On spheres with k points inside

joint work with
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Point sets

The sets in $\mathbb{R}^d$ that we are going to consider are **generic**

- finite point sets, or
- **thin Delone sets**.

Definition

A set $A \subseteq \mathbb{R}^d$ is called a **thin Delone set** if

- every ball contains finitely many points of A, and
- every halfspace contains at least one point of A.

Generic:

- no $d + 1$ points lie on the same hyperplane, and
- no $d + 2$ points lie on the same sphere.

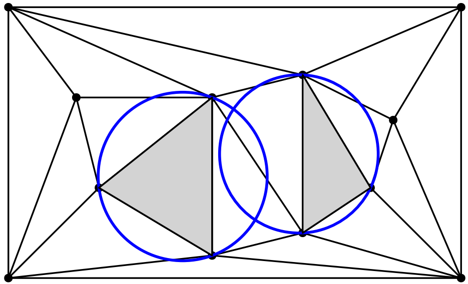
Delaunay triangulations

- Delone/Delaunay, "Sur la sphère vide", ICM 1924, Toronto.

Definition

For a set A, the **Delaunay triangulation** of A is the collection of simplices Δ with vertices in A such that the circumsphere of Δ has no points of A inside.

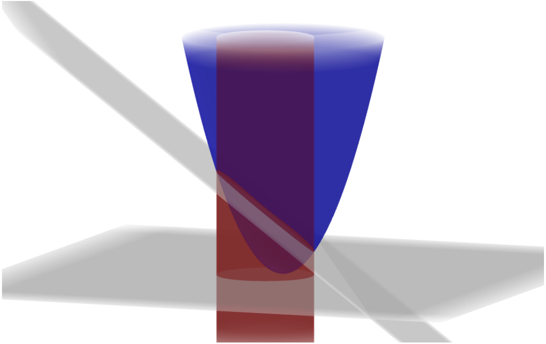
- This is a triangulation of $\operatorname{conv} A$ or the whole $\mathbb{R}^d$.



Lifting construction

Constructing Delaunay triangulation

- Lift every point of $A \subset \mathbb{R}^d$ to paraboloid $y = x_1^2 + \ldots + x_d^2$ by $a \mapsto (a, ||a||^2) \in \mathbb{R}^{d+1}$;
- Take convex hull of the lifted point set and project the (lower) boundary back to $\mathbb{R}^d$.



Motivation for this work

Definition

Let A be a finite or a thin Delone set in $\mathbb{R}^d$. A simplex Δ with vertices in A is called a ***k*-hefty** simplex of A if the circumsphere of A contains exactly k points of A inside.

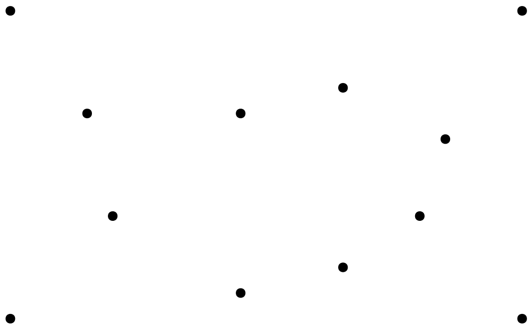
Question

What can we say about k-hefty simplices of A?

Let's play a game

Question

For $d = 2$ and $k = 1$, how many 1-hefty triangles of A contain a given **generic** *point x inside?*



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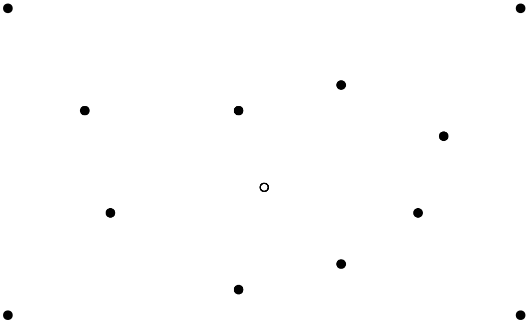

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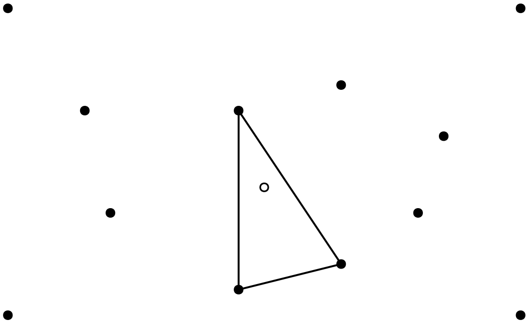

