Infinite Dimensional Backstepping–Style Feedback Transformations for a Heat Equation with an Arbitrary Level of Instability

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Abstract

We consider feedback transformations of the backstepping/feedback linearization type that have been prevalent in finite dimensional nonlinear stabilization, and, with the objective of ultimately addressing nonlinear PDE's, generate the first such transformations for a linear PDE that can have an arbitrary finite number of open-loop unstable eigenvalues. These transformations have the form of recursive relationships and the fundamental difficulty is that the recursion has an infinite number of iterations. Naive versions of backstepping lead to unbounded coefficients in those transformations. We show how to design them such that they are sufficiently regular (not continuous but L_∞). We then establish closed–loop stability, regularity of control, and regularity of solutions of the PDE.

1 Introduction

Motivation. In *finite dimensions*, stabilization problems for nonlinear systems are today most commonly solved using the methods of feedback linearization [20] and backstepping [22]. These methods apply diffeomorphic coordinate transformations that put the system equations in the form where the stabilization problem becomes easy (the control input has access to all the nonlinearities). The difference between the two methods is that feedback linearization was invented for systems with perfect models, while backstepping, developed later, allows some flexibility to deal with systems that contain perturbations, disturbances, and unmodeled dynamics. For the majority of nonlinear systems these are not only the most popular but the only stabilization methods available. It is therefore natural that, in attempting to solve stabilization problems for a broader class of *infinite dimensional* nonlinear systems, one first hopes that feedback linearization or backstepping can somehow be extended to infinite dimensions. Unfortunately, the chances that a simple solution to this problem exists are extremely slim. It is enough to look at what the coordinate transformations in feedback linearization and backstepping involve (repeated differentiation of system nonlinearities, combined with arithmetic operations on them) to realize that if such procedures take an infinite number of steps they will result in very problematic nonlinear operators for coordinate transformations, and also for control laws. This does not mean that proving some desirable properties for those transformations is impossible—it is just that, if possible, it will be highly nontrivial.

Because of potential significance of feedback linearization and backstepping for nonlinear infinite–dimensional systems, it is well worth starting the study of these methods on linear infinite–dimensional systems. It turns out that performing these recursive procedures in infinitely many steps is nontrivial even for linear systems. The first step in this direction was made by Boskovic and Krstic [6] who considered the same equation as in this paper (to be introduced below) but with parameters restricted so that the number of open–loop unstable eigenvalues is no greater than one. In this limited case they derived a closed–form and smooth coordinate transformation based on backstepping. This result is peculiar to the mild level of open–loop instability and cannot be extended to the same equation with an arbitrary level of instability. We stress that allowing an arbitrary level of instability is the whole point here. In finite dimensions backstepping can deal with systems where actually all the eigenvalues are unstable (and furthermore with finite–escape type instabilities).

The method we present here reveals a key issue for finding backstepping controls for arbitrarily unstable linear parabolic PDE systems. This key issue is the target system to which one is transforming the original system by coordinate transformation. For example, if one takes the standard feedback linearization route leading to the Brunovsky canonical form, or even the standard backstepping route leading to a tri–diagonal form, the resulting transformations, if thought of as integral transformations, end up with "kernels" that are not even finite. We show how to select the target system so that the the kernel is bounded and the solutions corresponding to the controlled problem are at least continuous.

Equation considered. The equation considered in this paper is

$$u_t(x,t) = \varepsilon u_{xx}(x,t) + \lambda(x)\,u(x,t)\,, \qquad x \in (0,1), \quad t > 0, \tag{1.1}$$

where ε is a positive constant and $\lambda(x) \in L_\infty(0,1)$, with initial condition $u(x,0) = u_0(x)$, for $x \in [0,1]$. The boundary condition at $x = 0$ is homogeneous Dirichlet,

$$u(0,t) \;=\; 0\,, \qquad t > 0\,, \tag{1.2}$$

and the boundary condition at the other end,

$$u(1,t) = \alpha(u(t))\,, \qquad t > 0\,, \tag{1.3}$$

is used as the control input, where α is a linear operator to be designed to achieve stabilization (the control law). For $\lambda(x) \equiv 0$ the open loop system (when $\alpha(u(t)) \equiv 0$) is the heat equation, which is asymptotically stable. However, it is unstable if $\ell = \min_{x\in[0,1]} \lambda(x)$ is large. The growth bound of the uncontrolled system is at least $\omega_0 = \ell - \varepsilon\pi^2$.

The physical motivation for considering equation (1.1) is that it represents the linearization of the class of reaction–diffusion equations that model many physical phenomena. An example is the problem of compressor rotating stall for which the most recent model due to Mezic [19] is $u_t = \varepsilon u_{xx} + u - u^3$, whose linearization is (1.1) with $\lambda(x) \equiv 1$. The Dirichlet boundary control problem that we are pursuing here corresponds to actuation via air injection on only a small interval of the compressor annulus. Control via air injectors distributed along the entire annulus was first reported in [3].

We use a backstepping method for the finite difference semi–discretized approximation of (1.1) to derive a boundary feedback control law that makes the infinite dimensional closed loop system stable with an arbitrary prescribed stability margin. We show that the integral kernel in the control law resides in the function space $L_\infty(0,1)$ and that solutions corresponding to the controlled problem are classical.

Prior work. The problem of boundary feedback stabilization of general parabolic equations is not new. In dimension higher than one Triggiani [30] and Lasiecka [23] considered a general framework for the structural assignment of eigenvalues in parabolic problems through the use of semigroup theory. In their approach the open loop system is separated into a finite dimensional unstable part and an infinite dimensional stable part. They applied feedback control that stabilizes the unstable part while leaving the stable part stable. A unified treatment of both interior and boundary observations/control generalized to semilinear problems can be found in [2]. Nambu [27] developed auxiliary functional observers to stabilize diffusion equations using boundary observation and feedback. Stabilizability by boundary control in the optimal control setting is discussed by Bensoussan et al. [4]. For the general Pritchard–Salamon class of state–space systems a number of frequency–domain results has been established on stabilization during the last decade (see, e.g. [15] and [26] for a survey). While these approaches give an answer to our stabilization problem in principle, ours offers an implementable, closed–form solution that avoids the additional steps of estimating eigenfunctions or solving operator Riccati equations, which are formidable tasks in the case when $\lambda(x)$ is not a constant.

The stabilization problem can be also approached using the abstract theory of boundary control systems developed by Fattorini [17] as described in [16, Section 3.3 and Exercise 5.25] and used in papers by Curtain and coworkers in the 1980's (e.g., [14]). While this approach results in a mathematically simple problem formulation, it has the disadvantage of producing a dynamical feedback as a result of the artificial state space introduced (see remarks in [16, Section 3.5]).

