The Voronoi conjecture

Convex polytopes that tile space with translations

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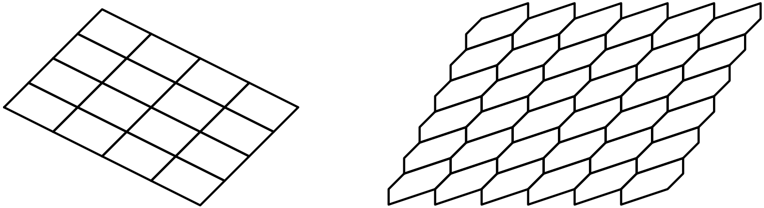
Parallelohedra

Definition

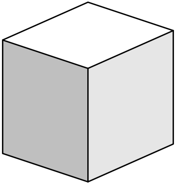
Convex d-dimensional polytope P is called **parallelohedron** if $\mathbb{R}^d$ can be (face-to-face) tiled into parallel copies of P.

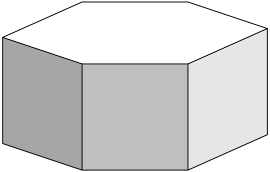


Two types of two-dimensional parallelohedra

Three-dimensional parallelohedra

In 1885 Russian crystallographer Fedorov listed all types of three-dimensional parallelohedra.





Parallelepiped and hexagonal prism with centrally symmetric base.

Three-dimensional parallelohedra

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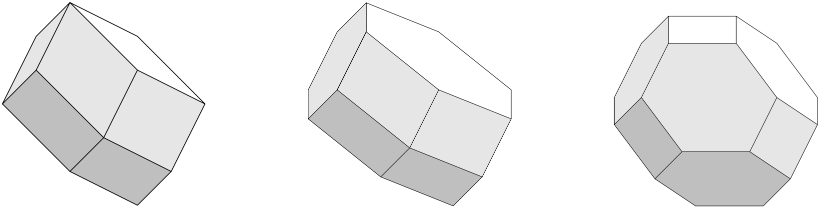
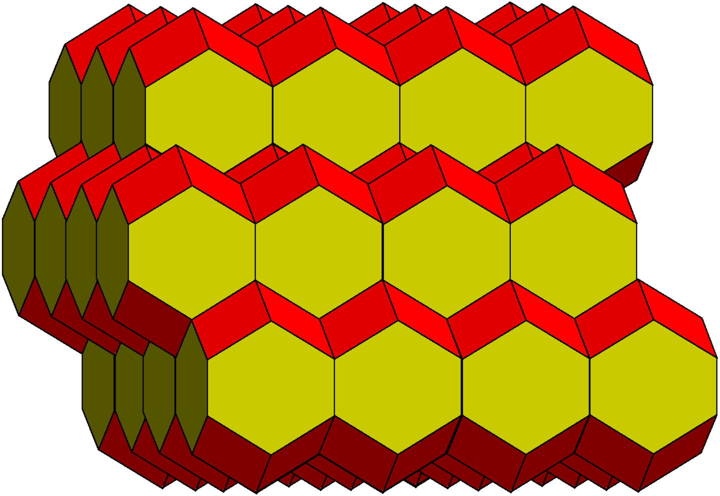


Rhombic dodecahedron, elongated dodecahedron, and truncated octahedron

Tiling by elongated dodecahedra (from Wikipedia)



Minkowski-Venkov conditions

Theorem (Minkowski, 1897; Venkov, 1954; and McMullen, 1980)

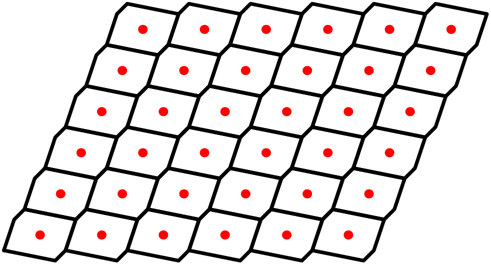
P is a d-dimensional parallelohedron iff it satisfies the following conditions:

1. *P is centrally symmetric;*
2. *Any facet of P is centrally symmetric;*
3. *Projection of P along any its $(d-2)$-dimensional face is parallelogram or centrally symmetric hexagon.*

- If P tiles $\mathbb{R}^d$ in any way (face-to-face or non face-to-face), then P satisfies the Minkowski-Venkov conditions;
- If P satisfies these conditions, then there is a face-to-face tiling of $\mathbb{R}^d$ with copies of P.

