Concrete polytopes may not tile the space

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Basic notions

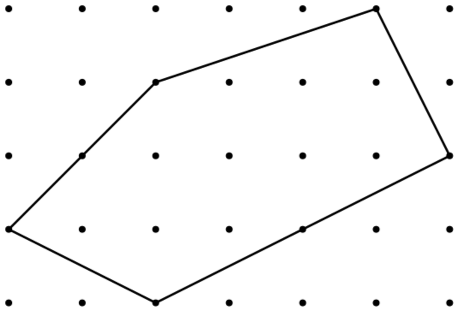
- Integer point: any point in $\mathbb{R}^d$ with integer coordinates
- $\mathbb{Z}^d$: the d-dimensional integer lattice that consists of all integer points in $\mathbb{R}^d$
- Lattice polygon or polytope: any convex polygon or polytope with vertices in $\mathbb{Z}^d$

Pick's formula

Theorem (Pick, 1899)

For every lattice polygon P,

$$\text{area}(P) = I + \frac{B}{2} - 1.$$



$I = 10 \qquad B = 7 \qquad \text{area} = 12.5$

Discrete area

At every lattice point x we place a small disk.

Definition

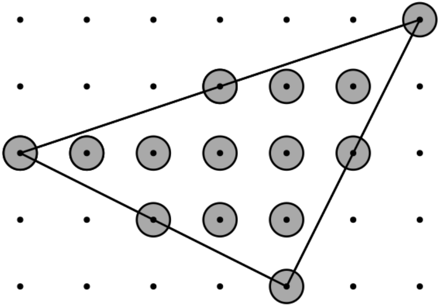
Let $w_P(x)$ be the solid angle of P at x which is the portion of the disk at x in P.

Definition

We define the **discrete area** of P as

$$\chi(P) = \sum_{x \in \mathbb{Z}^2} w_P(x).$$

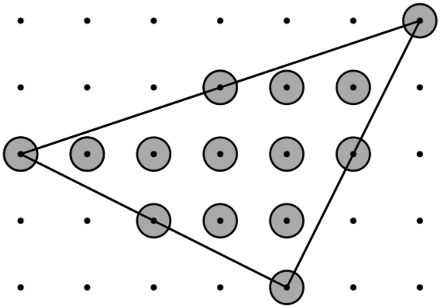
Discrete area, part 2



$$I + \frac{B}{2} - 1 = I + \frac{\pi(B-2)}{2\pi} = \chi(P)$$

- Pick's theorem claims that the discrete area is equal to the usual area.

Proof of Pick's theorem (for triangles)



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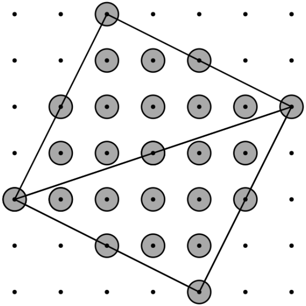
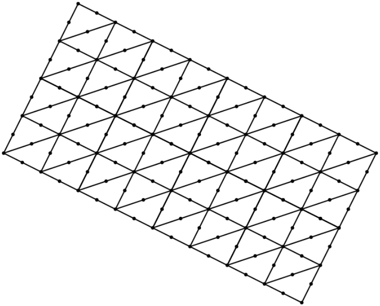


Proof of Pick's theorem (for triangles)



Both the covered area and the total number of covered circles grow as $C \cdot R^2$ but the error accumulates only on the boundary so the difference is at most linear.

Discrete volume

Definition

For every integer polytope P in $\mathbb{R}^d$ we define

- the **solid angle** at x as

$$w_P(x) := \frac{\mathrm{vol}(B_\varepsilon(x) \cap P)}{\mathrm{vol}(B_\varepsilon(x))}$$

  where $B_\varepsilon(x)$ is the ball of small radius ε centered at x, and

- **discrete volume** of P as the sum of solid angles at all integer points:

$$\chi(P) := \sum_{x \in \mathbb{Z}^d} w_P(x).$$

Concrete polytopes

Definition

We call polytope P **concrete** if

$$\chi(P) = \mathrm{vol}(P).$$

- Every polygon is concrete but not every polytope is concrete
- The family of Reeve's tetrahedra

$$R_n = \mathrm{conv}\{ \quad (0,0,0), \quad (1,0,0), \quad (0,1,0), \quad (0,0,n) \quad \}$$

contains tetrahedra with bounded discrete volume but arbitrarily large volume.