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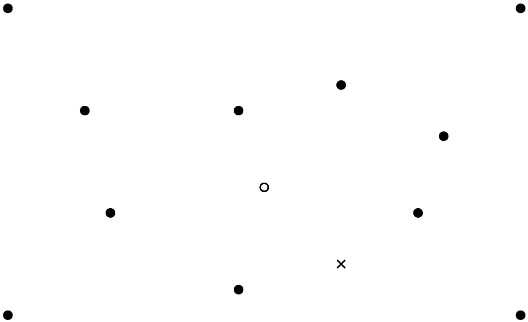

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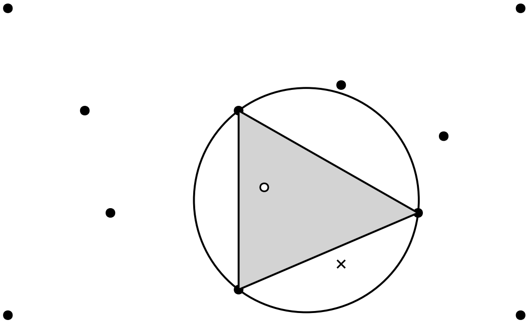

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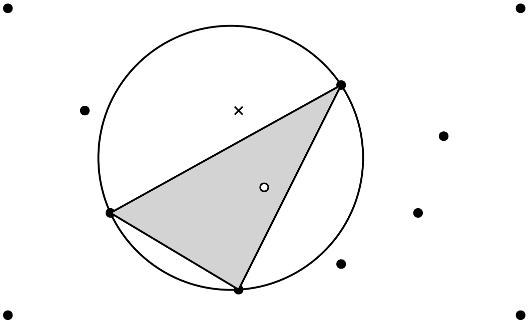

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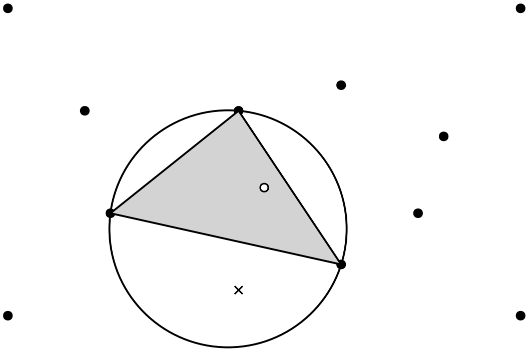

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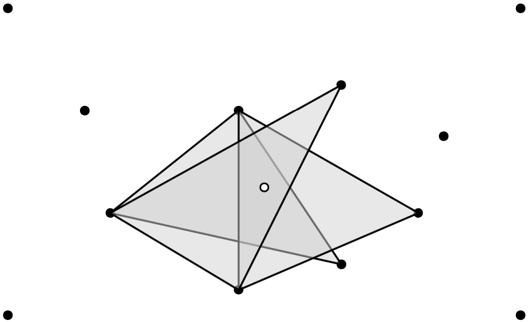
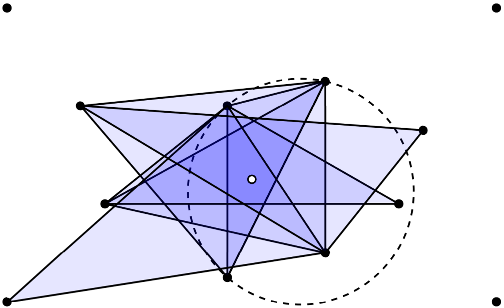


The main theorem

Theorem (Edelsbrunner, G., Saghafian, 2025)

*Let A be a thin Delone set in $\mathbb{R}^d$ and let x be generic point in $\mathbb{R}^d$. Then x belongs to **exactly** $\binom{d+k}{d}$ k-hefty simplices of A.*



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Proof.

- For every A, there is a “covering” constant.
- The constant does not depend on A.
- There is a set where the constant is obvious.

