# On spheres with $k$ points inside

joint work with
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# Point sets

The sets in $\mathbb{R}^d$ that we are going to consider are **generic**

- finite point sets, or
- **thin Delone sets**.

**Definition**

A set $A \subseteq \mathbb{R}^d$ is called a **thin Delone set** if

- every ball contains finitely many points of $A$, and
- every halfspace contains at least one point of $A$.

**Generic**:

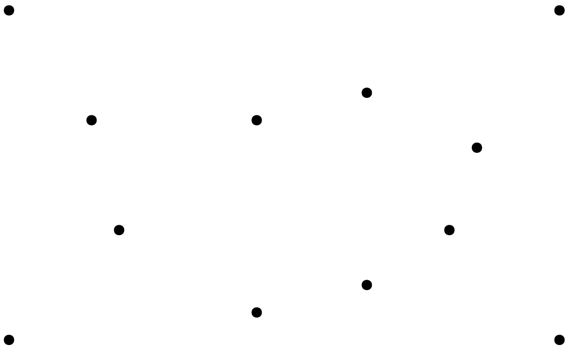
- no $d+1$ points lie on the same hyperplane, and
- no $d+2$ points lie on the same sphere.

# Delaunay triangulations

**Definition**

For a set $A$, the **Delaunay triangulation** of $A$ is the collection of simplices $\Delta$ with vertices in $A$ such that the circumsphere of $\Delta$ has no points of $A$ inside.

- This is a triangulation of $\operatorname{conv} A$ or the whole $\mathbb{R}^d$.



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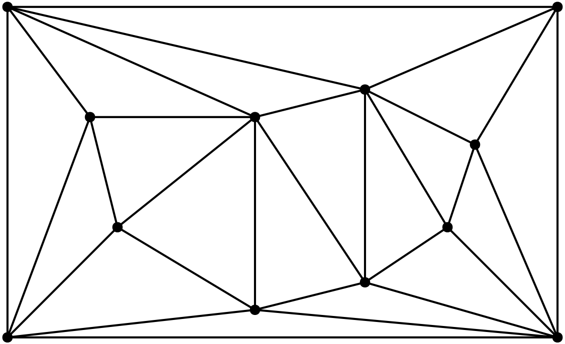

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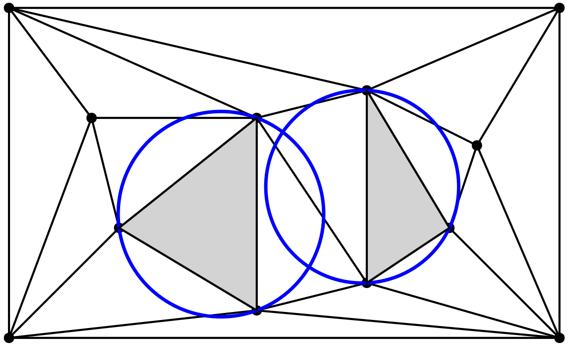

# Lifting construction

## Constructing Delaunay triangulation

- Lift every point of $A \subset \mathbb{R}^d$ to paraboloid $y = x_1^2 + \ldots + x_d^2$ by $a \mapsto (a, ||a||^2) \in \mathbb{R}^{d+1}$;
- Take convex hull of the lifted point set and project the (lower) boundary back to $\mathbb{R}^d$.

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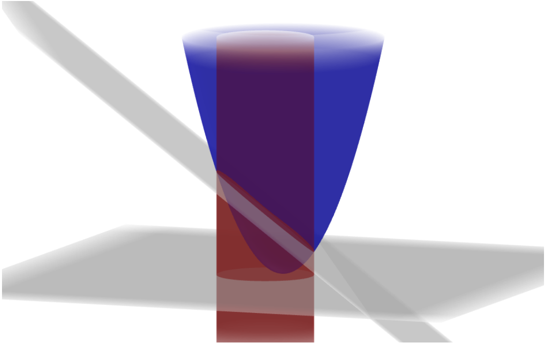


# Motivation for this work

**Definition**

Let $A$ be a finite or a thin Delone set in $\mathbb{R}^d$. A simplex $\Delta$ with vertices in $A$ is called a ***k*-heavy** simplex of $A$ if the circumsphere of $A$ contains exactly $k$ points of $A$ inside.