The conjecture

- Does being concrete mean anything?

The conjecture

- Does being concrete mean anything?

Conjecture (Brandolini, Colzani, Robins, Travaglini, 2020)

If $P \subset \mathbb{R}^d$ is a concrete polytope, then P **multitiles** *$\mathbb{R}^d$ using translations and finitely many reflections.*

- P **multitiles** $\mathbb{R}^d$ if there is an integer $k \geq 1$ and an infnite family of congruent copies of P such that every generic point belongs to exactly k copies.

Supporting data:

- All two-dimensional lattice polygons;
- All lattice zonotopes.

Main result

Theorem (G., Pak, 2020)

There exists a concrete polytope $P \subset \mathbb{R}^3$ which does not multitile the space.

- Moreover, for all N we can get such a concrete polytope P with more than N vertices/faces/edges.

Tools: volume defect

Definition

For lattice polytope P we define the **volume defect** as

$$\delta(P) = \chi(P) - \mathrm{vol}(P).$$

- The volume defect is additive for disjoint pieces

Tools: Dehn invariant

Definition

For polytope P we define the **Dehn invariant** as

$$\mathbb{D}(P) := \sum_{e \in E(P)} \text{length}(e) \otimes \text{angle}(e) \in \mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{R}/\pi\mathbb{Z})$$

where $E(P)$ is the set of edges of P.

Definition

Alternatively, for every **Kagan function** $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(a+b) = f(a) + f(b) \text{ and } f(\pi) = 0$$

we define

$$D_f(P) := \sum_{e \in E(P)} \text{length}(e) \cdot f(\text{angle}(e)) \in \mathbb{R}.$$

Dehn invariant and scissor congruence

- Originally, the Dehn invariant was used to answer Hilbert's third problem on scissor congruence in $\mathbb{R}^3$.
- It is also related to multitilings.

Proposition

- *If P multitiles $\mathbb{R}^3$, then $\mathbb{D}(P) = 0$;*
- *If P multitiles $\mathbb{R}^3$ by* **translations**, *then $\delta(P) = 0$.*

Idea of the proof, orginally by Debrunner (1980) and Mürner (1975).

If $\mathbb{D}(P) \neq 0$, then in a large ball of radius R, the total value of the Dehn invariant is $\Theta(R^3)\mathbb{D}(P)$. On the other hand, only the thin part next to the boundary contributes, so it must be $O(R^2)$. □

Idea of the counterexample

Task

Construct a lattice polytope P such that

$$\delta(P) = 0 \qquad \textit{and} \qquad \mathbb{D}(P) \neq 0.$$

Valuations

Definition

Function φ defined on some family of convex bodies is called a **valuation** if for all relevant P, Q,

$$\varphi(P) + \varphi(Q) = \varphi(P \cup Q) + \varphi(P \cap Q)$$

provided $\varphi(\emptyset) = 0$.

- Volume
- Discrete volume
- The number of integer points
- Dehn invariant
- D_f for every Kagan function f

McMullen's theory of lattice valuations

Definition

Minkowski sum of two polytopes P and Q is

$$P + Q := \{a + b \mid a \in P, b \in Q\}.$$

Theorem (McMullen, 1974)

Let φ be a valuation on lattice polytopes such that $\varphi(P+t) = \varphi(P)$ for every $t \in \mathbb{Z}^d$. Then for all polytopes P_i and non-negative integers t_i,

$$\varphi(t_1P_1 + \ldots + t_kP_k)$$

is a polynomial of degree at most d in t_i's.

- Mixed volumes, Ehrhart theory, etc.

Dehn invariant in $\mathbb{R}^3$ as valuation

Lemma

For every Kagan function f,

$$D_f(t_1P_1 + \ldots + t_kP_k) = t_1D_f(P_1) + \ldots + t_kD_f(P_k).$$

Proof outline.

- $D_f(tP) = tD_f(P)$;
- Since $D_f(\cdot)$ is a valuation,

  $$D_f(t_1P_1 + \ldots + t_kP_k)$$

  is a polynomial of degree at most 3;
- The restriction of that polynomial on every ray from the origin is linear, so the polynomial must be linear as well.

