Optimal Mixing Enhancement in 3D Pipe Flow^{*}

Andras Balogh^a, Ole Morten Aamo^b, and Miroslav Krstić^c

 $^a\mathrm{Department}$ of Mathematics, University of Texas – Pan American

Edinburg, TX 78541-2999, USA

^bDepartment of Engineering Cybernetics, Norwegian University of Science and Technology

N-7491 Trondheim, Norway

 $^c\mathrm{Department}$ of MAE, University of California at San Diego

La Jolla, CA 92093–0411, USA

Abstract

We design a Lyapunov based boundary feedback controller for achieving mixing in a 3D pipe flow governed by Navier-Stokes equations. We show that the control law maximizes a measure related to mixing (that incorporates stretching and folding of material elements), while at the same time minimizing the control effort and the sensing effort. The penalty on sensing results in a static *output*-feedback control law (rather than full-state feedback). We also derive a lower bound on the gain from the control effort to the mixing measure. Furthermore, we establish input/output-to-state-stability properties for the open-loop system. These results show a form of detectability of mixing in the interior of the pipe from the chosen outputs on the wall. The effectiveness of the optimal control in achieving mixing enhancement is demonstrated in numerical simulations.

^{*}This work was supported by the Office of Naval Research, the Air Force Office of Scientific Research, the National Science Foundation, and the Norwegian Research Council.

1 Introduction

The process of mixing is encountered frequently in engineering applications, with the mixing of air and fuel in combustion engines being a prime example [8, 3]. Approaches to analysing mixing range from experimental design and testing to modern applications of dynamical systems theory (see [24, 25] for thorough reviews). The latter was initiated by Aref [4], and followed by [5, 13, 17, 31, 28]. A framework in which to analyse mixing properties of (non-periodic) finite-time flow fields, was developed in [9, 10, 11, 12], and applied to geophysical flows in [27, 22]. Another method for identifying regions in a flow that have similar finite-time statistical properties based on ergodic theory was developed and applied in [19, 18, 20]. The relationship between the two methods mentioned, focusing on geometrical and statistical properties of particle motion, respectively, was examined in [26]. Rigorous application of control systems theory to problems in mixing appeared for the first time in [6, 7], and more recently in [23]. For a more elaborate review of these works, see [1].

In [1], we applied active feedback control in order to enhance existing instability mechanisms in a 2D model of plane channel flow. By applying boundary control intelligently in a feedback loop, mixing was considerably enhanced with relatively small control effort. Wall-normal suction and blowing was used for actuation, and the pressure difference between opposite points on the wall for sensing. The control law was decentralized and designed using Lyapunov stability analysis.

In the current work, these efforts are successfully extended to 3D pipe flow, which, in the uncontrolled case, has a parabolic steady state solution (known as Hagen-Poiseuille flow). With mixing in mind, we quantify the flow perturbations (away from the Hagen-Poiseuille flow) in terms of the L_2 -norm of their first order spatial derivatives. This norm is a volume integral over the entire flow domain. It explicitly incorporates stretching of material elements, and due to the boundedness of the domain, and the fact that the flow field satisfies the Navier-Stokes equations, folding is implicit in the measure. Since stretching and folding are key ingredients in mixing, the measure appears to be strongly related to mixing.

We design a Lyapunov based control law and show that it maximizes the measure of mixing described above, while at the same time minimizing the control effort and the sensing effort. The penalty on sensing results in a static *output*-feedback control law (rather than full-state feedback). We also derive a lower bound on the gain from the control effort to the mixing measure.

In separate results, we establish input/output-to-state-stability properties for the openloop system. These results show a form of detectability of mixing in the interior of the pipe from the chosen outputs on the wall.

The effectiveness of the optimal control in achieving mixing enhancement is demonstrated in numerical simulations of the full, nonlinear, Navier-Stokes equations for 3D pipe flow at Reynolds number 2100. To quantify mixing, massless particles are placed into the flow, simulating passive tracer dye. Visualizations compare perturbation energy, enstrophy, vorticity, and dye distribution for the uncontrolled and controlled cases.

The feedback system designed in this work stands a good chance of being realizable, due to its simplicity: sensing and actuation are restricted to the pipe wall; and the feedback law is decentralized and static. Furthermore, simulations show that the spatial changes in the control velocity are smooth and small, promising that a low number of actuators will suffice



Figure 1: Geometry of the pipe flow.

in practice.

This paper is organized as follows: in Section 2 we present the governing equations; in Section 3 we introduce our choices of sensing and actuation; in Section 4 we define two measures of the fluid flow field which are instrumental to the theoretical analysis; in Section 5 we provide an energy analysis resulting in two technical lemmas that are frequently used in the analysis; in Section 6 we present the main result on control design and optimality; in Section 7 we discuss detectability of mixing; in Section 8 numerical simulations are presented, and finally; in Section 9 we offer some concluding remarks.

2 Navier-Stokes Equations for 3D Pipe Flow

The non-dimensionalized Navier-Stokes equation is given by

$$\frac{\partial \mathbf{W}}{\partial t} + (\mathbf{W} \cdot \nabla) \mathbf{W} = -\nabla P + \frac{1}{Re} \Delta \mathbf{W}, \qquad (1)$$

$$\operatorname{div}(\mathbf{W}) = 0, \qquad (2)$$

where ∇ denotes the gradient operator, Δ denotes the Laplace operator, and div is short for divergence. $\mathbf{W}: \Omega \times \mathbb{R}_+ \to \mathbb{R}^3$ is the velocity of the fluid, $P: \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is the pressure, and Re is the Reynolds number. The Reynolds number is defined as $Re = \rho \breve{D} \breve{V} / \mu$, where ρ and μ are the density and viscosity of the fluid, respectively, and \check{D} and \check{V} are characteristic length and characteristic velocity of the problem. The pipe radius is chosen as characteristic length, and the center velocity of the steady flow (Hagen-Poiseuille flow, given by equation (3) below) is chosen as characteristic velocity. The domain, Ω , for the 3D pipe flow is most easily defined in terms of cylindrical coordinates (r, θ, z) , configured as shown in Figure 1. The domain is thus given by $\Omega = \{(r, \theta, z) \in [0, 1) \times [0, 2\pi) \times [0, L)\}$, where L is the length of the pipe. In the angular (θ) direction the boundary conditions are clearly periodic. In the streamwise (z) direction, we also use periodic boundary conditions. That is, we equate the flow quantities at $\theta = 0$ and $\theta = 2\pi$, and at z = 0 and z = L. In the radial direction (r) we impose the boundary conditions that the velocity be finite at r = 0, and at the wall (r = 1)we will eventually specify the flow velocity as a boundary control law, but for now we use no-slip. Under these boundary conditions, one may verify that the velocity field, W, and pressure \bar{P} , defined by

$$\left(\bar{\mathbf{W}},\bar{P}\right) = \left(\bar{V}_r,\bar{V}_\theta,\bar{V}_z,\bar{P}\right) = \left(0,0,1-r^2,-\frac{4}{Re}z\right),\tag{3}$$

is a steady state solution of (1)–(2), where \bar{V}_r , \bar{V}_θ , and \bar{V}_z are the velocity components in the radial (r), angular (θ), and streamwise (z) directions, respectively. Equation (1)–(2) in terms of perturbation variables, defined as

$$\mathbf{w} \triangleq \mathbf{W} - \bar{\mathbf{W}}, \text{ and } p \triangleq P - \bar{P},$$
(4)

becomes

$$\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{w} \cdot \nabla) \mathbf{w} + (\mathbf{w} \cdot \nabla) \bar{\mathbf{W}} + (\bar{\mathbf{W}} \cdot \nabla) \mathbf{w} = -\nabla p + \frac{1}{Re} \Delta \mathbf{w},$$
(5)

$$\operatorname{div}\left(\mathbf{w}\right) = 0. \tag{6}$$

In cylindrical coordinates, we will denote the perturbation variables

$$\mathbf{w} = (v_r, v_\theta, v_z) \triangleq \left(V_r, V_\theta, V_z - \bar{V}_z \right), \text{ and } p \triangleq P - \bar{P}.$$
(7)