Parallelohedra to Lattices

- Let P be a parallelohedron, i.e. a centrally symmetric convex polytope that satisfies the Minkowski-Venkov conditions;
- Let $\mathcal{T}(P)$ be the unique face-to-face tiling of $\mathbb{R}^d$ into parallel copies of P. Then the centers of the tiles form a lattice, i.e. the set of integer linear combinations of a basis.

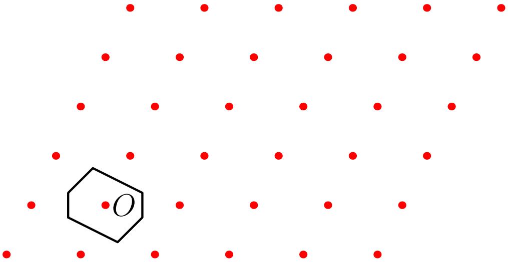


Lattices to Paralleohedra

- Let Λ be an arbitrary d-dimensional lattice and let O be a point of Λ.

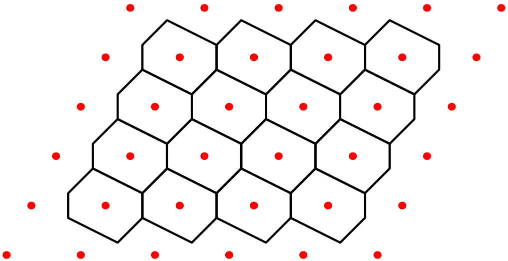
Lattices to Paralleohedra

- Let Λ be an arbitrary d-dimensional lattice and let O be a point of Λ.
- We construct the polytope consisting of points that are closer to O than to any other point of Λ (the Dirichlet-Voronoi polytope of Λ).



Lattices to Paralleohedra

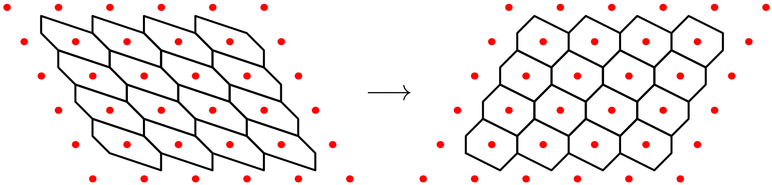
- Let Λ be an arbitrary d-dimensional lattice and let O be a point of Λ.
- We construct the polytope consisting of points that are closer to O than to any other point of Λ (the Dirichlet-Voronoi polytope of Λ).
- Then DV_Λ is a parallelohedron and the points of Λ are centers of the corresponding tiles.



The Voronoi conjecture

Conjecture (Voronoi, 1909)

For every parallelohedron P there exists a lattice Λ such that P is affinely equivalent to the Dirichlet-Voronoi polytope of Λ.



Voronoi conjecture in $\mathbb{R}^2$

- Each parallelogram can be transformed into some rectangle and all rectangles are Voronoi polygons.
- Each centrally-symmetric hexagon can be transformed into some hexagon inscribed in a circle. This transformation is **unique** modulo isometry and/or homothety. All centrally-symmetric hexagons inscribed in circles are Voronoi polygons.

The Voronoi conjecture: small dimensions

- $\mathbb{R}^2$: folklore.
- $\mathbb{R}^3$: kind of folklore. All three-dimensional parallelohedra are known due to Fedorov, and then one can **check** that they satisfy the Voronoi conjecture.

Theorem (Delone, 1929)

The Voronoi conjecture is true in $\mathbb{R}^4$.

Classification: there are 52 four-dimensional parallelohedra; Delone, 1929 and Stogrin, 1974.

Theorem (G., Magazinov, 2019+)

The Voronoi conjecture is true in $\mathbb{R}^5$.

Classification: there are 110244 five-dimensional (Voronoi) parallelohedra; Dutour Sikirić, G., Schürmann, and Waldmann, 2016.