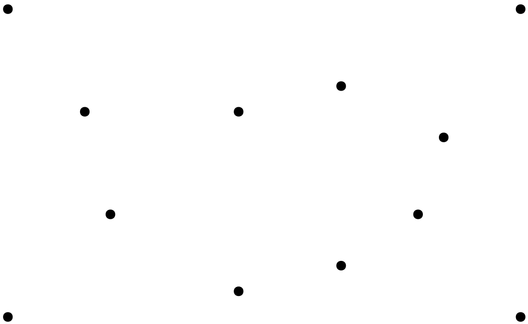
**Question**

*What can we say about k-heavy simplices of A?*

# Let's play a game

**Question**

*For $d = 2$ and $k = 1$, how many 1-heavy triangles of $A$ contain a given* **generic** *point $x$ inside?*



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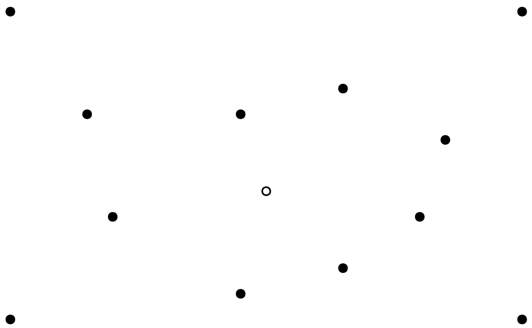

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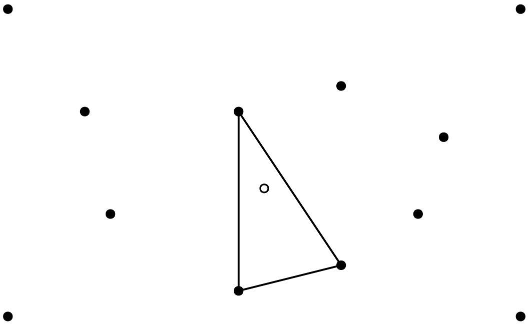

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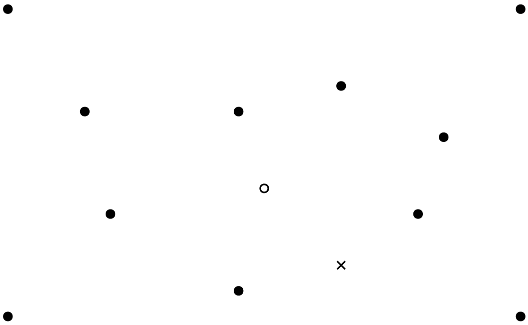

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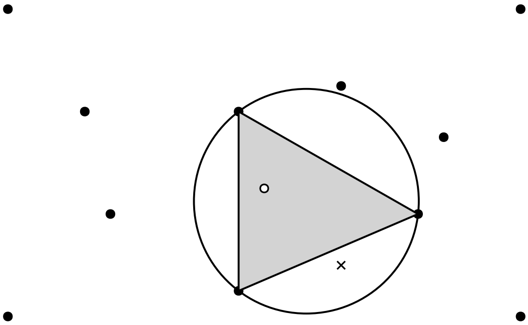

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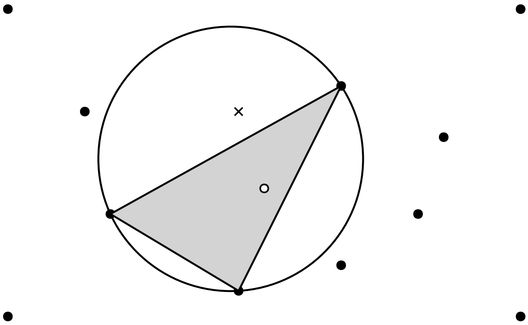