We will frequently need volume integrals over the domain Ω , as well as area integrals over the boundary, which we denote $\partial\Omega$. In view of the periodic boundary conditions in the angular and streamwise directions, the boundary is simply the pipe wall, $\partial\Omega = \{(r, \theta, z) : r = 1\}$. The volume and area integrals are denoted in the usual manner as

$$\int_{\Omega} (\cdot) \, dV, \text{ and } \int_{\partial\Omega} (\cdot) \, dA, \tag{8}$$

respectively. The equivalent integrals in cylindrical coordinates are

$$\int_{0}^{L} \int_{0}^{2\pi} \int_{0}^{1} (\cdot) r dr d\theta dz, \text{ and } \int_{0}^{L} \int_{0}^{2\pi} (\cdot) d\theta dz.$$

$$\tag{9}$$

3 Sensing and Actuation

As mentioned in the previous section, the boundary conditions on the wall of the pipe incorporate our actuation. The fluid velocity at the wall is restricted to be normal to the wall, that is, $v_r(1, \theta, z, t) = u(\theta, z, t)$, and $v_{\theta}(1, \theta, z, t) = v_z(1, \theta, z, t) = 0$, where $u(\theta, z, t)$ is the control input. Thus, $\mathbf{w} \cdot \mathbf{n} = u$ on $\partial \Omega$, where **n** is the outward pointing unit normal vector. We also impose on the control input that it satisfies

$$u(\theta, z, t) = -u(\theta + \pi, z, t), \qquad (10)$$

which states that if suction is applied at a point (θ, z) on the pipe wall, then an equal amount of blowing is applied at the opposite point $(\theta + \pi, z)$. This is illustrated in Figure 2. It is clear that condition (10) ensures a zero net mass flux across the pipe wall, and therefore it is a natural condition to impose from a mass balance point of view. The measurement available is the pressure drop, denoted y, from any point (θ, z) on the pipe wall to the opposite point $(\theta + \pi, z)$. That is,

$$y(\theta, z, t) \triangleq p(1, \theta, z, t) - p(1, \theta + \pi, z, t).$$
(11)

Figure 2: Actuation is symmetric about the pipe centerline.

We also define the instantaneous control effort and sensing effort as

$$U(t) = \int_{\partial\Omega} u^2 dA,$$
(12)

and

$$Y(t) = \int_{\partial\Omega} y^2 dA,$$
(13)

respectively.

4 Measures of Mixing

There are two key ingredients to effective mixing. The fluid flow field must inflict extensive stretching to material elements, and the stretching should be accompanied by folding. In this work, we define two measures of the fluid flow field that are instrumental to our development below. One is the kinetic energy of the perturbation, termed *turbulent kinetic energy* in the fluid mechanics literature, defined as

$$E(\mathbf{w}) \triangleq \frac{1}{2} \int_{\Omega} |\mathbf{w}|^2 dV = \frac{1}{2} \int_{0}^{L} \int_{0}^{2\pi} \int_{0}^{1} \left(v_r^2 + v_\theta^2 + v_z^2 \right) r dr d\theta dz,$$
(14)

and the other is a measure of spatial velocity gradients, defined as

$$m(\mathbf{w}) \triangleq \int_{\Omega} |\nabla \mathbf{w}|^2 dV = \int_{\Omega} Tr \left\{ \nabla \mathbf{w}^T \nabla \mathbf{w} \right\} dV$$
$$= \int_{0}^{L} \int_{0}^{2\pi} \int_{0}^{1} \left[\left(\frac{\partial v_r}{\partial z} \right)^2 + \left(\frac{\partial v_\theta}{\partial z} \right)^2 + \left(\frac{\partial v_z}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial v_z}{\partial \theta} \right)^2 + \left(\frac{\partial v_z}{\partial z} \right)^2 + \left(\frac{\partial v_\theta}{\partial r} \right)^2 + \left(\frac{\partial v_\theta}{\partial \theta} \right)^2 + \left(\frac{\partial v_\theta}{\partial r} \right)^2 \right] r dr d\theta dz.$$
(15)

The latter measure, (15), which is related to the dissipation function via the factor 1/Re, appears to be stronger connected to mixing. While it is clear that stretching of material elements is explicit in a measure of spatial gradients of the flow field, folding is implicit in the measure due to the boundedness of the flow domain, and the fact that \mathbf{w} satisfies the Navier-Stokes equations. It is recognized, that flow fields having poor mixing properties and large $m(\mathbf{w})$ exist, but we postulate that such flow fields will not be hydrodynamically stable under the control actuation to be designed below. Thus, our objective becomes that of designing a feedback control law, in terms of suction and blowing of fluid normally to the pipe wall, that is optimal with respect to some meaningful cost functional related to $m(\mathbf{w})$.

5 Energy Analysis

Before giving the main result on controller design and optimality, we state two key lemmas that are needed frequently in what follows. The first lemma is a Lyapunov type result and it relates the time derivative of $E(\mathbf{w}(t))$ to $m(\mathbf{w}(t))$. The second lemma provides a bound on the crossterm between the perturbation and the Hagen-Poiseuille steady state flow, originating from the nonlinear convective terms in the Navier-Stokes equation. This term is called the *instantaneous production* in the fluid mechanics literature. Proofs of the lemmas are provided in Appendix A.

Lemma 1 (Balance for the turbulent kinetic energy) For wall-normal actuation, satisfying (10),

$$\dot{E}(\mathbf{w}) = -\frac{1}{2} \int_{\partial\Omega} uy dA - \frac{1}{Re} U - \frac{1}{Re} m(\mathbf{w}) - \Gamma(\mathbf{w})$$
(16)

along solutions of system (5)-(6), where

$$\Gamma\left(\mathbf{w}\right) = \int_{\Omega} \left[\left(\mathbf{w} \cdot \nabla\right) \bar{\mathbf{W}} + \left(\bar{\mathbf{W}} \cdot \nabla\right) \mathbf{w} \right] \cdot \mathbf{w} dV.$$
(17)

Lemma 2 (Bound for the instantaneous production) If $v_z(1, \theta, z, t) = 0$, then solutions of system (5)–(6) satisfy

$$\Gamma\left(\mathbf{w}\right) \le \frac{a}{2}\left(1+b\right)U + \max\left\{\frac{a}{4}\left(1+\frac{1}{b}\right), \frac{1}{4a}\right\}m\left(\mathbf{w}\right),\tag{18}$$

for arbitrary positive constants a and b.

The conditions of Lemma 1 and 2 are assumed to hold throughout the analysis that follows, that is: actuation is wall-normal and satisfies (10).

6 Optimality

The following theorem incorporates the control design and optimality result.