Hilbert's 18th problem: lattices in $\mathbb{R}^d$

- Finiteness of the family of crystallographic groups in $\mathbb{R}^d$
- Existence of a polytope that tiles $\mathbb{R}^d$ but can't be obtained as a fundamental region of crystallographic group
- Densest (sphere) packings in $\mathbb{R}^3$ (Kepler conjecture)

Hilbert's 18th problem: lattices in $\mathbb{R}^d$

- Finiteness of the family of crystallographic groups in $\mathbb{R}^d$
  - Bieberbach, 1911-12;
- Existence of a polytope that tiles $\mathbb{R}^d$ but can't be obtained as a fundamental region of crystallographic group
  - Reinhardt, 1928 in $\mathbb{R}^3$ and Heesch, 1935 in $\mathbb{R}^2$;
- Densest (sphere) packings in $\mathbb{R}^3$ (Kepler conjecture)
  - Hales, 2005 and 2017.

Which (convex) polytopes may tile the space?

What if we do not restrict to translations only and allow all isometric copies?

- $\mathbb{R}^2$: If $n \geq 7$, then no convex n-gon can tile the plane;

  Rao (2017+): full classification of pentagons (15 types).

- $\mathbb{R}^3$: the maximal number of facets for **stereohedron** is unknown.

  Engel (1981): There exists a stereohedron with 38 facets;

  Santos et. al. (2001-2011): **Dirichlet stereohedron** (i.e. the Voronoi polytope of an orbit of crystallographic group) cannot have more than 92 facets.

Parallelohedra and lattice covering problem

Problem: for a given d, find the lattice that gives an **optimal** covering of $\mathbb{R}^d$ with balls of equal radii.

- $\mathbb{R}^2$: Kershner, 1939 and A_2^*;
- $\mathbb{R}^3$: Bambah, 1954 and A_3^* (the BCC lattice);
- $\mathbb{R}^4$: Delone and Ryshkov, 1963 and A_4^*;
- $\mathbb{R}^5$: Ryshkov and Baranovskii, 1976 and A_5^*;
- $\mathbb{R}^d$, $d = 6, 7, 8$: Schürmann and Vallentin, 2006 and lattices different from A_d^*. Best known lattices, no proof that they are actually the solutions.

The results in dimensions 4 through 8 rely on **reduction theory** for lattices, or (partial) classification of Voronoi parallelohedra.

SVP and CVP: using parallelohedra for lattice algorithms

- SVP (Shortest Vector Problem): find a shortest non-zero vector of a given lattice Λ;
- CVP (Closest Vector Problem): for a given target vector $\mathbf{t}$ and a lattice Λ, find the vector $\mathbf{x} \in \Lambda$ that minimizes $||\mathbf{t} - \mathbf{x}||$.

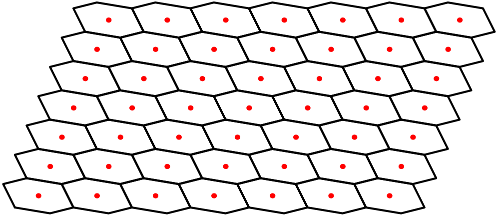
  - LLL-algorithm for lattice reduction and polynomial fatorization over $\mathbb{Q}$;
  - Solvability in radicals;
  - Cryptography;
  - Integer optimization.

The Voronoi conjecture

Conjecture (Voronoi, 1909)

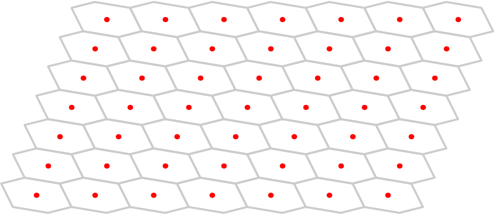
For every parallelohedron P there exists a lattice Λ such that P is affinely equivalent to the Dirichlet-Voronoi polytope of Λ.

Voronoi's generatrix



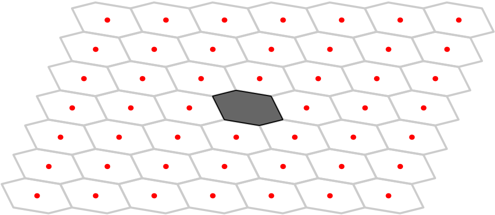
Voronoi's generatrix is one of the tools that can help checking the Voronoi conjecture for given P.