Volume defect in $\mathbb{R}^3$ as valuation

Lemma

$$\delta(t_1P_1 + \ldots + t_kP_k) = t_1\delta(P_1) + \ldots + t_k\delta(P_k).$$

Proof outline.

- Both $\mathrm{vol}(\cdot)$ and $\chi(\cdot)$ are lattice valuations so both $\mathrm{vol}(tP)$ and $\chi(tP)$ are cubic polynomials;
- Moreover, $\mathrm{vol}(tP)$ and $\chi(tP)$ are **odd** cubic polynomials (Macdonald, 1971) with the same leading coefficient;
- Thus,

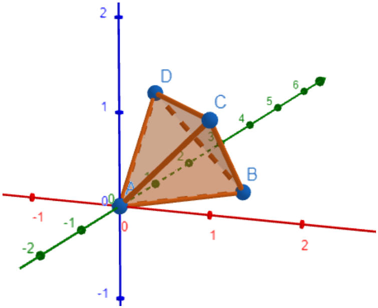
$$\delta(tP) = \chi(tP) - \mathrm{vol}(tP)$$

is linear and the claim follows.

□

Three tetrahedra: regular

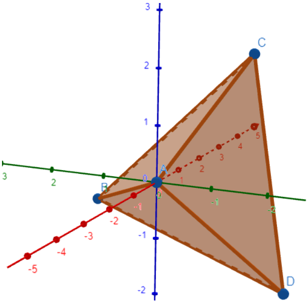
$$T_1 = \text{conv}\{ \quad (0,0,0), \quad (1,1,0), \quad (1,0,1), \quad (0,1,1) \quad \}$$



$\delta(T_1) = \frac{3\alpha}{\pi} - \frac{4}{3}$ and $D_f(T_1) = 6\sqrt{2}f(\alpha)$ where $\alpha = \arccos \frac{1}{3}$.

Three tetrahedra: standard

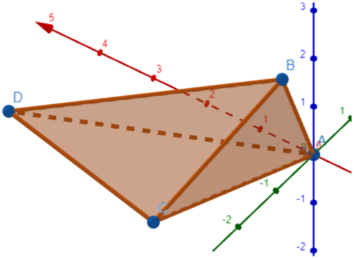
$$T_2 = \text{conv}\{ \quad (0,0,0), \quad (2,2,-1), \quad (2,-1,2), \quad (1,-2,-2) \quad \}$$



$\delta(T_2) = -\frac{5\alpha}{4\pi} - \frac{1}{2}$ and $D_f(T_2) = -\frac{9}{\sqrt{2}} f(\alpha)$ where $\alpha = \arccos\frac{1}{3}$.

Three tetrahedra: orthoscheme

$$T_3 = \text{conv}\{\quad (0,0,0), \quad (2,2,-1), \quad (3,0,-3), \quad (5,-1,-1) \quad \}$$



$$\delta(T_3) = \frac{2}{3} \text{ and } D_f(T_3) = 0.$$

Proof of the theorem

Theorem (G., Pak, 2020)

There exists a concrete polytope $P \subset \mathbb{R}^3$ which does not multitile the space.

- Moreover, for all N we can get such a concrete polytope P with more than N vertices/faces/edges.

Proof.

- Let
$$P := 5T_1 + 12T_2 + 19T_3.$$
Then $\delta(P) = 0$ and $D_f(P) \neq 0$ as long as $f(\alpha) \neq 0$.
- Let Q be a lattice zonotope with many vertices/edges/faces, then $P + Q$ works.

□

Higher dimensions and Hadwiger invariants

- **Hadwiger invariants** are generalizations of the Dehn invariant for higher dimensions;
- The orthogonal prism $P \times [0,1]$ has non-zero codimension 2 Hadwiger invariant and this can be generalized further;

Theorem (Sydler, Jessen)

For $d = 3, 4$, if all Hadwiger invariants of P are zeros, then P is scissor congruent with a d-cube.

- A similar conjecture is still open in $\mathbb{R}^d$ for $d \geq 5$.

Super conjecture

Definition

We call lattice polytope $P \subset \mathbb{R}^d$ **super concrete** if it is

- concrete, and
- scissor congruent with a d-cube.

Conjecture

For every $d \geq 3$, there exists a super concrete polytope $P \subset \mathbb{R}^d$ that cannot multitile the space.

THANK YOU!