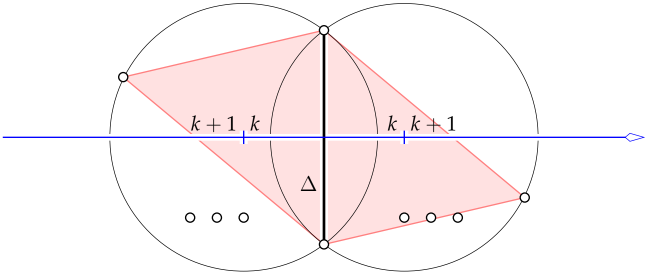
$\square$

Step 1: there is a constant

Lemma

For a given thin Delone set A and x, there is a constant $c(A)$ such that the number of k-simplices of A that contain x is $c(A)$.

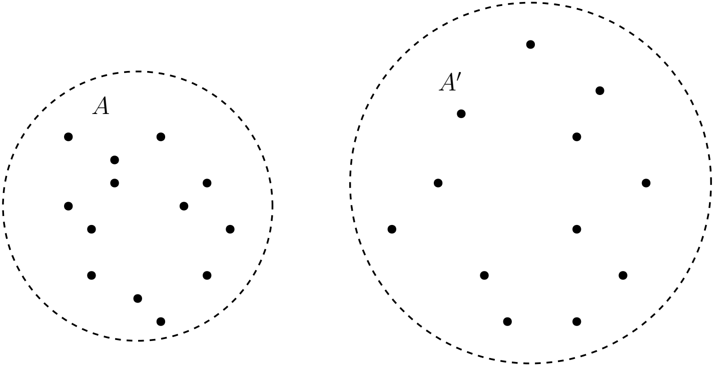
Let Δ be a simplex spanned by d points of A. We show that there are equal numbers of k-hefty simplices sharing Δ on both sides.



STEP 2: THE CONSTANT IS THE SAME FOR ALL SETS

Lemma

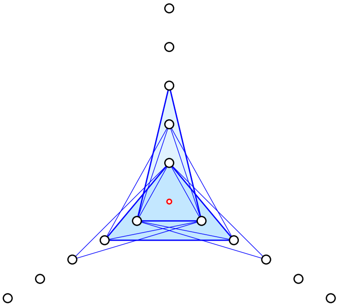
For two thin Delone sets A and A' in $\mathbb{R}^d$, $c(A) = c(A')$.



Step 3: there is a set where the constant is obvious

Lemma

For the **radial** *thin Delone set A, $c(A) = \binom{d+k}{d}$.*



$\binom{d+k}{d}$ is number of ways to put k points into $d+1$ boxes.

Finite sets

Theorem (Edelsbrunner, G., Saghafian, 2025)

For a finite set A in $\mathbb{R}^d$, generic point set $x \in \mathbb{R}^d$ belongs to **at most** $\binom{d+k}{d}$ *k-hefty simplices of A.*

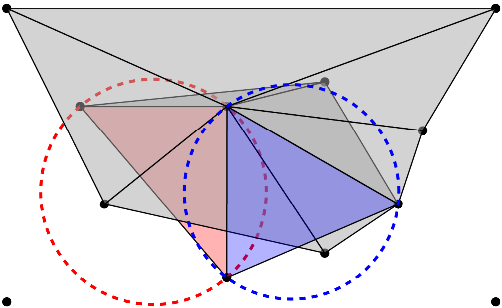
Theorem (Edelsbrunner, G., Saghafian, 2025)

If every hyperplane through x has at least $k+1$ points of A on both sides, then x belongs to **exactly** $\binom{d+k}{d}$ *k-hefty simplices of A.*

Local version

Theorem (Edelsbrunner, G., Saghafian, 2025)

Let A be a thin Delone set in $\mathbb{R}^d$ and let a be any point of A. Then the k-hefty simplices of A **incident** *to a cover a small neighborhood of A exactly $\binom{d+k-1}{d-1}$ times.*



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Proof.

$$\binom{d+k-1}{d-1} = \binom{d+k}{d} - \binom{d+k-1}{d}$$

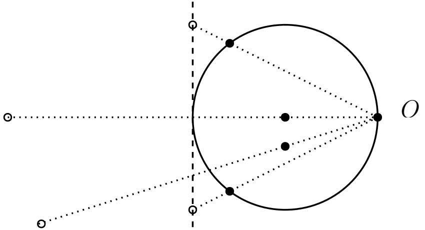
□

k-facets and spheres

Definition

Let A be a finite generic point set in $\mathbb{R}^d$. A subset $B \subset A$ of size d is called a **k-facet** of A if the hyperplane through B separates k points of A from the rest.