Our work is related to Burns, King and Rubio [9]. They already discovered there the applicability of boundary controls in the form of integral operators. Their result is quite different because the control objective is different (theirs is LQR optimal control, ours is stabilization), and their plant is open–loop stable but the spatial domain is of dimension higher than ours. Nonetheless the technical problem of proving some regularity of the gain kernel ties the two results together. In the paper [9] numerical evidence is presented that suggests that the gain kernel is an L_2 function with compact support concentrated near the boundary. We prove the existence of a non–smooth but bounded (L_∞) gain kernel.

Backstepping was applied to PDEs in [13, 25, 7] but in settings with only a finite number of steps. An approach for control of a fairly broad class of nonlinear parabolic PDEs based on approximate inertial manifolds was developed by Christofides [11, 12].

Organization. This paper is organized as follows. In Section 2 we formulate our problem and its discretization and we lay out our strategy for the solution of the stabilization problem. The precise formulation of our main theorem is contained in Section 3. In Lemma 1 of Section 4 we design a coordinate transformation for a semi–discretization of our system which maps it into an exponentially stable system. We show in Lemma 2 that the discrete coordinate transformation remains uniformly bounded as the grid gets refined and hence it converges to a coordinate transformation of the infinite dimensional system. The regularity $C_w([0,1], L_\infty(0,1))$ of the transformation is established in Lemma 3. The stability of the infinite dimensional controlled system is shown in Lemma 4 completing the proof of our main theorem. Finally we present numerical simulations in Section 5 showing, besides the effectiveness of our control, that reduced versions of the controller stabilizes the infinite dimensional system as well.

2 Motivation

The semi–discretized version of system (1.1)–(1.3) using central differencing in space is the finite dimensional system:

$$u_0 = 0, \tag{2.1}$$

$$\dot{u}_i = \varepsilon \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \lambda_i u_i, \qquad i = 1, \ldots, n, \tag{2.2}$$

$$u_{n+1} = \alpha_n(u_1, u_2, \ldots, u_n), \tag{2.3}$$

where $n \in \mathbb{N}$, $h = \frac{1}{n+1}$ and $u_i = u(ih,t)$, $\lambda_i = \lambda(ih)$ for $i = 0,\dots,n+1$. With u_{n+1} as control, this system is in the strict–feedback form and hence it is readily stabilizable by, for example backstepping, or by transforming the system into the Brunovsky form and applying pole placement, i.e., by "feedback linearization." However these naive control laws would have gains that grow unbounded as $n \to \infty$.

The problem with standard backstepping and feedback linearization is that they would not only attempt to stabilize the equation, but also place all of its poles, and thus as $n \to \infty$, change its parabolic character. Indeed, an infinite dimensional version of the Brunovsky form or the tridiagonal form in backstepping are not parabolic. Our approach will be to transform the system, but keep its parabolic character, i.e., keep the second spatial derivative in the transformed coordinates.

Towards this end, we start with a finite dimensional backstepping–style coordinate transformation

$$\begin{aligned} w_0 &= u_0 = 0, && (2.4)\\ w_i &= u_i - \alpha_{i-1}(u_1,\dots,u_{i-1}), \qquad i = 1,\dots,n, && (2.5)\\ w_{n+1} &= 0, && (2.6) \end{aligned}$$

for the discretized system (2.1)–(2.3), and seek the functions α_i such that the transformed system has the form

$$\begin{aligned} w_0 &= 0, && (2.7)\\ \dot{w}_i &= \varepsilon\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} - cw_i, \qquad i = 1,\dots,n, && (2.8)\\ w_{n+1} &= 0. && (2.9) \end{aligned}$$

The finite dimensional system (2.7)-(2.9) is the semi–discretized version of the infinite dimensional system

$$w_t(x,t) = \varepsilon w_{xx}(x,t) - cw(x,t), \qquad x \in (0,1), \quad t > 0, \qquad (2.10)$$

with boundary conditions

$$\begin{aligned} w(0,t) &= 0, && (2.11)\\ w(1,t) &= 0, && (2.12) \end{aligned}$$

which is exponentially stable for $c > -\varepsilon\pi^2$.

The backstepping coordinate transformation is obtained by combining (2.1)–(2.3), (2.4)–(2.6) and (2.7)–(2.9) and solving the resulting system for the α_i's. We obtain the recursive form

$$\begin{aligned} \alpha_i &= \frac{1}{\varepsilon}\left(2\varepsilon + ch^2\right)\alpha_{i-1} - \alpha_{i-2} - \frac{h^2}{\varepsilon}(\lambda_i + c)\,u_i + \frac{\partial\alpha_{i-1}}{\partial u_1}\left(u_2 - 2u_1 + \frac{1}{\varepsilon}h^2\lambda_1 u_1\right)\\ &\quad + \sum_{j=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial u_j}\left(u_{j+1} - 2u_j + u_{j-1} + \frac{1}{\varepsilon}h^2\lambda_j u_j\right), \qquad (2.13) \end{aligned}$$

for $i = 1,\dots,n$ with initial values $\alpha_0 = \alpha_{-1} = 0$. Writing the α_i's in the linear form

$$\alpha_i = \sum_{j=1}^{i} k_{i,j}u_j, \qquad i = 1,\dots,n \qquad (2.14)$$

and performing simple calculations we obtain the general recursive relationship

$$
\begin{aligned}
k_{i,1} &= \frac{h^2}{\varepsilon}(c+\lambda_1)\,k_{i-1,1}-k_{i-2,1}+k_{i-1,2}\,, && (2.15)\\
k_{i,j} &= \frac{h^2}{\varepsilon}\left(c+\lambda_j\right)k_{i-1,j}+k_{i-1,j-1}+k_{i-1,j+1}-k_{i-2,j}\,, \qquad j=2,\ldots,i-2, && (2.16)\\
k_{i,i-1} &= \frac{h^2}{\varepsilon}(c+\lambda_{i-1})\,k_{i-1,i-1}+k_{i-1,i-2}\,, && (2.17)\\
k_{i,i} &= k_{i-1,i-1}-\frac{h^2}{\varepsilon}(c+\lambda_i)\,. && (2.18)
\end{aligned}
$$

for $i=3,\ldots,n$ with initial conditions

$$
\begin{aligned}
k_{1,1} &= -\frac{h^2}{\varepsilon}(c+\lambda_1)\,, && (2.19)\\
k_{2,1} &= -\frac{h^4}{\varepsilon^2}(c+\lambda_1)^2\,, && (2.20)\\
k_{2,2} &= -\left(\frac{h^2}{\varepsilon}(c+\lambda_1)+\frac{h^2}{\varepsilon}(c+\lambda_2)\right), && (2.21)\\
k_{3,1} &= -\frac{h^6}{\varepsilon^3}(c+\lambda_1)^3-\frac{h^2}{\varepsilon}(c+\lambda_2)\,, && (2.22)\\
k_{3,2} &= -\frac{h^2}{\varepsilon}(c+\lambda_2)\left(\frac{h^2}{\varepsilon}(c+\lambda_1)+\frac{h^2}{\varepsilon}(c+\lambda_2)\right)-\frac{h^4}{\varepsilon^2}(c+\lambda_1)^2\,, && (2.23)\\
k_{3,3} &= -\left(\frac{h^2}{\varepsilon}(c+\lambda_1)+\frac{h^2}{\varepsilon}(c+\lambda_2)+\frac{h^2}{\varepsilon}(c+\lambda_3)\right). && (2.24)
\end{aligned}
$$