Voronoi's generatrix



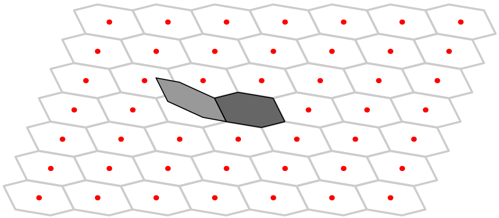
It is a **continuous** piecewise linear function $\mathcal{G} : \mathbb{R}^d \longrightarrow \mathbb{R}$ with constant gradient on each tile.

Voronoi's generatrix



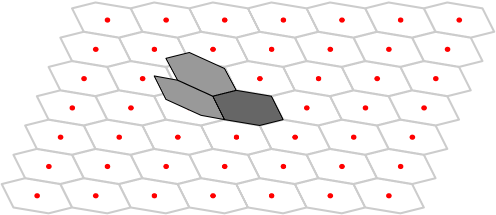
We put $\mathcal{G}$ equal to 0 on one of the tiles.

Voronoi's generatrix



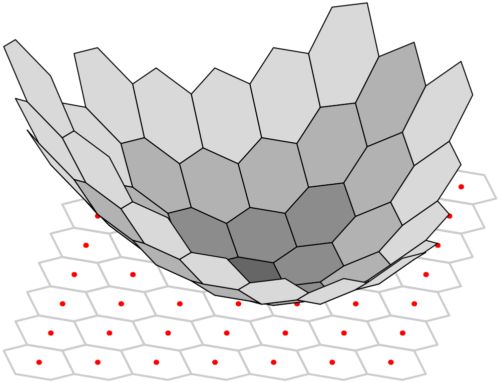
When we pass across one facet of the tiling, the gradient of $\mathcal{G}$ changes.

Voronoi's generatrix



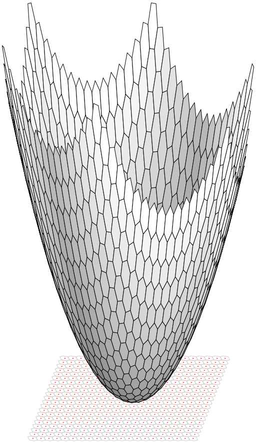
After that we are trying to “glue” adjacent shells. This can **always** be done “locally” assuming our polytope is not a direct product of two parallelohedra.

Voronoi's generatrix



If it can be done "globally", we obtain the graph of $\mathcal{G}$.

Voronoi's generatrix II



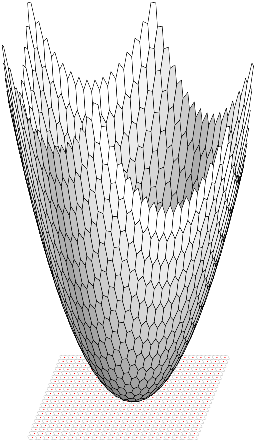
Properties of generatrix

- The graph of generatrix $\mathcal{G}$ looks like a "piecewise linear" paraboloid.
- And actually there is a paraboloid $y = \mathbf{x}^t Q \mathbf{x}$ for some positive definite quadratic form Q tangent to the generatrix at the centers of its shells.
- Moreover, if we consider an affine transformation $\mathcal{A}$ of this paraboloid into paraboloid $y = \mathbf{x}^t \mathbf{x}$ then the tiling by copies of P will transform into the Voronoi tiling for some lattice.

So to prove the Voronoi conjecture for P it is sufficient (and necessary) to show existence of a generatrix.

Constructing the Voronoi and Delone tilings

- **Lifting** construction for a point set X.
- Lift the points of X to paraboloid $y = \mathbf{x}^t\mathbf{x}$ in $\mathbb{R}^{d+1}$.
- Construct the tangent hyperplanes and take the intersection of the upper half-spaces; project this infinite polyhedron back to $\mathbb{R}^d$ to get the **Voronoi** tiling of X.
- Take the convex hull of points on $y = \mathbf{x}^t\mathbf{x}$ and project this (infinite) polyhedron back to $\mathbb{R}^d$ to get the **Delone** tiling of X.



Primitive parallelohedra

Definition

Let P be a d-dimensional parallelohedron. P is called **primitive**, if every vertex of the corresponding tiling belongs to exactly $d+1$ copies of P.