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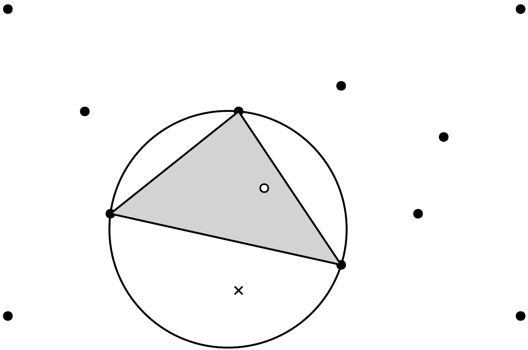

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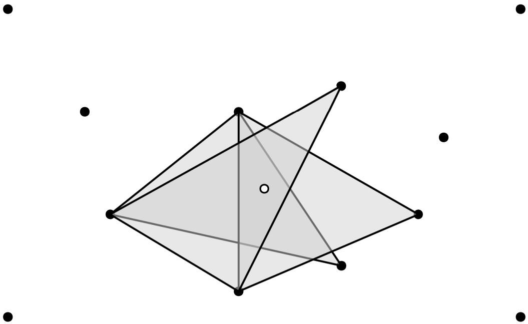

# The main theorem

**Theorem (Edelsbrunner, G., Saghafian, 2024+)**

*Let $A$ be a thin Delone set in $\mathbb{R}^d$ and let $x$ be generic point in $\mathbb{R}^d$. Then $x$ belongs to* **exactly** $\binom{d+k}{d}$ *$k$-heavy simplices of $A$.*

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**Proof.**

- For every $A$, there is a “covering” constant.
- The constant does not depend on $A$.
- There is a set where the constant is obvious.

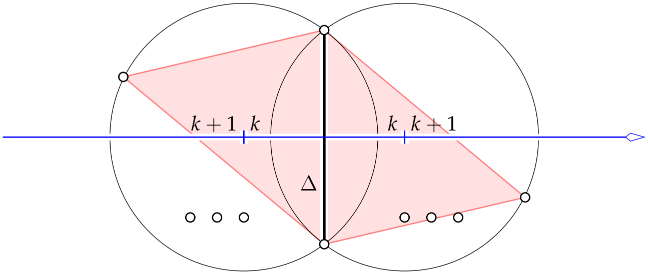
□

# Step 1: there is a constant

**Lemma**

*For a given thin Delone set $A$ and $x$, there is a constant $c(A)$ such that the number of $k$-simplices of $A$ that contain $x$ is $c(A)$.*

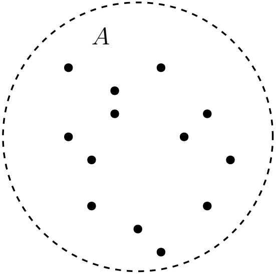
Let $\Delta$ be a simplex spanned by $d$ points of $A$. We show that there are equal numbers of $k$-heavy simplices sharing $\Delta$ on both sides.



# Step 2: the constant is the same for all sets

**Lemma**

*For two thin Delone sets $A$ and $A'$ in $\mathbb{R}^d$, $c(A) = c(A')$.*



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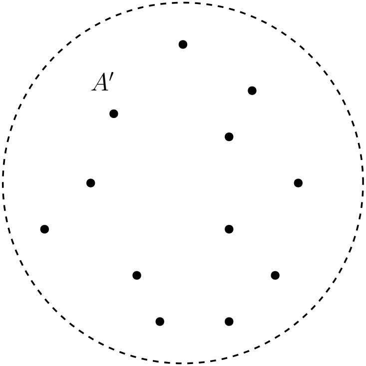