For the simple case when $\lambda(x)\equiv\lambda=$constant, equations (2.15)–(2.24) can be solved explicitly to obtain

$$
k_{i,i-j}=-\begin{pmatrix} i \\ j+1 \end{pmatrix}\left(\frac{(c+\lambda)}{\varepsilon(n+1)^2}\right)^{j+1}-(i-j)\sum_{l=1}^{[j/2]}\frac{1}{l}\begin{pmatrix} j-l \\ l-1 \end{pmatrix}\begin{pmatrix} i-l \\ j-2l \end{pmatrix}\left(\frac{(c+\lambda)}{\varepsilon(n+1)^2}\right)^{j-2l+1} \tag{2.25}
$$

for $i=1,\ldots,n$, $j=1,\ldots,i$.

Regarding the infinite dimensional system (1.1)–(1.3), the linearity of the control law in (2.14) suggests a stabilizing boundary feedback control of the form

$$
\alpha(u)=\int_0^1 k(x)\,u(x)\,dx, \tag{2.26}
$$

where the function $k(x)$ is obtained as a limit of $\left\{(n+1)\,k_{n,j}\right\}_{j=1}^{n}$ as $n\to\infty$. From the complicated expression (2.25) it is not clear if such limit exists. A quick numerical simulation (see Figure 1) shows that the coefficients $\left\{(n+1)\,k_{n,j}\right\}_{j=1}^{n}$ remain bounded but it also shows their oscillation, and increasing n only increases the oscillation (see Figure 2). Clearly, there is no hope for pointwise convergence to a continuous kernel $k(x)$. However, as we will see in the next sections, there is weak* convergence in L_∞ as we go from the finite dimensional case to the infinite dimensional one. As a result, we obtain a solution to our stabilization problem (1.1)–(1.3).

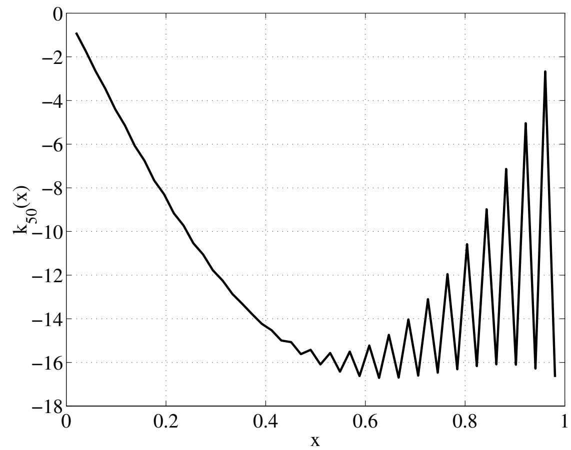

Figure 1: Oscillation of the approximating kernel for $n = 50$, $\lambda = 17$, $\varepsilon = 1$, $c = 0$.

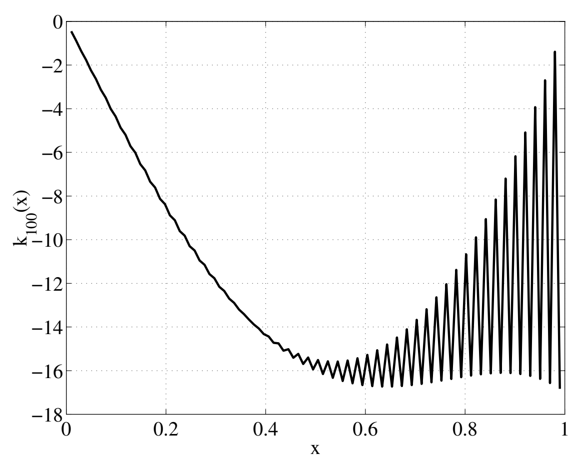

Figure 2: Oscillation of the approximating kernel for $n = 100$, $\lambda = 17$, $\varepsilon = 1$, $c = 0$.

3 Main Result

As we stated earlier, we use a backstepping scheme for the semi–discretized finite difference approximation of system (1.1)–(1.3), (2.26) to derive a linear boundary feedback control law that makes the infinite dimensional closed loop system stable with an arbitrary prescribed stability margin. The precise formulation of our main result is given by the following theorem.

Theorem 1. *For any* $\lambda(x) \in L_\infty(0,1)$ *and* ε, $c > 0$ *there exists a function* $k \in L_\infty(0,1)$ *such that for any* $u_0 \in L_\infty(0,1)$ *the unique classical solution* $u(x,t) \in C^1\left((0,\infty);C^2(0,1)\right)$ *of system (1.1)–(1.3), (2.26) is exponentially stable in the* $L_2(0,1)$ *and maximum norms with decay rate* c*. The precise statements of stability properties are the following: There exists a positive constant* M[a] *such that for all* $t > 0$

$$\|u(t)\| \le M\|u_0\|e^{-ct} \tag{3.1}$$

and

$$\max_{x\in[0,1]}|u(t,x)| \le M \sup_{x\in[0,1]}|u_0(x)|e^{-ct}. \tag{3.2}$$

Remark 1. For a given integral kernel $k \in L_\infty(0,1)$ the existence and regularity results for the corresponding solution $u(x,t)$ follows from trivial modifications in the proof of [24, Thm 4.1]. See also [18]. □

4 Proof of Main Result

As it was already mentioned in the introduction, the proof of Theorem 1 requires four lemmas.

Lemma 1. *The elements of the sequence* $\{k_{i,j}\}$ *defined in (2.15)–(2.24) satisfy*

$$\left|k_{i,i-j}\right| \le \binom{i}{j+1}\left(\frac{h^2}{\varepsilon}(\lambda+c)\right)^{j+1} + (i-j)\sum_{l=1}^{[j/2]}\frac{1}{l}\binom{j-l}{l-1}\binom{i-l}{j-2l}\left(\frac{h^2}{\varepsilon}(\lambda+c)\right)^{j-2l+1} \tag{4.1}$$

where $\lambda = \max_{x\in[0,1]}|\lambda(x)|$.

Remark 2. There is equality in (4.1) when $\lambda(x) \equiv \lambda = constant$ with a minus sign replacing the absolute value sign. □

[a] M grows with c, λ and $1/\varepsilon$.