Theorem 1 The control

$$u = -ky, \tag{19}$$

with $k \in (0, \frac{Re}{4})$ and Re arbitrary, maximizes the cost functional

$$J(u) = \lim_{t \to \infty} \left[2\beta E(\mathbf{w}(t)) + \int_{0}^{t} h(\mathbf{w}(\tau)) d\tau \right]$$
(20)

where

$$\beta = \frac{2k}{\left(1 - \frac{4}{Re}k\right)}$$

and

$$h\left(\mathbf{w}\right) = \frac{2\beta}{Re}m\left(\mathbf{w}\right) + 2\beta\Gamma\left(\mathbf{w}\right) - \left(\frac{\beta}{2}\right)^{2}\left(1 + \frac{2\beta}{Re}\right)^{-1}Y - U.$$
(21)

Moreover, solutions of system (5)-(6) satisfy

$$h\left(\mathbf{w}\right) \le c_1 m\left(\mathbf{w}\right) - c_2 Y - \frac{1}{2}U \tag{22}$$

for arbitrary values of the control u, and with

$$c_1 = \frac{2\beta}{Re} + \max\left(\frac{1}{4}, 2\beta^2\right) > 0 \text{ and } c_2 = \left(\frac{\beta}{2}\right)^2 \left(1 + \frac{2\beta}{Re}\right)^{-1} > 0.$$
 (23)

Proof. By Lemma 1, we can write (21) as

$$h(\mathbf{w}) = -2\beta \dot{E}(\mathbf{w}) - \beta \int_{\partial\Omega} uy dA - \frac{2\beta}{Re} U - \left(\frac{\beta}{2}\right)^2 \left(1 + \frac{2\beta}{Re}\right)^{-1} Y - U.$$
(24)

Inserting (24) into (20) we get

$$J(u) = \lim_{t \to \infty} \left[2\beta E\left(\mathbf{w}\left(t\right)\right) + \int_{0}^{t} \left(\left(-2\beta \dot{E}(\mathbf{w}\left(\tau\right))\right) - \beta \int_{\partial\Omega} uy dA - \left(1 + \frac{2\beta}{Re}\right) U - \left(\frac{\beta}{2}\right)^{2} \left(1 + \frac{2\beta}{Re}\right)^{-1} Y \right) d\tau \right]$$

$$= \lim_{t \to \infty} \left[2\beta E\left(\mathbf{w}\left(t\right)\right) - 2\beta \int_{0}^{t} \dot{E}(\mathbf{w}\left(\tau\right)) d\tau - \left(1 + \frac{2\beta}{Re}\right) \int_{0}^{t} \int_{\partial\Omega} \left(u^{2} + \beta \left(1 + \frac{2\beta}{Re}\right)^{-1} uy + \left(\frac{\beta}{2}\right)^{2} \left(1 + \frac{2\beta}{Re}\right)^{-2} y^{2}\right) dAd\tau \right]$$

$$= 2\beta E\left(\mathbf{w}\left(0\right)\right) - \left(1 + \frac{2\beta}{Re}\right) \lim_{t \to \infty} \int_{0}^{t} \int_{\partial\Omega} \left(u + \frac{\beta}{2} \left(1 + \frac{2\beta}{Re}\right)^{-1} y\right)^{2} dAd\tau.$$
(25)

The maximum of (20) is achieved when the integral in (25) is zero. Thus, (19) is the optimal control. Inequality (22) is obtained by applying Lemma 2 with $a = \frac{1}{4\beta}$ and b = 1, to (21).

The objective of applying the control input (19) is to increase the value of $m(\mathbf{w})$. That this objective is targeted in the cost functional (20), is clear from inequality (22), which gives an upper bound on $h(\mathbf{w})$ in terms of $m(\mathbf{w})$. Thus, $h(\mathbf{w})$ cannot be made large without making $m(\mathbf{w})$ large, so the cost functional (20) is meaningful with respect to our objective. The cost functional also puts penalty on the output. Since the output is fed back to the control input, the output penalty works in conjunction with the input penalty to minimize control effort.

The next theorem writes the result of Theorem 1 on a form that puts emphasis on signal gains.

Theorem 2 For all Re and $t \ge 0$, solutions of system (5)–(6) satisfy

$$\max_{\substack{u \\ E(\mathbf{w}(0)) \neq 0}} \left\{ \lim_{t \to \infty} \frac{2\beta E(\mathbf{w}(t)) + \int_{0}^{t} g(\mathbf{w}(\tau)) d\tau}{2\beta E(\mathbf{w}(0)) + c_{2} \int_{0}^{t} Y(\tau) d\tau + \frac{1}{2} \int_{0}^{t} U(\tau) d\tau} \right\} = 1,$$
(26)

where

$$g(\mathbf{w}) \triangleq h(\mathbf{w}) + c_2 Y + \frac{1}{2}U,$$
(27)

and

$$g\left(\mathbf{w}\right) \le c_1 m\left(\mathbf{w}\right). \tag{28}$$

Furthermore, the maximum is achieved with the optimal control (19), for which solutions of the closed-loop system satisfy

$$2\beta E\left(\mathbf{w}\left(t\right)\right) + c_{1} \int_{0}^{t} m\left(\mathbf{w}\left(\tau\right)\right) d\tau \ge 2\beta E\left(\mathbf{w}\left(0\right)\right) + \left(\frac{3}{2} + \frac{2\beta}{Re}\right) \int_{0}^{t} U(\tau) d\tau.$$
(29)

Proof. Integration of (27) with respect to time, and adding $2\beta E(\mathbf{w}(t))$ to each side, gives

$$2\beta E\left(\mathbf{w}\left(t\right)\right) + \int_{0}^{t} g\left(\mathbf{w}\left(\tau\right)\right) d\tau = 2\beta E\left(\mathbf{w}\left(t\right)\right) + \int_{0}^{t} h\left(\mathbf{w}\left(\tau\right)\right) d\tau + c_{2} \int_{0}^{t} Y(\tau) d\tau + \frac{1}{2} \int_{0}^{t} U(\tau) d\tau.$$
(30)

The two first terms on the right hand side of (30) is J(u) (without the limit), so inserting (25) we get

$$2\beta E\left(\mathbf{w}\left(t\right)\right) + \int_{0}^{t} g\left(\mathbf{w}\left(\tau\right)\right) d\tau = 2\beta E\left(\mathbf{w}\left(0\right)\right) \\ - \left(1 + \frac{2\beta}{Re}\right) \int_{0}^{t} \int_{\partial\Omega} \left(u + \frac{\beta}{2} \left(1 + \frac{2\beta}{Re}\right)^{-1} y\right)^{2} dA d\tau \\ + c_{2} \int_{0}^{t} Y(\tau) d\tau + \frac{1}{2} \int_{0}^{t} U(\tau) d\tau.$$

$$(31)$$

Dividing both sides of (31) by

$$2\beta E(\mathbf{w}(0)) + c_2 \int_{0}^{t} Y(\tau) d\tau + \frac{1}{2} \int_{0}^{t} U(\tau) d\tau,$$

assuming $E(\mathbf{w}(0)) \neq 0$, taking the limit as $t \to \infty$, and then taking the maximum value over u, we obtain

$$\max_{\substack{u \\ E(\mathbf{w}(0)) \neq 0}} \left\{ \lim_{t \to \infty} \frac{2\beta E(\mathbf{w}(t)) + \int_{0}^{t} g(\mathbf{w}(\tau)) d\tau}{2\beta E(\mathbf{w}(0)) + c_{2} \int_{0}^{t} Y(\tau) d\tau + \frac{1}{2} \int_{0}^{t} U(\tau) d\tau} \right\} \\
= \max_{\substack{u \\ E(\mathbf{w}(0)) \neq 0}} \left\{ 1 - \lim_{t \to \infty} \frac{\left(1 + \frac{2\beta}{Re}\right) \int_{0}^{t} \int_{\partial\Omega} \left(u + \frac{\beta}{2} \left(1 + \frac{2\beta}{Re}\right)^{-1} y\right)^{2} dA d\tau}{2\beta E(\mathbf{w}(0)) + c_{2} \int_{0}^{t} Y(\tau) d\tau + \frac{1}{2} \int_{0}^{t} U(\tau) d\tau} \right\}. \quad (32)$$

