Counting tiles in substitution tilings

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Why counting matters?

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Given a tiling (or a point set), how many tiles (or points) are there in a large part of the space?

- We count the tiles in a ball of radius R and are interested in the asymptotics of the counting function;
- The leading term of the function may give us information about density of tiles or points;
- The second term says how "well" the tiles or points are distributed.

Examples: Gauss circle problem, Mass transport, Bounded remainder sets.

What will we count?

- $\mathcal{A}$ is an alphabet
- ρ is a substitution on $\mathcal{A}$, so for every $x \in \mathcal{A}$, $\rho(x)$ is a non-empty word with letters from $\mathcal{A}$;
- We fix a letter $a \in \mathcal{A}$ and study the function

$$L(n) = \#(\rho^n(a))$$

that counts the number of letters in $\rho^n(a)$;
- In the "nice" situations

$$L(n) = \text{"Main part"} \quad \pm \quad \text{"Error"}$$

- **Goal:** quantify the error

Thue-Morse sequence

$$\rho = \begin{cases} a & \mapsto & ab \\ b & \mapsto & ba \end{cases}$$

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Thue-Morse sequence

$$\rho = \begin{cases} a & \mapsto & ab \\ b & \mapsto & ba \end{cases}$$

Then

- $L(n) = \#(\rho^n(a)) = 2^n$
- This is the “main part” and there is no error

Fibonacci sequence

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Then

- $L(n)$ is the $(n+1)$st Fibonacci number in the sequence $1, 1, 2, 3, 5, 8, \ldots$, and
- $$L(n) \approx \frac{1}{\sqrt{5}}\varphi^{n+1} \qquad \text{where} \qquad \varphi = \frac{1+\sqrt{5}}{2}$$
- and the error is
$$\frac{1}{\sqrt{5}}(1-\varphi)^{n+1} = \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

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- Let $M = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}$ be the **substitution matrix** that counts the letters in $\rho(a)$ and $\rho(b)$, then
-

$$L(n) = \begin{pmatrix} 1 & 1 \end{pmatrix} M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and this can be expressed through the eigenvalues of M.

Crazy substitution

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- The substitution matrix

$$M = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 4 & 3 \\ 3 & 6 & 10 \end{pmatrix}$$

has eigenvalues $\lambda = 13, 1, 1$ with a non-trivial Jordan block.

Finite alphabets and the Perron-Frobenius theory

- Let ρ be a **primitive** substitution on a finite alphabet $\mathcal{A}$;
- Let M be the corresponding substitution matrix
  - primitivity means that for some N, M^N has only positive entries;
- Let λ be the largest eigenvalue of M, and λ' be the second in absolute value;
- Then

$$L(n) = \mathbf{1} \cdot M^n \cdot \mathbf{e}_1 = \quad c_1 \cdot \lambda^n \quad + \quad O(|\lambda'|^n \cdot \text{“polynomial”})$$

- The constant c_1 can be found from the point set density of the geometric version of the substitution

Infinite alphabet setting

We approach similar questions in the case of infinite alphabets described by Dan and Dirk in the previous two talks.

We fix an appropriate sequence $\mathbf{a} = a_0, a_1, a_2, \ldots$

$$\rho_{\mathbf{a}} = \begin{cases} [0] & \mapsto & [0]^{a_0}[1] \\ [i] & \mapsto & [0]^{a_i}[i-1][i+1] \text{ for } i \geq 1 \end{cases}$$

The associated infinite "substitution matrix" is

$$\mathbf{A} = \begin{pmatrix} a_0 & a_1+1 & a_2 & a_3 & a_4 & \ldots \\ 1 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 1 & \ldots \\ 0 & 0 & 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Density estimates

Since the frequencies and "lengths" of all letters are known,

$$L(n) = \#(\rho_{\mathbf{a}}^{n}([0])) = c_1 \cdot \lambda^n + o(\lambda^n)$$

where

- $\lambda = \mu + \dfrac{1}{\mu}$ is the inflation factor with μ defined by

$$\frac{1}{\mu} = \sum_{i=0}^{\infty} a_i \mu^i$$

- c_1 is the density of the associated geometric substitution, or the reciprocal of the average tile length assuming the length of $[0]$ is 1.

Goal: get better estimates for the error, a.k.a. the **discrepancy** function $d_{\mathbf{a}}(n)$

The "simplest" case $\mathbf{a} = 1, 1, 1, 1, 1, \ldots$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & \ldots \\ 1 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 1 & \ldots \\ 0 & 0 & 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\frac{1}{\mu} = \sum_{i=0}^{\infty} \mu^i = \frac{1}{1-\mu}, \qquad \text{so} \qquad \mu = \frac{1}{2} \qquad \text{and} \qquad \lambda = \frac{5}{2}$$

- lengths are $2 - \frac{1}{2^i}$ and frequencies are $\frac{1}{2^{i+1}}$
- $$L(n) = \frac{3}{4} \cdot \left(\frac{5}{2}\right)^n + d_{\mathbf{a}}(n)$$

Finding discrepancy, part I

Idea: pretend that the same linear algebra works

- We start from writing the couting function in a vector form

$$L(n) = (1,1,1,\ldots)\mathbf{A}^n(1,0,0,\ldots)^t = \frac{3}{4}\cdot\left(\frac{5}{2}\right)^n + d_{\mathbf{a}}(n)$$

  In other words, we are interested in $[(1,1,1,\ldots)\mathbf{A}^n]_0$, the 0th term of that vector.

