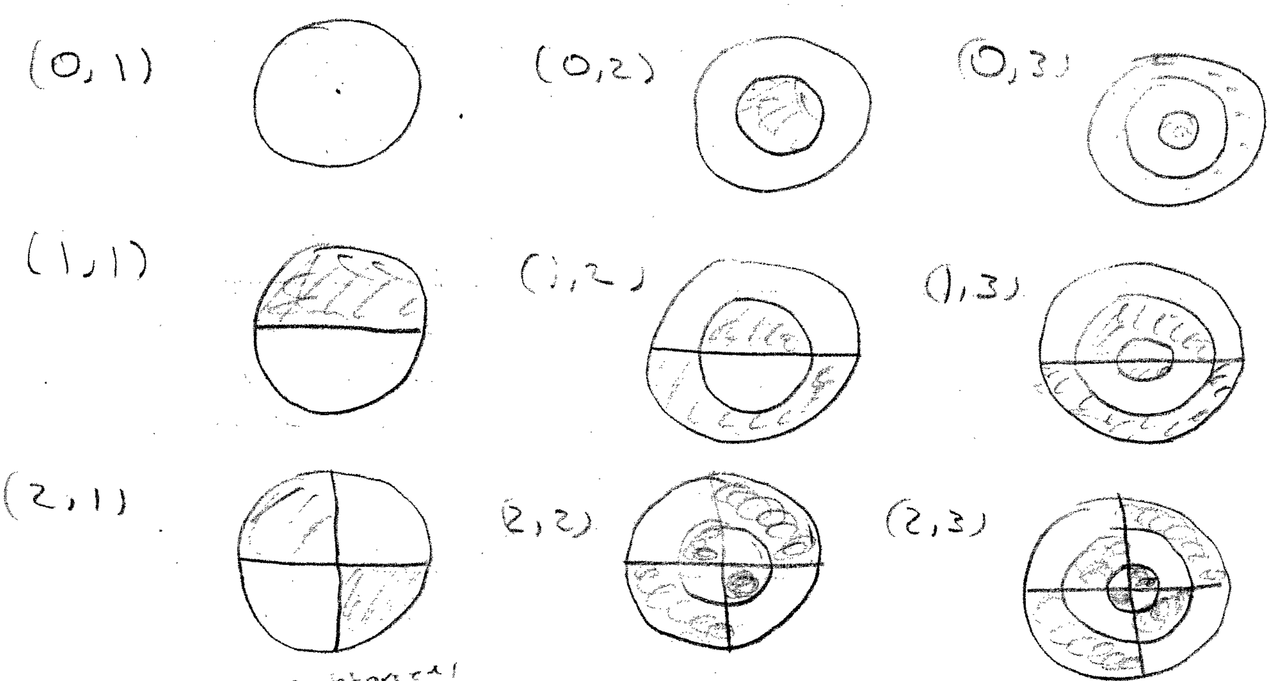
n = number of circular nodal lines
(including rim)

see Fig 4.4.1

(0,1) (0,2) (0,3)

(1,1) (1,2) (1,3)

(2,1) (2,2) (2,3)

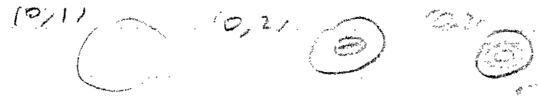


any general (unforced) motion can be written as a sum of these natural (normal) modes

(nonsymmetric modes not eff. sound gen.)

finish up by discussing purely radial modes

$m = 0$ then



$$\bar{Y}_{0,n}(r,\theta,t) = \bar{A}_{0,n} J_0(k_{0,n} r) \cos(\gamma_{0,n}) e^{j\omega_{0,n} t}$$

or $\qquad \bar{A}_n = A_n e^{j\phi_n}$

$$y_n(r,t) = A_n J_0(k_n r) \cos(\omega_n t + \phi_n)$$

$$k_n = \frac{j_{0,n}}{a} = \frac{\omega_n}{c}$$

general solution

$$y(r,t) = \sum_{n=1}^{\infty} A_n J_0(k_n r) \cos(\omega_n t + \phi_n)$$

since $J_0(0) = 1$, $A_n \Rightarrow$ amplitude of n^{th} mode at $r = 0$

average displacement amplitude for n^{th} mode

$$\langle \Psi_n \rangle_s = \frac{1}{\pi a^2} \int_0^a A_n J_0(k_n r) \underbrace{2\pi r \, dr}_{dA}$$

$\rightarrow dr\,(2\pi r)$

now $\alpha J_0(\alpha) = \dfrac{d[\alpha J_1(\alpha)]}{d\alpha}$ App A4

$$\alpha = k_n r \Rightarrow r = \frac{\alpha}{k_n} \quad dr = \frac{d\alpha}{k_n}$$

$$\langle \Psi_n \rangle_s = \frac{1}{\pi a^2} \int_0^{k_n a} A_n J_0(\alpha) \, 2\pi \frac{\alpha}{k_n} \frac{d\alpha}{k_n}$$

$$= \frac{1}{\pi a^2} 2\pi A_n \frac{1}{k_n^2} \int_0^{k_n a} \alpha J_0(\alpha) \, d\alpha$$

$$= \frac{1}{\cancel{\pi} a^2} 2\cancel{\pi} A_n \frac{1}{k_n^2} (k_n a) J_1(k_n a)$$

$$\langle \Psi_n \rangle_S = \frac{2A_n}{(k_n a)} J_1(k_n a) \qquad Eq\ (4.5.4)$$

$$= \frac{2A_n}{j_{0,n}} J_1(j_{0,n})$$

find $$\langle \Psi_1 \rangle_S = \frac{2A_1}{2.40} J_1(2.40) = \frac{2(0.5202)}{2.40} A_1$$

$$= 0.434\, A_1$$

-0.3400

$$\langle \Psi_2 \rangle_S = \frac{2A_2}{5.52} J_1(5.52)$$

$$= -0.123\, A_2$$

0.2712

$$\langle \Psi_3 \rangle_S = \frac{2A_3}{8.65} J_1(8.65)$$

$$= 0.0627\, A_3$$

etc.

higher order modes displace smaller volume of surrounding fluid, less acoustic emission

any last question about Chap 4, transverse vibrations of stretched membrane?

rest of Chap 4 ⇒ vibrations
surface waves in finite plate
etc.

Intro. to
Lecture #17 Acoustics of Fluids
Read Secs. 5.6 -> 5.7

- return homework

- have exams graded, class as a whole did o.k.
  will return and discuss near end of period

- now => Chap 5 Intro. to Acoustics of Fluids