Since the numerator of the last term in (32) is non-negative, and the denominator is strictly positive, the maximum on the right hand side of (32) is attained when the numerator is zero, which is for the optimal control (19). Thus, we obtain (26). Inequality (28) follows from (27) and (22). Inserting the optimal control into (31) by writing y in terms of u using (19), we obtain

$$2\beta E\left(\mathbf{w}\left(t\right)\right) + \int_{0}^{t} g\left(\mathbf{w}\left(\tau\right)\right) d\tau = 2\beta E\left(\mathbf{w}\left(0\right)\right) + c_{2}\left(\frac{2}{\beta}\right)^{2} \left(1 + \frac{2\beta}{Re}\right)^{2} \int_{0}^{t} U(\tau) d\tau + \frac{1}{2} \int_{0}^{t} U(\tau) d\tau.$$
(33)

Inserting for c_2 , as defined in (23), and using (28), we get (29).

The result (26) was inspired by the work on optimal destabilization of linear systems reported in [21]. In view of (28), by maximizing the ratio in the curly brackets of (26), we make sure that the input and output signals are small compared to the internal states. This is equivalent to obtaining a large closed-loop gain. In addition, the theorem gives a lower bound on the states in terms of the control input for system (5)–(6) in closed loop with (19). Thus, it establishes the fact that the states cannot be small without the control input being small, and the control input cannot be made large without making the states large. As we shall see in our simulation study, this will lead to good mixing with low control effort.

7 Detectability of Mixing

Achieving optimality with static output feedback of y is remarkable. In this section we explain why this special output is strongly related to mixing and allows its enhancement. The next theorem establishes an open-loop property of system (5)–(6) that is reminiscent of an integral variant of input/output-to-state-stability (IOSS) for finite dimensional nonlinear systems.

Theorem 3 If $Re \in (0,4)$, then solutions of system (5)–(6) satisfy

$$c_3 \int_0^t m\left(\mathbf{w}\left(\tau\right)\right) d\tau \le 2\beta E\left(\mathbf{w}\left(0\right)\right) + \beta^2 \left(1 + \frac{2\beta}{Re}\right)^{-1} \int_0^t Y(\tau) d\tau + c_4 \int_0^t U(\tau) d\tau, \qquad (34)$$

for all $t \geq 0$ and for arbitrary values of the control u, with

$$c_3 = \frac{\beta}{4} \left(\frac{4 - Re}{Re} \right) > 0 \text{ and } c_4 = 1 + \beta \left(\frac{4 + Re}{4 - Re} \right) > 0.$$

Proof. From (20), (21) and (25), we get for all $t \ge 0$:

$$2\beta E\left(\mathbf{w}\left(t\right)\right) + \int_{0}^{t} \frac{2\beta}{Re} m\left(\mathbf{w}\left(\tau\right)\right) d\tau \leq 2\beta E\left(\mathbf{w}\left(0\right)\right) - 2\beta \int_{0}^{t} \Gamma(\mathbf{w}(\tau)) d\tau + \left(\frac{\beta}{2}\right)^{2} \left(1 + \frac{2\beta}{Re}\right)^{-1} \int_{0}^{t} Y(\tau) d\tau + \int_{0}^{t} U(\tau) d\tau.$$
(35)

Using Lemma 2 , we obtain

$$\begin{split} &2\beta E(\mathbf{w}(t)) + \int_{0}^{t} \frac{2\beta}{Re} m(\mathbf{w}(\tau)) d\tau \\ &\leq 2\beta E(\mathbf{w}(0)) + 2\beta \int_{0}^{t} \left(\frac{a}{2}(1+b)U(\tau) + \max\left\{\frac{a}{4}\left(1+\frac{1}{b}\right), \frac{1}{4a}\right\} m(\mathbf{w}(\tau))\right) d\tau \\ &+ \left(\frac{\beta}{2}\right)^{2} \left(1 + \frac{2\beta}{Re}\right)^{-1} \int_{0}^{t} Y(\tau) d\tau + \int_{0}^{t} U(\tau) d\tau, \end{split}$$

so it follows that

$$2\beta E(\mathbf{w}(t)) + \int_{0}^{t} \frac{2\beta}{Re} m(\mathbf{w}(\tau)) d\tau$$

$$\leq 2\beta E(\mathbf{w}(0)) + \int_{0}^{t} \frac{\beta}{2} \max\left(a\left(1+\frac{1}{b}\right), \frac{1}{a}\right) m(\mathbf{w}(\tau)) d\tau$$

$$+ \left(\frac{\beta}{2}\right)^{2} \left(1+\frac{2\beta}{Re}\right)^{-1} \int_{0}^{t} Y(\tau) d\tau + (1+\beta a(1+b)) \int_{0}^{t} U(\tau) d\tau$$

Rearranging the terms, we obtain

$$2\beta E\left(\mathbf{w}\left(t\right)\right) + \left(\frac{2\beta}{Re} - \frac{\beta}{2}\max\left(a\left(1+\frac{1}{b}\right), \frac{1}{a}\right)\right) \int_{0}^{t} m\left(\mathbf{w}\left(\tau\right)\right) d\tau$$
$$\leq 2\beta E\left(\mathbf{w}\left(0\right)\right) + \left(\frac{\beta}{2}\right)^{2} \left(1+\frac{2\beta}{Re}\right)^{-1} \int_{0}^{t} Y(\tau) d\tau + \left(1+\beta a\left(1+b\right)\right) \int_{0}^{t} U(\tau) d\tau,$$

which is (34) for a = 1 and $b = 2\left(\frac{Re}{4-Re}\right)$.

The significance of inequality (34) is that it provides a notion of detectability of internal states from the output y. In particular, if $m(\mathbf{w})$ is large, y must be large as well, or if y is small, so is $m(\mathbf{w})$. This is reminiscent of an integral variant of the IOSS property for finite-dimensional nonlinear systems, as presented in [15] (and motivated by earlier results in [29, 30]). In the case of (34) we have an integral-to-integral property (iiIOSS) with $m(\mathbf{w})$ as a measure of the states, so the "energy" of the states is bounded above by the "energy" of the input and output signals. With $E(\mathbf{w})$ as a measure of the states, we can also find a uniform upper bound (as opposed to an "energy" upper bound) in terms of the input and output signals. That is, system (5)–(6) has the IOSS property, as stated formally in the next theorem.