- Then eliminate the leading term

$$\begin{aligned} 2d_{\mathbf{a}}(n+1) - 5d_{\mathbf{a}}(n) &= \\ &= \left[(1,1,1,\ldots)\left(2\mathbf{A} - 5\mathbf{I}\right)\mathbf{A}^n\right]_0 = \\ &= [(-1,1,1,\ldots)\mathbf{A}^n]_0 \end{aligned}$$

Finding discrepancy, part II

- Then we choose a nicer basis in an ivariant subspace of $\mathbf{A}$

$$\begin{aligned} \mathbf{e}_0 &= (-1, 1, 1, 1, \dots) \\ \mathbf{e}_1 &= (1, -2, 0, 0, \dots) \\ \mathbf{e}_2 &= (0, 1, -2, 0, \dots) \\ \mathbf{e}_3 &= (0, 0, 1, -2, \dots) \end{aligned}$$

  and so on.
- In this basis, the right multiplication by $\mathbf{A}$ has the matrix

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Finding discrepancy, part III

- In this setting

$$\mathbf{B}^n(1,0,0,0,\ldots)^t = \text{“sorted binomial coefficients”}$$

- and therefore

$$2d_{\mathbf{a}}(n+1) - 5d_{\mathbf{a}}(n) = \text{ difference between two largest}$$

- or alternatively

$$2d_{\mathbf{a}}(n+1) - 5d_{\mathbf{a}}(n) = \begin{cases} -C_k & \text{if } n = 2k \\ 0 & \text{if } n = 2k+1 \end{cases}$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the **kth Catalan number**

Finding discrepancy, part IV

- Using the initial terms and the **Catalan series**

$$\frac{1-\sqrt{1-4x}}{2x} = \sum_{i=0}^{\infty} C_i x^i$$

we get an expression for $d_{\mathbf{a}}(n)$ as the (scaled) remainder of the series at $x = 4/25$.

- Namely,

$$d_{\mathbf{a}}(2k+1) = \left(\frac{25}{4}\right)^k \sum_{i=k+1}^{\infty} C_i \left(\frac{4}{25}\right)^i \quad \text{and} \quad d_{\mathbf{a}}(2k) = \frac{5}{2} d_{\mathbf{a}}(2k-1)$$

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Theorem (Frettlöh, G., Mañibo, 2022+)

$$d_{\mathbf{a}}(n) = \Theta(C_{2k}) = \Theta\left(\frac{2^n}{n^{3/2}}\right)$$

What about other sequences **a**?

If **a** stabilizes on some positive number, then

- μ and λ are algebraic, and
- it is possible to employ a similar strategy (even getting the same matrix **B** in a new basis) and get that for some coefficents

$$\alpha_0 d_{\mathbf{a}}(n) + \alpha_1 d_{\mathbf{a}}(n+1) + \ldots + \alpha_p d_{\mathbf{a}}(n+p) = \\ = \begin{cases} \beta_0 C_k + \ldots + \beta_q C_{k+q} & \text{if } n = 2k, \\ \gamma_0 C_k + \ldots + \gamma_q C_{k+q} & \text{if } n = 2k+1. \end{cases}$$

Discrepancy for stabilizing $\mathbf{a}$

Theorem (Frettlöh, G., Mañibo, 2022+)

There is a non-negative integer t such that a subsequence of $d_{\mathbf{a}}(n)$ grows at least as fast as $\Omega\left(\frac{2^n}{n^{t+3/2}}\right)$.

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How is this connected to the general theory?

Theorem (Mañibo, Rust, Walton)

Under some assumptions on the substitution on a compact alphabet, the discrepancy does not exceed

$$\theta(n) \cdot |\lambda'|^n$$

where θ is a function with $\lim \sqrt[n]{\theta(n)} = 1$ and λ' is the second largest element in the spectrum of $\mathbf{A}$.

Final remarks

- Same "infinite-dimensional linear algebra" approach can be used to count not only letters in supertiles of [0] but "things" in any supertiles;
- We expect that similar growth rates appear there;
- As Dirk said, we expect that $|\lambda'| = 2$ for all appropriate sequences $\mathbf{a}$ and in this case our lower bound "coincides" with the upper bound from the theorem of Mañibo, Rust, and Walton.

THANK YOU!