Proof. The right hand side of equations (2.19)–(2.24) can be estimated to obtain estimates for the initial values of k's

$$
\begin{aligned}
|k_{1,1}| &\leq \frac{h^2}{\varepsilon}(c+\lambda) , && (4.2)\\
|k_{2,1}| &\leq \frac{h^4}{\varepsilon^2}(c+\lambda)^2 , && (4.3)\\
|k_{2,2}| &\leq 2\frac{h^2}{\varepsilon}(c+\lambda) , && (4.4)\\
|k_{3,1}| &\leq \frac{h^6}{\varepsilon^3}(c+\lambda)^3+\frac{h^2}{\varepsilon}(c+\lambda) , && (4.5)\\
|k_{3,2}| &\leq 3\frac{h^4}{\varepsilon^2}(c+\lambda)^2 , && (4.6)\\
|k_{3,3}| &\leq 3\frac{h^2}{\varepsilon}(c+\lambda) . && (4.7)
\end{aligned}
$$

We then go from $j=i$ backwards to obtain from (2.18)

$$|k_{i,i}| \leq i\frac{h^2}{\varepsilon}(c+\lambda) , \tag{4.8}$$

$$|k_{i,i-1}| \leq \frac{i(i-1)}{2}\frac{h^4}{\varepsilon^2}(c+\lambda)^2 . \tag{4.9}$$

Finally we obtain inequality (4.1) of Lemma 1 using the general identity (2.16) and mathematical induction. □

In order to prove that the finite dimensional coordinate transformation (2.4), (2.5), (2.14) converges to an infinite dimensional one that is well–defined, we show the uniform boundedness of $(n+1)\,k_{i,j}$ with respect to $n\in\mathbb{N}$ as $i=1,\ldots,n,\ j=1,\ldots,i$. Note that the binomial coefficients of equation (4.1) are monotone increasing in i and hence it is enough to show the boundedness of terms $(n+1)\left|k_{n,n-j}\right|$. Also, we introduce notations

$$E=\frac{\lambda+c}{\varepsilon} \tag{4.10}$$

and

$$q=\frac{j}{n}\in[0,1] , \tag{4.11}$$

so that we can write

$$\frac{h^2}{\varepsilon}(c+\lambda)=\frac{E}{(n+1)^2} \tag{4.12}$$

and

$$
\begin{aligned}
\left|k_{n,n-j}\right| &= \left|k_{n,n-qn}\right| \\
&\leq \begin{pmatrix} n \\ qn+1 \end{pmatrix}\left(\frac{E}{(n+1)^2}\right)^{qn+1} \\
&\quad +(n-qn)\sum_{i=1}^{[qn/2]}\frac{1}{i}\begin{pmatrix} qn-i \\ i-1 \end{pmatrix}\begin{pmatrix} n-i \\ qn-2i \end{pmatrix}\left(\frac{E}{(n+1)^2}\right)^{qn-2i+1}
\end{aligned} \tag{4.13}
$$

Lemma 2. *The sequence* $\left\{(n+1)\,k_{n,j}\right\}_{j=1,\dots,n,\,n\geq 1}$ *remains bounded uniformly in n and j as* $n\to\infty$.

Proof. We can write, according to (4.13),

$$\begin{aligned}(n+1)\left|k_{n,n-qn}\right| \;\leq\; & (n+1)\begin{pmatrix} n \\ qn+1\end{pmatrix}\left(\frac{E}{(n+1)^2}\right)^{qn+1} \\ & +(n+1)\,(n-qn)\sum_{l=1}^{[qn/2]}\frac{1}{l}\begin{pmatrix} qn-l \\ l-1\end{pmatrix}\begin{pmatrix} n-l \\ qn-2l\end{pmatrix}\left(\frac{E}{(n+1)^2}\right)^{qn-2l+1}\end{aligned}\tag{4.14}$$

The first term in (4.14) can be estimated as

$$\begin{aligned}(n+1)\begin{pmatrix} n \\ qn+1\end{pmatrix}\left(\frac{E}{(n+1)^2}\right)^{qn+1} \;&\leq\; (n+1)^{qn+2}\left(\frac{E}{n+1}\right)^{qn}\frac{E}{(n+1)^{qn+2}} \\ &\leq\; E\left(\frac{E}{n}\right)^{qn} \\ &\leq\; Ee^{E/e},\end{aligned}\tag{4.15}$$

where the last line shows that the bound is uniform in n and also in q.

In the following steps we will use the simple inequalities

$$\frac{(n-l)!}{(n-qn+l)!}\leq\frac{n}{n-qn+2l}\,\frac{n-1}{n-qn+2l-1}\cdots\frac{n-l+1}{n-qn+l+1}\,\frac{(n-l)!}{(n-qn+l)!}=\frac{n!}{(n-qn+2l)!}\tag{4.16}$$

and

$$\frac{(qn-l)!}{l!\,(qn-2l+1)!}\left(\frac{1}{n+1}\right)^{qn-2l}\;\leq\; q\tag{4.17}$$

with this we obtain

$$\begin{aligned}&(n+1)\,(n-nq)\sum_{l=1}^{[qn/2]}\frac{1}{l}\begin{pmatrix} qn-l \\ l-1\end{pmatrix}\begin{pmatrix} n-l \\ qn-2l\end{pmatrix}\left(\frac{E}{(n+1)^2}\right)^{qn-2l+1} \\ &\leq E\frac{(n+1)\,n}{(n+1)^2}\sum_{l=1}^{[qn/2]}\frac{(qn-l)!}{l!\,(qn-2l+1)!}\left(\frac{1}{n+1}\right)^{qn-2l}\frac{n!}{(qn-2l)!\,(n-qn+2l)!}\left(\frac{E}{n+1}\right)^{qn-2l} \\ &\leq Eq\sum_{s=1}^{nq}\begin{pmatrix} n \\ s\end{pmatrix}\left(\frac{E}{n}\right)^{s}1^{n-s} \\ &\leq Eq\left(1+\frac{E}{n}\right)^{nq} \\ &\leq Ee^{E}.\end{aligned}$$

Here in the last step we used the fact that the convergence $\left(1+\frac{E}{n}\right)^{n}\xrightarrow{n\to\infty}e^{E}$ is monotone increasing and $q\in[0,1]$. This proves the lemma. ∎

As a result of the above boundedness, we obtain a sequence of piecewise constant functions

$$k_n(x,y) = (n+1)\sum_{i=1}^{n}\sum_{j=1}^{i} k_{i,j}\chi_{I_{i,j}}(x,y), \qquad (x,y)\in[0,1]\times[0,1], \qquad n\geq 1, \tag{4.18}$$

where

$$I_{i,j} = \left[\frac{i}{n+1},\frac{i+1}{n+1}\right]\times\left[\frac{j}{n+1},\frac{j+1}{n+1}\right], \qquad j=1,\dots,i, \quad i=1,\dots,n, \quad n\geq 1. \tag{4.19}$$

The sequence (4.18) is bounded in $L_\infty([0,1]\times[0,1])$. The space $L_\infty([0,1]\times[0,1])$ is the dual space of $L_1([0,1]\times[0,1])$ hence, it has a corresponding weak*–topology. Since the space $L_1([0,1]\times[0,1])$ is separable, it follows now by Alaoglu's theorem, see, e.g. [21, pg. 140] or [28, Theorem 6.62], that (4.18) converges in the weak*–topology to a function $\tilde{k}(x,y)\in L_\infty([0,1]\times[0,1])$. The uniform in $p\in\mathbb{N}$ weak convergence in each $L_p([0,1]\times[0,1])\supset L_\infty([0,1]\times[0,1])$, immediately follows.