Theorem 4 For $Re \in (0, 4)$, solutions of system (5)–(6) satisfy

$$E(\mathbf{w}(t)) \le E(\mathbf{w}(0)) e^{-c_5 t} + \frac{1}{4c_5} \sup_{\tau \in [0,t]} Y(\tau) + \frac{c_6}{c_5} \sup_{\tau \in [0,t]} U(\tau)$$
(36)

for all $t \ge 0$ and for arbitrary values of the control u, with

$$c_5 = 2\max\left(\frac{4}{3Re} - 1, \frac{4 - Re}{4 + Re}\right) > 0 \text{ and } c_6 = \max\left(\frac{1}{4}, \frac{1}{4} + \frac{5Re - 4}{Re(4 - Re)}\right) > 0.$$

Proof. From (62), (63) and a similar derivation for v_{θ} , we have

$$2E\left(\mathbf{w}\right) \leq \frac{1}{2}\left(1+b\right)U + \frac{1}{4}\int_{0}^{L}\int_{0}^{2\pi}\int_{0}^{1}\left(\left(1+\frac{1}{b}\right)\left(\frac{\partial v_{r}}{\partial r}\right)^{2} + \left(\frac{\partial v_{\theta}}{\partial r}\right)^{2} + \left(\frac{\partial v_{z}}{\partial r}\right)^{2}\right)rdrd\theta dz, \quad (37)$$

and therefore

$$2E(\mathbf{w}) \le \frac{1}{2}(1+b)U + \frac{1}{4}\left(1+\frac{1}{b}\right)m(\mathbf{w}).$$
(38)

From Lemma 1 and (38) we get

$$\dot{E}(\mathbf{w}) \le -2\left(\frac{4}{Re}\left(\frac{b}{1+b}\right) - 1\right)E(\mathbf{w}) - \frac{1}{2}\int_{\partial\Omega} uydA + \frac{2b-1}{Re}U,$$

so that

$$\dot{E}(\mathbf{w}) \le -2\left(\frac{4}{Re}\left(\frac{b}{1+b}\right) - 1\right)E(\mathbf{w}) + \frac{1}{4}\int_{\partial\Omega}\left(u^2 + y^2\right)dA + \frac{2b-1}{Re}U.$$

Setting

$$b = \max\left(\frac{1}{2}, \frac{2Re}{4 - Re}\right)$$

we obtain

$$\dot{E}(\mathbf{w}) \le -c_5 E(\mathbf{w}) + \frac{1}{4}Y + c_6 U \tag{39}$$

with

$$c_5 = 2 \max\left(\frac{4}{3Re} - 1, \frac{4 - Re}{4 + Re}\right)$$
, and $c_6 = \max\left(\frac{1}{4}, \frac{1}{4} + \frac{5Re - 4}{Re(4 - Re)}\right)$

Inequality (36) now follows from the comparison principle [16, Lemma C.5] (and the triangle inequality applied to the two last terms in (39)).

In Theorem 4, the notation $\sup_{[0,t]}$ denotes the essential supremum taken over the finite time interval [0,t]. The detectability properties stated in Theorems 3 and 4 indicate that our choice of sensing, y, is appropriate.

8 Numerical Simulations

8.1 Computational scheme

The simulations are performed using a flow solver that is based on a second-order staggered grid discretization, second-order time advancement, and a Poisson equation for pressure, based on a scheme designed by Akselvoll and P. Moin [2]. The length of the cylinder is $L = 3\pi$ and the radius is R = 1. The grid is structured, single-block with cylindrical coordinates. It is uniform and periodic in z and θ with Fourier-modes 64 and 128 respectively, and linearly spaced with ratio 8 : 1 in the radial direction in order to achieve high resolution at the wall. The adaptive time step was in the range of 0.01-0.08 with constant CFL number 0.5 and constant 1 volume flux per unit span. Numerical results corresponding to three different Reynolds numbers are presented here. The lowest Reynolds number is Re = 10 which is a stable, and perhaps nonphysical flow, but it is a good test case for the effectiveness of the control design. The second Reynolds number we used was Re = 2100 which is slightly higher than the limiting number Re = 2000 for self-sustained turbulence. Finally for the case of Re = 5000 we compare natural and controlled turbulence.

We started both the controlled and the uncontrolled case from a statistically steady state flow field with control gain k = 0.1 in the controlled case. The initial flow field was obtained from a random perturbation of the parabolic profile over a large time interval using the uncontrolled case. All the measured quantities were scaled to unit surface or unit volume, whichever was appropriate.

8.2 Measuring mixing

We start our comparison with the perturbation energy and the enstrophy. While the perturbation energy is part of the cost functional (20) and it is also one of the simplest quantities to measure in our numerical simulation, enstrophy provides us with a measurement that is more closely related to mixing. Figure 3 shows that in the Re = 2100 case our control results in an approximately 60% increase in the perturbation energy and 100% almost instantaneous increase in the enstrophy. For the Re = 5000 case these numbers are smaller: approximately 13% and 16% respectively. One sees the largest control effect in the Re = 10 case: "infinite % increase in the perturbation energy and 140% increase in the enstrophy. Intuitively these differences are easy to understand. In a laminar flow (Re = 10) there is no mixing, hence this case can be dramatically improved with control. The Re = 2100 case corresponds to

self-sustained turbulence already, but it is not far from the laminar (Re = 2000) case. In this border case it is expected that our control can easily tip the scale toward increased turbulence and mixing. In the fully turbulent (Re = 5000) case we can't expect dramatic changes, but the increase in the measured quantities would be considered significant in applications.

The instantaneous streamwise vorticity along a cross section of the pipe (Figure 4) also shows some promise for increased mixing with higher values of vorticity and more complex vortex structures in the controlled case than in the uncontrolled case. Vorticity is increased not only near the wall but everywhere in the pipe.

The method we use to quantify and visualize mixing is the tracking of dye in the flow. We consider the problem of mixing of a single fluid (or similar fluids) governed by the stretching and folding of material elements. We introduce passive tracer dye along the center of the pipe represented by a set of 100 particles (Figure 5). We trace the position of these particles using a particle–line method [14, 32]. The distance between neighboring particles is kept under 0.1 by introducing new particles to halve the distance if necessary to obtain a connected dye surface at all time. As shown in Figure 6, the number of particles, that is, the length of the dye, increases in the controlled case at a much higher rate than in the uncontrolled case. Adding particles is not feasible computationally for an extended period of time. We stopped adding particles when their number reached two million (t = 4 in the controlled case and t = 8 in the uncontrolled case), but we continued tracing them. Figure 7 shows the distribution of particles inside the pipe. In the controlled case we obtain more uniform particle distribution even for smaller time.

We compare the nine different parts of functional $m(\mathbf{w})$ for different Reynolds numbers in Figures 8, 9 and 10. In these figures we omit the integral signs as well as the square signs in order to simplify notations. In all the cases the terms can be divided into two groups: one dominant group and another group with terms more than a magnitude smaller.

We start the comparison with the large Reynolds number cases Re = 2100 and Re = 5000(see Figures 8 and 9), because they have similar features. In both of these cases there are two parts that are significantly larger than the rest: the parts containing $\frac{\partial v_{\theta}}{\partial r}$ and $\frac{\partial v_z}{\partial r}$. The next large term corresponds to the expression $\frac{1}{r}\frac{\partial v_z}{\partial \theta}$ and it is 4 to 6 times smaller than the two largest terms. The rest of the terms are about 30 to 200 times smaller than the two largest terms.

For Reynolds number Re = 10 some of the terms have large values and their behavior is chaotic for the initial transient time $t \in [0, 6]$. We only make comparison for larger time in Figure 10. The dominant terms are the ones containing $\frac{\partial v_r}{\partial r}$, $\frac{v_{\theta}}{r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta}$, $\frac{v_r}{r} + \frac{1}{r} \frac{\partial v_{\theta}}{\partial \theta}$ and $\frac{\partial v_z}{\partial r}$. Only the last one of these terms, the terms containing $\frac{\partial v_z}{\partial r}$ belongs to the dominant group for larger Reynolds numbers. The term containing $\frac{1}{r} \frac{\partial v_z}{\partial \theta}$ is one magnitude smaller than these four dominant terms. The rest of the terms are all four magnitude smaller and are omitted from the figure.