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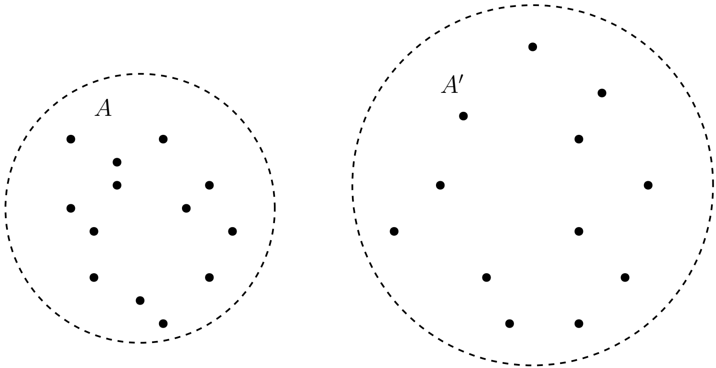
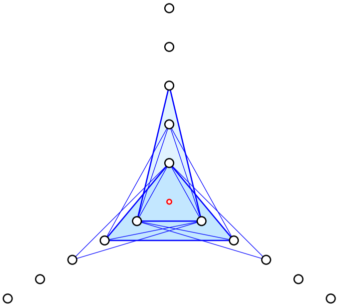


# STEP 3: THERE IS A SET WHERE THE CONSTANT IS OBVIOUS

**Lemma**

*For the* **radial** *thin Delone set $A$, $c(A) = \binom{d+k}{d}$.*



$\binom{d+k}{d}$ is number of ways to put $k$ points into $d+1$ boxes.

# Finite sets

**Theorem (Edelsbrunner, G., Saghafian, 2024+)**

*For a finite set $A$ in $\mathbb{R}^d$, generic point set $x \in \mathbb{R}^d$ belongs to* **at most** $\binom{d+k}{d}$ *k-heavy simplices of $A$.*

**Theorem (Edelsbrunner, G., Saghafian, 2024+)**

*If every hyperplane through $x$ has at least $k+1$ points of $A$ on both sides, then $x$ belongs to* **exactly** $\binom{d+k}{d}$ *k-heavy simplices of $A$.*

# Local version

**Theorem (Edelsbrunner, G., Saghafian, 2024+)**

*Let $A$ be a thin Delone set in $\mathbb{R}^d$ and let $a$ be any point of $A$. Then the $k$-heavy simplices of $A$* **incident** *to $a$ cover a small neighborhood of $A$ exactly $\binom{d+k-1}{d-1}$ times.*

# Hypersimplices and Eulerian numbers

- The **order-$k$ $d$-dimensional hypersimplex** $\Delta_d^{(k)}$ is defined as

$$\Delta_d^{(k)} := \operatorname{conv}\left(\sum_{i\in I} e_i \mid I \subset \{1,\ldots,d+1\}, |I| = k\right) \subset \mathbb{R}^{d+1}.$$

  Let $v(d,k) := d! \cdot \operatorname{vol} \Delta_d^{(k)}$.

- Eulerian number $A(d,k)$ is the number of permutations of $\{1,\ldots,d\}$ with $k$ descents.

**Theorem (de Laplace, 1886)**

$$v(d,k) = A(d,k-1).$$

- The covering constant allows us to give a new proof.

# Lifting construction again

- Let $A$ be a ~~thin Delone set~~ **perturbation of $\mathbb{Z}^d$**. We lift every point $a \in A$ to the paraboloid $a \mapsto (a, ||a||^2) \in \mathbb{R}^{d+1}$ to get a lifted set $A'$.

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- Let $n$ be a positive integer. For every subset of $n$ points of $A'$, take the average point.

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- Let $n$ be a positive integer. For every subset of $n$ points of $A'$, take the average point.
- For the set of averages, take the convex hull and project the boundary back onto $\mathbb{R}^d$.

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- Let $n$ be a positive integer. For every subset of $n$ points of $A'$, take the average point.
- For the set of averages, take the convex hull and project the boundary back onto $\mathbb{R}^d$.
- The resulted tiling (which is called the order-$n$ Delaunay triangulation of $A$) uses hypersimplices of orders $1, 2, \ldots, d$ originating from $(n-1)$-, $(n-2)$-, …, $(n-d)$-heavy simplices of $A$, respectively.