Remark 3. Alternatively, using the Eberlein–Shmulyan theorem see, e.g., [32, pg. 141], one finds that (4.18) has a weekly convergent subsequence in each $L_p([0,1]\times[0,1])$ space for $1<p<\infty$ with L^p–norms bounded uniformly in p. Using diagonal process we choose a subsequence $m(n)\in\mathbb{N}$ such that $\{k_{m(n)}(x,y)\}_{n\geq 1}$ converges weakly to the same function $\tilde{k}(x,y)$ in each of the spaces $L_p([0,1]\times[0,1])$, $p\in\mathbb{N}$. The function $\tilde{k}(x,y)$ along with $\{k_{m(n)}(x,y)\}_{n\geq 1}$ is uniformly bounded in all these L_p–spaces with the same bound for all $p\in\mathbb{N}$. □

Remark 4. In the case of constant λ we have equality in (4.1). The right hand side is strictly monotone increasing in i, which results in $\tilde{k}\in C([0,1];L_\infty(0,1))$. □

Lemma 3. *The map* $\tilde{k}:[0,1]\to L_\infty(0,1)$ *is weakly continuous.*

Proof. From the uniform boundedness in i of (4.1) we obtain that

$$\sum_{j=1}^{[nx]} k_{[nx],j}u_j = \sum_{j=1}^{[nx]}\left((n+1)k_{[nx],j}\right)u_j\frac{1}{n+1} \xrightarrow{n\to\infty} \int_0^x \tilde{k}(x,\xi)\,u(\xi)\,d\xi \qquad \forall u\in L_1(0,1), \quad \forall x\in[0,1]. \tag{4.20}$$

Here $[nx]$ denotes the largest integer not larger than nx and the convergence is uniform in x, meaning that for all $\varepsilon>0$ there exists $N(\varepsilon)\in\mathbb{N}$ such that

$$\left|\int_0^x \tilde{k}(x,\xi)\,u(\xi)\,d\xi - \sum_{j=1}^{[nx]} k_{[nx],j}u_j\right| < \varepsilon \qquad \forall x\in[0,1], \quad \forall n>N.$$

For an arbitrary $x\in[0,1]$ we now fix an $n>N(\varepsilon/2)$ and choose a $\delta>0$ such that $[nx]=[n(x+\delta)]$. We obtain

$$\begin{aligned}
&\left|\int_0^1 \tilde{k}(x,\xi)\,u(\xi)\,d\xi - \int_0^1 \tilde{k}(x+\delta,\xi)\,u(\xi)\,d\xi\right| \\
&\leq \left|\int_0^x \tilde{k}(x,\xi)\,u(\xi)\,d\xi - \sum_{j=1}^{[nx]} k_{[nx],j}u_j\right| + \left|\sum_{j=1}^{[nx]} k_{[nx],j}u_j - \sum_{j=1}^{[n(x+\delta)]} k_{[n(x+\delta)],j}u_j\right| \\
&+ \left|\sum_{j=1}^{[n(x+\delta)]} k_{[n(x+\delta)],j}u_j - \int_0^{x+\delta} \tilde{k}(x+\delta,\xi)\,u(\xi)\,d\xi\right| \\
&< \varepsilon/2 + 0 + \varepsilon/2 = \varepsilon
\end{aligned} \tag{4.21}$$

which proves the weak continuity, i.e.

$$\tilde{k} \in C_w([0,1]; L_\infty(0,1)). \tag{4.22}$$

□

The following lemma shows how norms change under the above transformation.

Lemma 4. *Suppose that two functions $w(x) \in L_\infty(0,1)$ and $u(x) \in L_\infty(0,1)$ satisfy the relationship*

$$w(x) = u(x) - \int_0^x \tilde{k}(x,\xi)\, u(\xi)\, d\xi \qquad \forall x \in [0,1], \tag{4.23}$$

where

$$\tilde{k} \in C_w([0,1]; L_\infty(0,1)). \tag{4.24}$$

Then there exist positive constants m and M, whose sizes depend only on $\tilde{k}$, such that

$$m\|w\|_\infty \le \|u\|_\infty \le M\|w\|_\infty$$

and

$$m\|w\| \le \|u\| \le M\|w\|.$$

Proof. Clearly

$$\|w\|_\infty \le \left(1 + \left\|\tilde{k}\right\|_\infty\right)\|u\|_\infty. \tag{4.25}$$

Let us choose a positive constant

$$\delta = \min\left\{1, 1/\left(2\left\|\tilde{k}\right\|_\infty\right)\right\} \tag{4.26}$$

so that $\delta\left\|\tilde{k}\right\|_\infty < 1/2$, and let us denote

$$\|u\|_{\infty,i\delta} = \operatorname*{ess\,sup}_{x\in[(i-1)\delta,i\delta]} |u(x)| \qquad i = 1,\ldots,\left[\frac{1}{\delta}\right] \tag{4.27}$$

and for $N_\delta = [1/\delta] + 1$

$$\|u\|_{\infty,N_\delta\delta} = \operatorname*{ess\,sup}_{x\in[(N_\delta-1)\delta,1]} |u(x)| \tag{4.28}$$

which is zero in the special case when $1/\delta$ is an integer. We have that

$$\|u\|_\infty \le \sum_{i=1}^{N_\delta} \|u\|_{\infty,i\delta} \le N_\delta \|u\|_\infty. \tag{4.29}$$

We have, from (4.23)

$$\|u\|_{\infty,1\delta} \quad\le\quad \|w\|_{\infty,1\delta} + \delta\|u\|_{\infty,1\delta}\left\|\tilde{k}\right\|_\infty \tag{4.30}$$

and then

$$\|u\|_{\infty,1\delta} \le \frac{1}{1-\delta\left\|\tilde{k}\right\|_\infty}\|w\|_{\infty,1\delta}. \tag{4.31}$$