8.3 Measure of control effort

In the previous subsection our numerical simulations show significant mixing results with our feedback boundary control law. This control law is optimal with respect to the cost functional (20). In this subsection we measure how big the control effort is relative to natural quantities in the pipe flow.

Figure 11 compares the natural pressure power $(\int_0^{2\pi} \int_0^1 pv_z r \, dr d\theta)$ that propels the flow through the channel to the power of the control actuation $(\int_0^{2\pi} \int_0^L pu \, dz d\theta)$. The actuation power is four magnitude smaller than the natural pressure power for all three Reynolds numbers. The negative sign of the actuation power shows that the actuation acts against high pressure on the wall by blowing inward the pipe.

We compare the pipe flux $(\int_0^{2\pi} \int_0^1 v_z \, dr d\theta)$ and the actuation flux $(\int_0^{2\pi} \int_0^L |u| \, dr d\theta)$ with the help of Figure 12. As we stated earlier, the pipe flux is set to constant one per unit span. The actuation flux is one quarter of the pipe flux for Re = 10 and about one eighth of the pipe flux for Re = 2100 and Re = 5000. The maximum velocity at the wall is 10^{-3} for Re = 10 and 5×10^{-4} for Re = 2100 and for Re = 5000.

8.4 Actuator distribution and bandwidth for Re = 2100

Figure 13 shows the instantaneous pressure field in a cross section of the pipe along with the boundary velocity that is magnified 500 times for visualization. The control "blows in" when wall pressure is high and "sucks out" when wall pressure is low. Spatial changes in the control velocity are smooth and small, promising that low number of actuators will suffice in practice. In order to investigate the density and bandwidth of sensors and actuators needed we calculate the power spectral densities of the control. The spectral plots alongside with the original signals are shown in Figure 14. Figures 14(a,b) show that only about 10-15 actuators/sensors are needed along the pipe length. Similarly, in the angular direction (see Figures 14(c,d)) we need at most 15-20 actuators/sensors. That results in approximately 200 micro-actuators/sensors for the whole pipe surface. The time-frequency analysis (Figures 14(e) and (f)) shows a bandwidth required for sensing/actuation of only 1.5Hz.

9 Concluding Remarks

We have shown that mixing in 3D pipe flow is considerably enhanced by applying small amounts of blowing and suction across the pipe wall. With the L_2 -norm of first order spatial derivatives of the flow perturbations as a measure of mixing, we have designed a Lyapunov based control law that maximizes this measure, while at the same time minimizing the control effort and the sensing effort. The penalty on sensing resulted in a static *output*-feedback control law (rather than full-state feedback). A lower bound on the gain from the control effort to the mixing measure was also derived. For the open-loop system, input/output-tostability properties were established, which show a form of detectability of mixing in the interior of the pipe from the chosen outputs on the wall.



Figure 3: Perturbation Energy and Enstrophy.



Figure 4: Streamwise vorticity at Re = 2100.



Figure 5: Initial particle distribution.



Figure 6: Length of dye as a function of time.





Controlled, t = 19



Controlled, t = 38



Figure 7: Particle distribution, Re = 2100.



Figure 8: Parts of functional $m(\mathbf{w})$ for Re = 2100.



Figure 9: Parts of functional $m(\mathbf{w})$ for Re = 5000.



Figure 10: Parts of functional $m(\mathbf{w})$ for Re = 10.

The effectiveness of the optimal control in achieving mixing enhancement was demonstrated in numerical simulations of the 3D pipe flow at Reynolds numbers 10, 2100 and 5000. Massless particles placed into the flow, simulating passive tracer dye, indicated considerable mixing enhancement as a result of the control. Simulation results also showed that the spatial changes in the control velocity were smooth and small, promising that a low number of actuators will suffice in practice.

Acknowledgment

We thank Charles Pierce for providing the code for the uncontrolled case.

References

- O.M. Aamo, M. Krstić and T.R. Bewley. "Control of mixing by boundary feedback in 2D channel flow," *Automatica*, vol. 39, no. 9, 2003.
- [2] K. Akselvoll and P. Moin, "An efficient method for temporal integration of the Navier–Stokes equations in confined axisymmetric geometry," *Journal of Computational Physics*, Vol. 125, pp. 454–463, 1996.
- [3] A.M. Annaswamy and A.F. Ghoniem. "Active control in combustion systems," *IEEE Control Systems*, vol. 15, no. 6, pp. 49–63, 1995.

Figure 11: Pressure power and actuation power.

Figure 12: Actuation flux.

Figure 13: Instantaneous pressure field with controlled velocity (magnified) in a cross section of the pipe, Re = 2100.

- [4] H. Aref. "Stirring by chaotic advection," Journal of Fluid Mechanics, 143, pp. 1–21, 1984.
- [5] W.-L. Chien, H. Rising and J.M. Ottino. "Laminar mixing and chaotic mixing in several cavity flows," *Journal of Fluid Mechanics*, vol. 170, pp. 355–377, 1986.
- [6] D. D'Alessandro, M. Dahleh and I. Mezić. "Control of fluid mixing using entropy methods," *Proceedings of the American Control Conference*, Philadelphia, Pennsylvania, 1998.
- [7] D. D'Alessandro, M. Dahleh and I. Mezić. "Control of mixing in fluid flow: a maximum entropy approach," *IEEE Transactions on Automatic Control*, vol. 44, no. 10, pp.1852– 1863, 1999.
- [8] A.F. Ghoniem and K.K. Ng. "Numerical study of the dynamics of a forced shear layer," *Phys. Fluids*, vol. 30, no. 3, pp. 706–721, 1987.
- [9] G. Haller and A.C. Poje. "Finite time transport in aperiodic flows," *Physica D*, 119, pp.352–380, 1998.
- [10] G. Haller. "Finding finite-time invariant manifolds in two-dimensional velocity fields," *Chaos*, vol.10, no. 1, 2000.
- [11] G. Haller and G. Yuan. "Lagrangian coherent structures and mixing in two-dimensional turbulence," *Physica D*, 147, pp. 352–370, 2000.

Figure 14: Spectral analysis of the control, Re = 2100.