# Worpitzky identity

From a "volume argument" for all tiles in a large ball,

$$\sum_{p=1}^{d} v(d,p)\binom{d+n-p}{d} = n^d.$$

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Which is a rewritten Worpitzky identity for Eulerian numbers

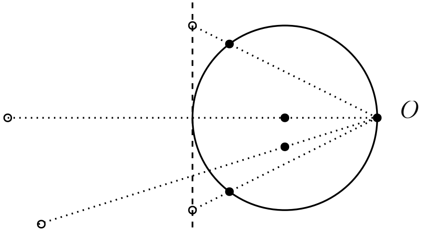
$$\sum_{k=0}^{d-1} A(d,k)\binom{x+k}{d} = x^d.$$

# $k$-facets and spheres

**Definition**

Let $A$ be a finite generic point set in $\mathbb{R}^d$. A subset $B \subset A$ of size $d$ is called a ***k*-facet** of $A$ if the hyperplane through $B$ separate $k$ points of $A$ from the rest.

**Observation**. After an inversion with respect to sphere with center $O$, $k$-facet turns into a $k$-heavy simplex of the set $A \cup \{O\}$ provided $O$ is on the right side of the hyperplane.



# An application to the planar $k$-facets

**Theorem (Alon, Györi, 1986)**

*For every generic point set $A$ in $\mathbb{R}^2$ with $n$ points, the total number of 0-, 1-, ..., $k$-facets is at most $(k+1)n$.*

**New proof for $k \leq \frac{n}{3}$.**

- Pick a point $O$ that is on a right side of all $i$-facets for $i \leq k$.
- Perform inversion with respect to a circle centered at $O$.
- Count how many $i$-heavy simplices for $i = 0, 1, \ldots, k$ could be so they cover a neighborhood of $A$ at most $\frac{(k+1)(k+2)}{2}$ times.

□

# Lovász lemma

## Theorem (Bárány, Füredi, Lovász, 1990)

*Let $A \subset \mathbb{R}^d$ be a generic set with $2n + d$ points. Let $\ell$ be a line generic with respect to $A$. Then $\ell$ intersects interiors of at most $O(n^{d-1})$* **halving** *facets of $A$.*

## Theorem ("Exact Lovász lemma" by Welzl, 2001)

*In a similar setting, the number of* **directed** *$k$-facets intersected by a line is at most $\binom{d+k-1}{d-1}$.*

## New proof.

Pick a point $O$ far on $\ell$. After inversion, $k$-facets intersected by $\ell$ turn into $k$-heavy simplices incident to $O$ and intersected by $\ell$, and the local covering gives us the bound. □

# $\mathbb{H}^d$ and $\mathbb{S}^d$

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**Definition**

A set $A \subset \mathbb{S}^d$ is called ***k*-balanced** if every open hemisphere contains at least $k+1$ points of $A$.

**Theorem** (Edelsbrunner, G., Saghafian, 2024+)

*Let $A$ be a k-balanced finite generic point set in $\mathbb{S}^d$. Then*

1. *Every generic point of $\mathbb{S}^d$ is covered by $\binom{d+k}{d}$ k-heavy simplices of $A$;*
2. *For every $a \in A$, a small neighborhood of $a$ is covered by k-heavy simplices of $A$ incident to $a$ in $\binom{d+k-1}{d-1}$ layers.*

# Weighted point sets

- We assign weights to points of a thin Delone set $A$;
- One can think of weighted points as spheres with radii defined by weights;
- Circumspheres are spheres orthogonal to weighted points.

**Theorem (Edelsbrunner, G., Saghafian, 2024+)**

*Let $(A, w)$ be a weighted generic thin Delone set in $\mathbb{R}^d$ with a bounded weight function $w$. Then every generic point of $\mathbb{R}^d$ is covered by $\binom{d+k}{d}$ $k$-heavy simplices of $(A, w)$.*

# THANK YOU!