Similarly

$$\begin{aligned} \|u\|_{\infty,2\delta} &\leq \|w\|_{\infty,2\delta} + \int_0^{2\delta} \left|\tilde{k}(x,\xi)\, u(\xi,t)\right| d\xi \\ &\leq \|w\|_{\infty,2\delta} + \delta \left\|\tilde{k}\right\|_\infty \|u\|_{\infty,1\delta} + \delta \left\|\tilde{k}\right\|_\infty \|u\|_{\infty,2\delta} \end{aligned} \tag{4.32}$$

and from here

$$\begin{aligned} \|u\|_{\infty,2\delta} &\leq \frac{1}{1-\delta\left\|\tilde{k}\right\|_\infty} \left(\|w\|_{\infty,2\delta} + \frac{\delta\left\|\tilde{k}\right\|_\infty}{1-\delta\left\|\tilde{k}\right\|_\infty} \|w\|_{\infty,1\delta} \right) \\ &= \frac{1}{1-\delta\left\|\tilde{k}\right\|_\infty} \|w\|_{\infty,2\delta} + \frac{\delta\left\|\tilde{k}\right\|_\infty}{\left(1-\delta\left\|\tilde{k}\right\|_\infty\right)^2} \|w\|_{\infty,1\delta}\,. \end{aligned} \tag{4.33}$$

Similarly

$$\begin{aligned} \|u\|_{\infty,3\delta} &\leq \|w\|_{\infty,3\delta} + \int_0^{3\delta} \left|\tilde{k}(x,\xi)\, u(\xi,t)\right| d\xi \\ &= \|w\|_{\infty,3\delta} + \int_0^{\delta} \left|\tilde{k}(x,\xi)\, u(\xi,t)\right| d\xi + \int_\delta^{2\delta} \left|\tilde{k}(x,\xi)\, u(\xi,t)\right| d\xi + \int_{2\delta}^{3\delta} \left|\tilde{k}(x,\xi)\, u(\xi,t)\right| d\xi \\ &\leq \|w\|_{\infty,3\delta} + \delta\left\|\tilde{k}\right\|_\infty \|u\|_{\infty,1\delta} + \delta\left\|\tilde{k}\right\|_\infty \|u\|_{\infty,2\delta} + \delta\left\|\tilde{k}\right\|_\infty \|u\|_{\infty,3\delta} \end{aligned} \tag{4.34}$$

resulting in

$$\begin{aligned} \|u\|_{\infty,3\delta} &\leq \frac{1}{\left(1-\delta\left\|\tilde{k}\right\|_\infty\right)} \left(\|w\|_{\infty,3\delta} + \delta\left\|\tilde{k}\right\|_\infty \|u\|_{\infty,1\delta} + \delta\left\|\tilde{k}\right\|_\infty \|u\|_{\infty,2\delta} \right) \\ &\leq \frac{1}{1-\delta\left\|\tilde{k}\right\|_\infty} \|w\|_{\infty,3\delta} + \frac{\delta\left\|\tilde{k}\right\|_\infty}{\left(1-\delta\left\|\tilde{k}\right\|_\infty\right)^2} \|w\|_{\infty,1\delta} \\ &\quad + \frac{\delta\left\|\tilde{k}\right\|_\infty}{\left(1-\delta\left\|\tilde{k}\right\|_\infty\right)^2} \|w\|_{\infty,2\delta} + \frac{\left(\delta\left\|\tilde{k}\right\|_\infty\right)^2}{\left(1-\delta\left\|\tilde{k}\right\|_\infty\right)^3} \|w\|_{\infty,1\delta}\,, \\ &= \frac{1}{1-\delta\left\|\tilde{k}\right\|_\infty} \|w\|_{\infty,3\delta} + \frac{\delta\left\|\tilde{k}\right\|_\infty}{\left(1-\delta\left\|\tilde{k}\right\|_\infty\right)^3} \|w\|_{\infty,1\delta} + \frac{\delta\left\|\tilde{k}\right\|_\infty}{\left(1-\delta\left\|\tilde{k}\right\|_\infty\right)^2} \|w\|_{\infty,2\delta} \end{aligned} \tag{4.35}$$

and by induction

$$\|u\|_{\infty,i\delta} \leq \frac{1}{1-\delta\left\|\tilde{k}\right\|_\infty} \|w\|_{\infty,i\delta} + \delta\left\|\tilde{k}\right\|_\infty \sum_{j=0}^{i-1} \|w\|_{\infty,j\delta} \frac{1}{\left(1-\delta\left\|\tilde{k}\right\|_\infty\right)^{i-j+1}} \tag{4.36}$$

for $i = 1,\ldots,N_\delta$ with the convention that $\|w\|_{\infty,0\delta} = 0$. Using the definition of δ we obtain from inequality (4.36) that

$$
\begin{aligned}
\|u\|_\infty &\le \sum_{i=1}^{N_\delta} \|u\|_{\infty,i\delta} \\
&\le \frac{1}{1-\delta\left\|\tilde{k}\right\|_\infty} \sum_{i=1}^{N_\delta} \|w\|_{\infty,i\delta} + \delta\left\|\tilde{k}\right\|_\infty \sum_{i=1}^{N_\delta}\sum_{j=0}^{i-1} \|w\|_{\infty,j\delta} \frac{1}{\left(1-\delta\left\|\tilde{k}\right\|_\infty\right)^{i-j+1}} \\
&\le 2N_\delta \|w\|_\infty + \sum_{j=1}^{N_\delta-1} \|w\|_{\infty,j\delta} \sum_{i=j+1}^{N_\delta} 2^{i-j} \\
&= 2N_\delta \|w\|_\infty + \sum_{j=1}^{N_\delta-1} \|w\|_{\infty,j\delta} \left(2^{N_\delta-j+1}-2\right) \\
&\le 2^{N_\delta} N_\delta \|w\|_\infty
\end{aligned}
\tag{4.37}
$$

Inequality (4.25) together with (4.37) results in the relationship

$$
\frac{1}{1+\left\|\tilde{k}\right\|_\infty} \|w\|_\infty \le \|u\|_\infty \le 2^{N_\delta} N_\delta \|w\|_\infty . \tag{4.38}
$$

For the L_2–norms the inequality

$$
\frac{1}{1+\left\|\tilde{k}\right\|} \|w\| \le \|u\| \le 2^{N_\delta}\sqrt{N_\delta}\,\|w\| \tag{4.39}
$$

can be proven in a similar way. Taking

$$
m = \frac{1}{1+\left\|\tilde{k}\right\|_\infty} \tag{4.40}
$$

and

$$
M = 2^{N_\delta} N_\delta \tag{4.41}
$$

we obtain the statement of the lemma. □

Proof of Theorem 1. We now complete the proof of Theorem 1 by combining the results of Lemmas 1–4. In Lemma 1 we derived a coordinate transformation that transforms the finite dimensional system (2.1)–(2.3) into the finite dimensional system (2.7)–(2.9). As a result of the uniform boundedness of the transformation (shown in Lemma 2) we obtained the coordinate transformation (4.23) that transforms the unstable heat equation (1.1) with zero Dirichlet boundary conditions into the stable heat equation (2.10)–(2.12). Due to the weak continuity proven in Lemma 3 the infinite dimensional coordinate transformation results in the specific boundary condition

$$
u(1,t) = \alpha(u) = \int_0^1 k(\xi)\, u(\xi,t)\, d\xi, \tag{4.42}
$$

where

$$
k(\xi) = \tilde{k}(1,\xi), \qquad \xi \in [0,1] \tag{4.43}
$$

with $k \in L_\infty(0,1)$.