- [12] G. Haller. "Distinguished material surfaces and coherent structures in three-dimensional fluid flows," *Physica D*, 149, pp. 248–277, 2001.
- [13] D.V. Khakhar, H. Rising and J.M. Ottino. "Analysis of chaotic mixing in two model systems," *Journal of Fluid Mechanics*, vol. 172, pp. 419–451, 1986.
- [14] T.S. Krasnopolskaya, V.V. Meleshko, G.W.M. Peters and H.E.H. Meier, "Mixing in Stokes flow in an annular wedge cavity," *Eur. J. Mech. B/Fluids*, Vol. 18, pp. 793–822, 1999.
- [15] M. Krichman, E.D. Sontag and Y. Wang. "Input-output-to-state stability," SIAM Journal on Control and Optimization, vol. 39, no. 6, pp. 1874–1928, 2001.
- [16] M. Krstić, I. Kanellakopoulos and P. Kokotović. Nonlinear and Adaptive Control Design, John Wiley & Sons, Inc., 1995.
- [17] C.W. Leong and J.M. Ottino. "Experiments on mixing due to chaotic advection in a cavity," *Journal of Fluid Mechanics*, vol. 209, pp. 463–499, 1989.
- [18] N. Malhotra, I. Mezić and S. Wiggins. "Patchiness: a new diagnostic for Lagrangian trajectory analysis in time-dependent fluid flows," *International Journal of Bifurcation* and Chaos, vol. 8, no. 6, pp. 1053–1093, 1998.
- [19] I. Mezić. On geometrical and statistical properties of dynamical systems: theory and applications, Ph.D. thesis, California Institute of Technology, 1994.
- [20] I. Mezić and S. Wiggins. "A method for visualization of invariant sets of dynamical systems based on the ergodic partition," *Chaos*, vol. 9, no. 1, pp.213–218, 1999.
- [21] I. Mezić. "Nonlinear dynamics and ergodic theory methods in control of fluid flows: theory and applications," *Proceedings of the 2001 AFOSR Workshop on Dynamics and Control*, 2001.
- [22] P.D. Miller, C.K.R.T. Jones, A.M. Rogerson and L.J. Pratt. "Quantifying transport in numerically generated velocity fields," *Physica D*, 110, pp. 105–122, 1997.
- [23] B.R. Noack, I. Mezić and A. Banaszuk. "Controlling vortex motion and chaotic advection," *Proceedings of the 39th IEEE Conference on Decision and Control*, Sydney, Australia, December 11–15, 2000.
- [24] J.M. Ottino. The Kinematics of Mixing: Stretching, Chaos, and Transport, Cambridge University Press, 1989.
- [25] J.M. Ottino. "Mixing, chaotic advection, and turbulence," Annu. Rev. Fluid Mech., vol. 22, pp. 207–253, 1990.
- [26] A.C. Poje, G. Haller and I. Mezić. "The geometry and statistics of mixing in aperiodic flows," *Physics of Fluids*, vol. 11, no. 10, 1999.

- [27] A.C. Poje, G. Haller. "Geometry of cross-stream mixing in a double-gyre ocean model," *Journal of Physical Oceanography*, vol. 29, pp. 1469–1665, 1999.
- [28] V. Rom-Kedar, A. Leonard and S. Wiggins. "An analytical study of transport, mixing and chaos in an unsteady vortical flow," *Journal of Fluid Mechanics*, vol. 214, pp. 347– 394, 1990.
- [29] E.D. Sontag and Y. Wang. "Output-to-state stability and detectability of nonlinear systems," Systems & Control Letters, 29, 279–290, 1997.
- [30] E.D. Sontag. "Comments on integral variants of ISS," Systems & Control Letters, vol. 34, pp. 93–100, 1998.
- [31] P.D. Swanson and J.M. Ottino. "A comparative computational and experimental study of chaotic mixing of viscous fluids," *Journal of Fluid Mechanics*, vol. 213, pp. 227–249, 1990.
- [32] A.A. Ten, Y.Y. Podladchikov, D.A. Yuan, T.B. Larsen and A.V. Malevsky, "Comparison of mixing properties in convection with the particle–line method," *Geophys. Res. Lett.*, 25, pp. 3205–3208, 1998.

A Proofs of Technical Lemmas

A.1 Proof of Lemma 1

The time derivative of $E(\mathbf{w})$ along trajectories of (5)–(6) is

$$\dot{E}(\mathbf{w}) = \int_{\Omega} \frac{\partial \mathbf{w}}{\partial t} \cdot \mathbf{w} dV, \tag{40}$$

so inserting (5) we have

$$\dot{E}(\mathbf{w}) = -\int_{\Omega} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{w} dV - \int_{\Omega} \left[(\mathbf{w} \cdot \nabla) \, \bar{\mathbf{W}} + (\bar{\mathbf{W}} \cdot \nabla) \, \mathbf{w} \right] \cdot \mathbf{w} dV - \int_{\Omega} \nabla p \cdot \mathbf{w} dV + \frac{1}{Re} \int_{\Omega} \Delta \mathbf{w} \cdot \mathbf{w} dV.$$
(41)

We will now integrate term by term using the divergence theorem of Gauss, and the following three formulas from vector differential calculus (can be found in any book on calculus)¹

$$\operatorname{div}(f\mathbf{w}) = \mathbf{w} \cdot \nabla f, \tag{42}$$

$$\Delta(fg) = g\Delta f + 2\nabla f \cdot \nabla g + f\Delta g, \qquad (43)$$

$$\Delta f = \operatorname{div}(\nabla f). \tag{44}$$

 1 In (42), we have used (6).

The convective term

$$-\int_{\Omega} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{w} dV = -\frac{1}{2} \int_{\Omega} \nabla(\mathbf{w} \cdot \mathbf{w}) \mathbf{w} dV = -\frac{1}{2} \int_{\Omega} \operatorname{div}((\mathbf{w} \cdot \mathbf{w}) \mathbf{w}) dV, \qquad (45)$$

so, by the divergence theorem of Gauss, we get

$$-\int_{\Omega} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{w} dV = -\frac{1}{2} \int_{\partial \Omega} [(\mathbf{w} \cdot \mathbf{w}) \mathbf{w}] \cdot \mathbf{n} dA.$$
(46)

 $\mathbf{w} \cdot \mathbf{n} = u$ on $\partial \Omega$, so the integrand in the last integral of (46) is simply u^3 . Since u has the form (10), we have for odd n

$$\int_{0}^{2\pi} u^{n} d\theta = \int_{-\pi}^{0} u^{n} d\theta + \int_{0}^{\pi} u^{n} d\theta = -\int_{-\pi}^{0} u^{n} \left(-\theta, z, t\right) d\theta + \int_{0}^{\pi} u^{n} \left(\theta, z, t\right) d\theta$$

and by a change of variables in the first integral $(\theta^* = -\theta)$, we get

$$\int_{0}^{2\pi} u^{n} d\theta = -\int_{0}^{\pi} u^{n} \left(\theta^{*}, z, t\right) d\theta^{*} + \int_{0}^{\pi} u^{n} \left(\theta, z, t\right) d\theta = 0.$$

So we obtain

$$-\int_{\Omega} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{w} dV = 0.$$
(47)

The crossterm

$$-\int_{\Omega} \left[\left(\mathbf{w} \cdot \nabla \right) \bar{\mathbf{W}} + \left(\bar{\mathbf{W}} \cdot \nabla \right) \mathbf{w} \right] \cdot \mathbf{w} dV = -\Gamma(\mathbf{w}), \tag{48}$$

by definition of $\Gamma(\mathbf{w})$.

The pressure term

$$-\int_{\Omega} \nabla p \cdot \mathbf{w} dV = -\int_{\Omega} \operatorname{div}(p\mathbf{w}) dV, \tag{49}$$

so by the divergence theorem of Gauss, and the fact that $\mathbf{w} \cdot \mathbf{n} = u$ on $\partial \Omega$,

$$-\int_{\Omega} \nabla p \cdot \mathbf{w} dV = -\int_{\partial \Omega} p u dA.$$
⁽⁵⁰⁾

From (10) and (11) we have

$$\int_{0}^{2\pi} [up]_{r=1} d\theta = \int_{0}^{\pi} [up]_{r=1} d\theta + \int_{\pi}^{2\pi} [up]_{r=1} d\theta$$

$$= \int_{0}^{\pi} [up]_{r=1} d\theta - \int_{0}^{\pi} u(\theta, z, t) p(1, \theta + \pi, z, t) d\theta$$

$$= \int_{0}^{\pi} u(\theta, z, t) y d\theta = \frac{1}{2} \int_{0}^{2\pi} u(\theta, z, t) y d\theta,$$
(51)

so we obtain

$$-\int_{\Omega} \nabla p \cdot \mathbf{w} dV = -\frac{1}{2} \int_{\partial \Omega} uy dA.$$
(52)