Observation. After an inversion with respect to sphere with center O, k-facet turns into a k-hefty simplex of the set $A \cup \{O\}$ provided O is on the right side of the hyperplane.



k-facets in the plane

Theorem (Alon, Györi, 1986)

For every generic point set A in $\mathbb{R}^2$ with n points, the total number of 0-, 1-, ..., k-facets is at most $(k+1)n$.

New proof for $k \leq \frac{n}{3}$.

- Pick a point O that is on a right side of all i-facets for $i \leq k$.
- Perform inversion with respect to a circle centered at O.
- Count how many i-hefty simplices for $i = 0, 1, \ldots, k$ could be so they cover a neighborhood of A at most $\frac{(k+1)(k+2)}{2}$ times.

□

Lovász lemma

Theorem (Bárány, Füredi, Lovász, 1990)

Let $A \subset \mathbb{R}^d$ be a generic set with $2n + d$ points. Let ℓ be a line generic with respect to A. Then ℓ intersects interiors of at most $O(n^{d-1})$ **halving** *facets of A.*

Theorem ("Exact Lovász lemma" by Welzl, 2001)

In a similar setting, the number of **directed** *k-facets intersected by a line is at most $\binom{d+k-1}{d-1}$.*

New proof.

Pick a point O far on ℓ. After inversion, k-facets intersected by ℓ turn into k-hefty simplices incident to O and intersected by ℓ, and the local covering gives us the bound. $\square$

Hypersimplices and Eulerian numbers

- The **order-k d-dimensional hypersimplex** $\Delta_d^{(k)}$ is

$$\Delta_d^{(k)} := \operatorname{conv}\left(\sum_{i\in I} e_i \mid I \subset \{1,\dots,d+1\}, |I| = k\right) \subset \mathbb{R}^{d+1}.$$

- Eulerian number $A(d,k)$ is the number of permutations of $\{1,\dots,d\}$ with k descents.

Theorem (de Laplace, 1886, also Stanley, 1977)

$$d! \cdot \operatorname{vol} \Delta_d^{(k)} = A(d,k-1).$$

- The covering constant allows us to give a new proof using a volume argument for order-k Delaunay triangulations.

Lifting construction again

- Let A be a ~~thin Delone set~~ **perturbation of $\mathbb{Z}^d$**. We lift every point $a \in A$ to the paraboloid $a \mapsto (a, ||a||^2) \in \mathbb{R}^{d+1}$ to get a lifted set A'.

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- Let n be a positive integer. For every subset of n points of A', take the average point.
- For the set of averages, take the convex hull and project the boundary back onto $\mathbb{R}^d$.
- The resulted tiling (which is called the order-n Delaunay triangulation of A) uses hypersimplices of orders $1, 2, \ldots, d$ originating from $(n-1)$-, $(n-2)$-, $\ldots$, $(n-d)$-hefty simplices of A, respectively.

Worpitzky identity

From a "volume argument" for all tiles in a large ball,

$$\sum_{p=1}^{d} v(d,p)\binom{d+n-p}{d} = n^d.$$

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Which is a rewritten Worpitzky identity for Eulerian numbers

$$\sum_{k=0}^{d-1} A(d,k)\binom{x+k}{d} = x^d.$$

$\mathbb{H}^d$ and $\mathbb{S}^d$

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Definition

A set $A \subset \mathbb{S}^d$ is called ***k*-balanced** if every open hemisphere contains at least $k+1$ points of A.

Theorem (Edelsbrunner, G., Saghafian, 2025)

Let A be a k-balanced finite generic point set in $\mathbb{S}^d$. Then

1. *Every generic point of $\mathbb{S}^d$ is covered by $\binom{d+k}{d}$ k-hefty simplices of A;*
2. *For every $a \in A$, a small neighborhood of a is covered by k-hefty simplices of A incident to a in $\binom{d+k-1}{d-1}$ layers.*

Weighted point sets

- We assign weights to points of a thin Delone set A;
- One can think of weighted points as spheres with radii defined by weights;
- Circumspheres are spheres orthogonal to weighted points.

Theorem (Edelsbrunner, G., Saghafian, 2025)

Let (A, w) be a weighted generic thin Delone set in $\mathbb{R}^d$ with a bounded weight function w. Then every generic point of $\mathbb{R}^d$ is covered by $\binom{d+k}{d}$ k-hefty simplices of (A, w).

THANK YOU!