It is important to note that the function $k(x)$ is not necessarily smooth, not even continuous. This non–smoothness can be seen numerically in Figure 5 and analytically when we consider $k(x)$ as the limit of its finite difference approximation $\left\{(n+1)\,k_{n,j}\right\}_{j=1,\ldots,n,\,n\geq 1}$. For example for the case $\lambda(x) \equiv \lambda > 0$ we have from (2.18)

$$\lim_{n\to\infty} (n+1)\,k_{n,n} \quad = \quad -\frac{(\lambda+c)}{\varepsilon}, \tag{4.44}$$

which is a negative constant, while (2.17) provides us with

$$\lim_{n\to\infty} (n+1)\,k_{n,n-1} \quad = \quad 0. \tag{4.45}$$

The convergence in Sobolev spaces $W_2^{2,1}$ (see, e.g. [1]) of the finite difference approximations obtained from (2.1)–(2.3) and (2.7)–(2.9) to the solutions of (1.1)–(1.3) and (2.10)–(2.12) respectively is obtained using interpolation techniques (see, e.g. [5].) Using Green's function and fixed point method as it was done in [24], we see that solutions to (1.1)–(1.3), (4.42) are, in fact, classical solutions. The well known (see, e.g. [10]) stability properties of solution w to the heat equation (2.10)–(2.12) along with Lemma 4 proves the stability statements of Theorem 1. $\square$

5 Numerical Demonstration

In accordance with the derivation of our control we use a second order finite difference scheme in our numerical simulations. In space the discretization is exactly the one used in the previous section. The time discretization is based on a low–storage, three time step, third–order Runge–Kutta/Crank–Nicolson scheme (see [29]). Consider system (1.1)–(1.3) with $\lambda(x) \equiv \lambda = 17$, $\varepsilon = 0.1$ and with initial condition $u_0(x) = -0.01e^{6.7x}\sin 8\pi x$. In this case the number of unstable eigenvalues is 4 and the growth bound of the open loop system is $\omega_0 \approx 16$ (see Figure 3). Using the method developed in Lemma 1 we obtain for $c = 1$, $n = 400$ a kernel function $k(x) \approx k_n(x)$ displayed in Figure 4. For a smaller value of ε Figures 1 and 2 of Section 2 already showed the oscillation of the function $k_n(x)$. This tells us that the limiting kernel function $k(x)$ is not continuous. Due to the high growth bound ($\omega_0 \approx 16$) of the open loop system in the present case the gain values are quite high and hence similar oscillation can be seen only after enlarging some part of function $k_n(x)$ in Figure 5. As Figures 6 and 7 show, the obtained control effectively stabilizes system (1.1)–(1.3). Next, we keep the high resolution ($n = 400$) in the discretization of (1.1) but reduce the number of points n_k used in the feedback control (1.3) with still uniformly distributed observation points $x_k = \dfrac{k}{n_k+1}$, $k = 1,\ldots,n_k$. As Figure 8 shows, the case $n_k = 100$ virtually agrees with the "full" observation case $n_k = 400$. By increasing the resolution in the finite difference approximation of the system to $n = 1000$ and $dt = 10^{-5}$ we were able to decrease the number of observation points down to $n_k = 5$ before losing the stabilizing effect.

Remark 5. *1.* By increasing n further above 1000 it should be possible to reduce n_k to 4.

2. Another possible way to reduce the number of measurements even below the very low $n_k = 5$ is to use a low–dimensional observer based on Galerkin's method as in [12].

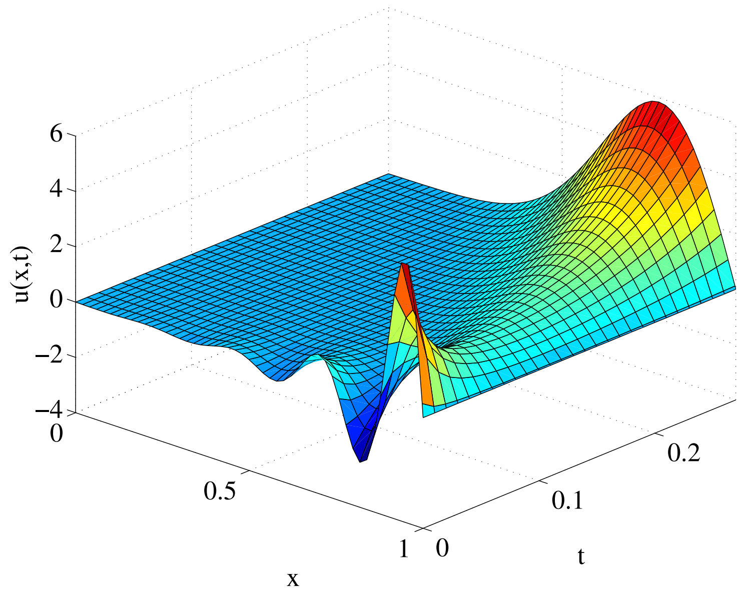

Figure 3: Uncontrolled

3. We use equispaced observation points in the derivation of the kernel function $k(x)$. Even though numerically this is not necessarily the most optimal choice, it is a choice that allows to establish regularity of the kernel and of the closed–loop PDE system.

$\square$

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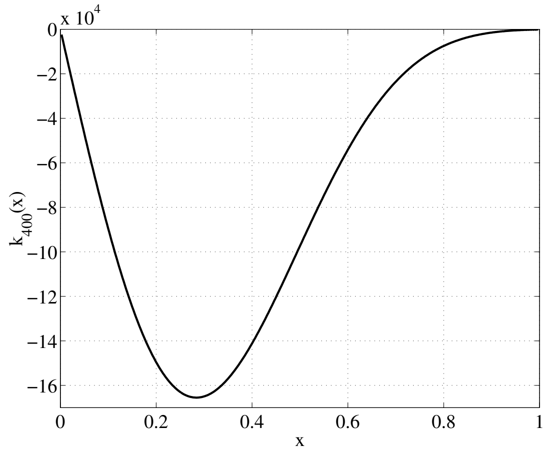

Figure 4: Kernel function $k_n(x)$ for $n = 400$.