The dissipation term

$$\frac{1}{Re} \int_{\Omega} \Delta \mathbf{w} \cdot \mathbf{w} dV = \frac{1}{Re} \int_{\Omega} (\Delta w_i) w_i dV = \frac{1}{Re} \int_{\Omega} \left[\Delta(w_i w_i) - 2\nabla w_i \cdot \nabla w_i - w_i (\Delta w_i) \right] dV, \quad (53)$$

where we have adopted the Einstein summation notation. From (53) we get

$$\frac{1}{Re} \int_{\Omega} \Delta \mathbf{w} \cdot \mathbf{w} dV = \frac{1}{2Re} \int_{\Omega} \Delta(w_i w_i) dV - \frac{1}{Re} \int_{\Omega} |\nabla \mathbf{w}|^2 dV.$$
(54)

The integrand of the first integral on the right hand side of (54) can be written as $\operatorname{div}(\nabla(|\mathbf{w}|^2))$, so by the divergence theorem of Gauss, we get

$$\int_{\Omega} \Delta(w_i w_i) dV = \int_{\partial \Omega} (\nabla |\mathbf{w}|^2) \cdot \mathbf{n} dA.$$
(55)

Since the control is wall-normal, the integrand on the right hand side of (55) reduces to

$$2v_r \frac{\partial v_r}{\partial r}.$$
(56)

We now note from incompressibility (6), which in cylindrical coordinates reads

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0,$$
(57)

that

$$\frac{\partial v_r}{\partial r}(1,\theta,z,t) = -v_r(1,\theta,z,t).$$
(58)

Thus, by inserting (58) into (56), (56) into (55), and (55) into (54), we obtain

$$\frac{1}{Re} \int_{\Omega} \Delta \mathbf{w} \cdot \mathbf{w} dV = -\frac{1}{Re} \int_{\Omega} |\nabla \mathbf{w}|^2 \, dV - \frac{1}{Re} \int_{\partial \Omega} u^2 dA.$$
(59)

Substituting the terms (47), (48), (52), and (59) into (41) yields (16).

A.2 Proof of Lemma 2

Using cylindrical coordinates, we have that

$$\Gamma(\mathbf{w}) = \int_{0}^{L} \int_{0}^{2\pi} \int_{\varepsilon}^{1} v_z v_r \frac{\partial \bar{V}_z}{\partial r} r dr d\theta dz.$$
(60)

Since $\sup_{\Omega} \left| \frac{\partial \bar{V}_z}{\partial r} \right| = 2$, we have

$$\int_{0}^{L} \int_{0}^{2\pi} \int_{\varepsilon}^{1} v_{z} v_{r} \frac{\partial \bar{V}_{z}}{\partial r} r dr d\theta dz \leq 2 \int_{0}^{L} \int_{0}^{2\pi} \int_{\varepsilon}^{1} |v_{z}| |v_{r}| r dr$$

$$\leq a \int_{0}^{L} \int_{0}^{2\pi} \int_{\varepsilon}^{1} v_{r}^{2} r dr d\theta dz + \frac{1}{a} \int_{0}^{L} \int_{0}^{2\pi} \int_{\varepsilon}^{1} v_{z}^{2} r dr d\theta dz. \quad (61)$$

We write

$$v_r(r, \theta, z, t) = u - \int_r^1 \frac{\partial v_r}{\partial r} dr$$

so that

$$v_r^2(r,\theta,z,t) = \left(u - \int_r^1 \frac{\partial v_r}{\partial r} dr\right)^2 \le (1+b)u^2 + \left(1 + \frac{1}{b}\right) \left(\int_r^1 \frac{\partial v_r}{\partial r} dr\right)^2.$$

By the Schwartz inequality,

$$\left(\int_{r}^{1} \frac{1}{\sqrt{r}} \sqrt{r} \frac{\partial v_r}{\partial r} dr\right)^2 \leq -\ln r \int_{r}^{1} r \left(\frac{\partial v_r}{\partial r}\right)^2 dr,$$

so we have that

$$rv_r^2\left(r,\theta,z,t\right) \le \left(1+b\right)ru^2 - \left(1+\frac{1}{b}\right)r\ln r\int\limits_{\varepsilon}^1 r\left(\frac{\partial v_r}{\partial r}\right)^2 dr,$$

where we have set $r = \varepsilon$ in the lower integral limit. We now get

$$\int_{0}^{L} \int_{0}^{2\pi} \int_{\varepsilon}^{1} v_{r}^{2} r dr d\theta dz \leq \int_{0}^{L} \int_{0}^{2\pi} \int_{\varepsilon}^{1} \left((1+b) r u^{2} - \left(1+\frac{1}{b}\right) r \ln r \int_{\varepsilon}^{1} r \left(\frac{\partial v_{r}}{\partial r}\right)^{2} dr \right) dr d\theta dz$$

$$= \frac{1}{2} \left(1-\varepsilon^{2}\right) (1+b) \int_{0}^{L} \int_{0}^{2\pi} u^{2} d\theta dz$$

$$+ \frac{1}{4} \left(1-\varepsilon^{2}+2\varepsilon^{2} \ln \varepsilon\right) \left(1+\frac{1}{b}\right) \int_{0}^{L} \int_{0}^{2\pi} \int_{\varepsilon}^{1} r \left(\frac{\partial v_{r}}{\partial r}\right)^{2} dr d\theta dz. \quad (62)$$

For v_z , we have

$$v_z(r,\theta,z,t) = -\int_r^1 \frac{\partial v_z}{\partial r} dr,$$

so we get

$$v_z^2(r,\theta,z,t) = \left(\int_r^1 \frac{1}{\sqrt{r}} \sqrt{r} \frac{\partial v_z}{\partial r} dr\right)^2 \le -\ln r \int_r^1 r \left(\frac{\partial v_z}{\partial r}\right)^2 dr.$$

and, finally

$$\int_{0}^{L} \int_{0}^{2\pi} \int_{\varepsilon}^{1} v_{z}^{2} r dr d\theta dz \leq \int_{0}^{L} \int_{0}^{2\pi} \int_{\varepsilon}^{1} -r \ln r \int_{\varepsilon}^{1} r \left(\frac{\partial v_{z}}{\partial r}\right)^{2} dr dr d\theta dz$$

$$= \frac{1}{4} \left(1 - \varepsilon^{2} + 2\varepsilon^{2} \ln \varepsilon\right) \int_{0}^{L} \int_{0}^{2\pi} \int_{\varepsilon}^{1} r \left(\frac{\partial v_{z}}{\partial r}\right)^{2} dr d\theta dz. \quad (63)$$

Inserting (62) and (63) into (61), and letting $\varepsilon \to 0,$ yield

$$\int_{0}^{L} \int_{0}^{2\pi} \int_{0}^{1} v_{z} v_{r} \frac{\partial \bar{V}_{z}}{\partial r} r dr d\theta dz \leq \frac{a}{4} \left(1 + \frac{1}{b}\right) \int_{0}^{L} \int_{0}^{2\pi} \int_{0}^{1} \left(\frac{\partial v_{r}}{\partial r}\right)^{2} r dr d\theta dz + \frac{1}{4a} \int_{0}^{L} \int_{0}^{2\pi} \int_{0}^{1} \left(\frac{\partial v_{z}}{\partial r}\right)^{2} r dr d\theta dz.$$
(64)

(18) now follows from (60), (64), and (15).