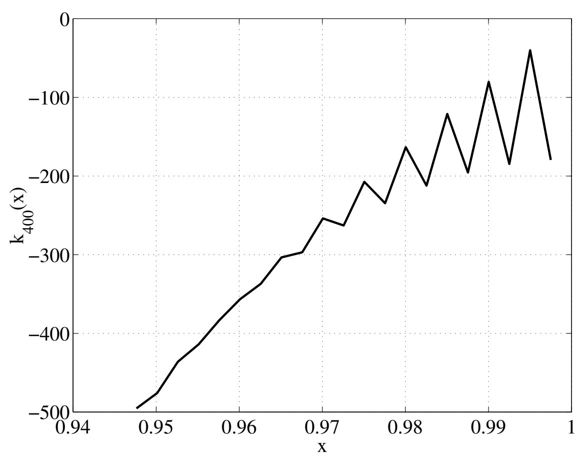

Figure 5: Oscillation of the approximating kernel function.

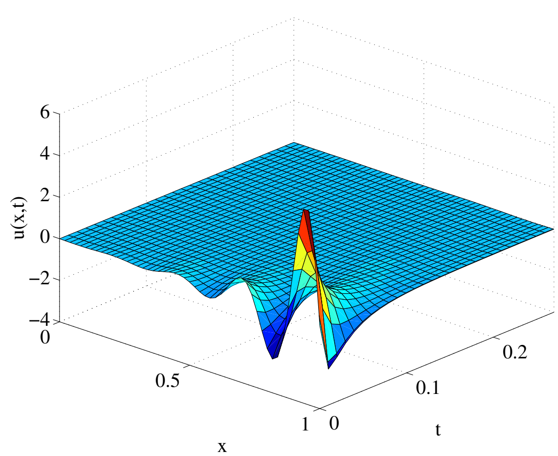

Figure 6: Approximation of controlled system for $n = 400$.

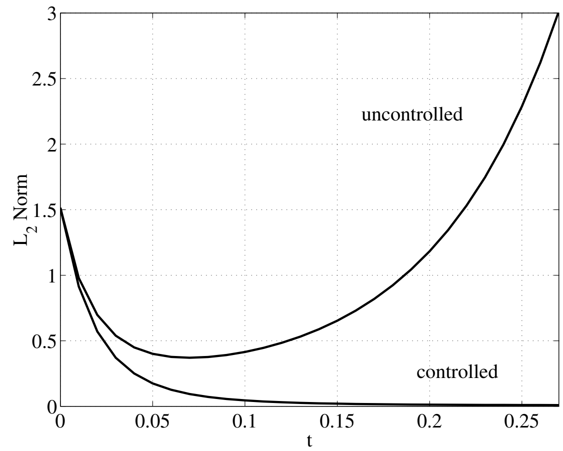

Figure 7: Comparison of L^2–norms

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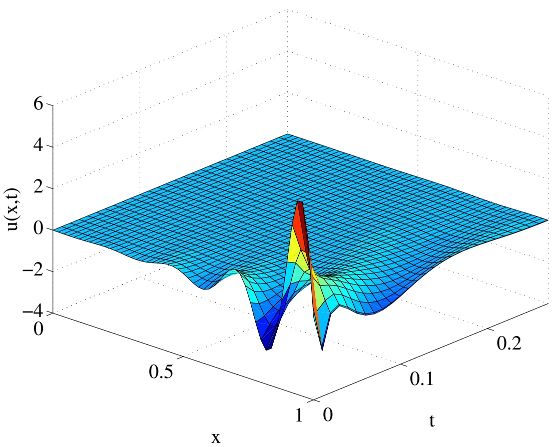
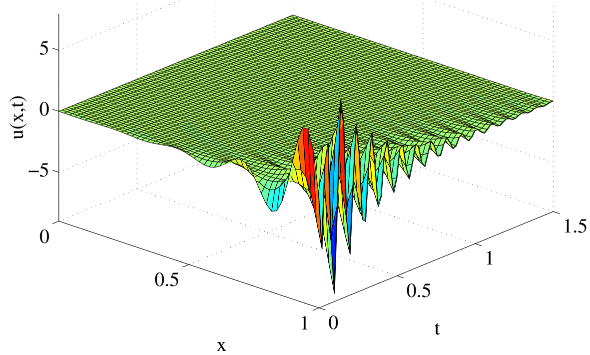
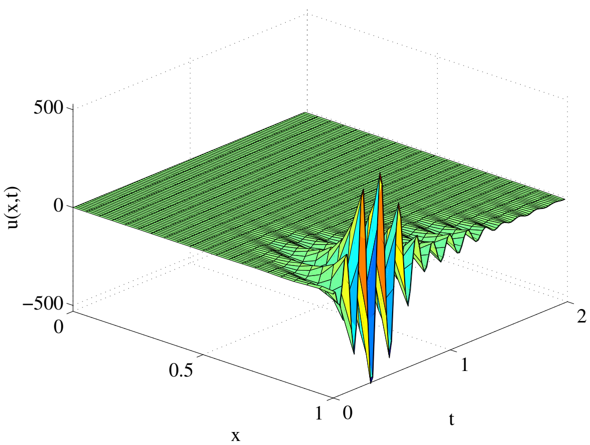


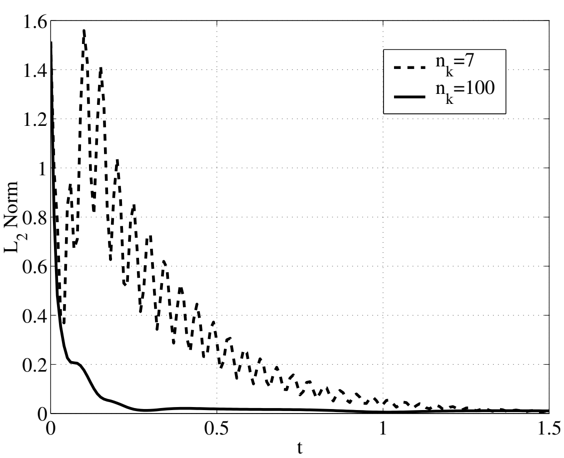
$n_k = 100$



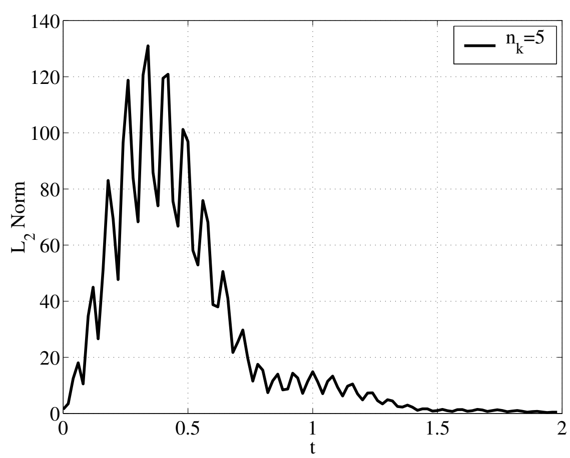
$n_k = 7$



$n_k = 5$



$n = 400$



$n = 1000$

Figure 8: Approximation of controlled system using reduced controllers.

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