Primitive parallelohedra appear exactly as dual to Delone triangulations (not arbitrary Delone tilings).

Theorem (Voronoi, 1909)

The Voronoi conjecture is true for primitive parallelohedra.

Primitive parallelohedra II

Definition

Let P be a d-dimensional parallelohedron. P is called **k-primitive** if every k-face of the corresponding tiling belongs to exactly $d+1-k$ copies of P.

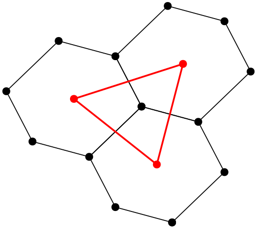
Theorem (Zhitomirskii, 1929)

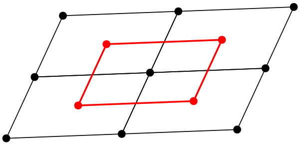
The Voronoi conjecture is true for $(d-2)$-primitive d-dimensional parallelohedra. Or the same, it is true for parallelohedra with all projections along $(d-2)$-faces being hexagons.

DUAL CELLS

Definition

The **dual cell** $\mathcal{D}(F)$ of a face F of given parallelohedral tiling is the set of all centers of parallelohedra that share F.
If F is $(d-k)$-dimensional then the corresponding cell is called ***k*-cell**.





Dual 3-cells and 4-dimensional parallelohedra

Lemma (Delone, 1929)

There are five types of three-dimensional dual cells.

- *tetrahedron,*
- *octahedron,*
- *quadrangular pyramid,*
- *triangular prism, and*
- *parallelepiped.*

Theorem (Ordine, 2005)

The Voronoi conjecture is true for parallelohedra without cubical or prismatic dual 3-cells.

Topological help

Question

Are there any **topological** *reasons that will prevent us to "glue" the graph of a generatrix locally?*

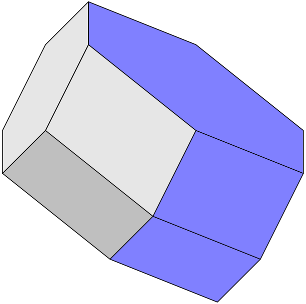
Definition

Let P_π, the **π-surface** of P, be the manifold obtained from the surface of P by removing non-primitive $(d-2)$-faces and identifying opposite points.

- We can track the gradient of $\mathcal{G}$ along every curve on P_π and the generatrix exists if and only if the values are **consistent** along every closed curve on P_π.

Half-belt cycles

- Any **half-belt cycle** which starts at the center of a facet and ends at the center of the opposite facet crossing only three parallel primitive $(d-2)$-faces gives consistent gradient values.



GGM condition

Theorem (G., Gavrilyuk, Magazinov, 2015)

If the group of one-dimensional homologies $H_1(P_\pi, \mathbb{Q})$ of the π-surface of parallelohedron P is generated by half-belt cycles then the Voronoi conjecture is true for P.

Which parallelohedra satisfy this condition?

- All 5 parallelohedra in $\mathbb{R}^3$.
- All 52 parallelohedra in $\mathbb{R}^4$.
- All 110244 **Voronoi** parallelohedra in $\mathbb{R}^5$ (Dutour-Sikirić, G., and Magazinov, 2020).
- All **Voronoi** parallelohedra for "small perturbations" of the dual root lattices D_n^*, E_6^*, E_7^*, and E_8^* (G., 2021+).

Zonotopes and free directions

Theorem (Erdahl, 1999)

The Voronoi conjecture is true for space-filling zonotopes.

- The space-filling zonotopes and corresponding lattices give rise to regular oriented matroids and vice versa.
- For $d \leq 3$, all parallelohedra are zonotopes, but for $d \geq 4$ the non-zonotopes outnumber the zonotopes.

Definition

Let I be a segment. If $P + I$ and P are both parallelohedra, then I is called a **free** direction for P.

- If I is a free direction for P, then the Voronoi conjecture holds (or doesn't hold) for P and for $P + I$ simultaneously (Grishukhin, 2004; Végh, 2015; Magazinov, 2015).

Proof of the Voronoi conjecture in $\mathbb{R}^5$

Theorem (G., Magazinov, 2019+)

The Voronoi conjecture is true in $\mathbb{R}^5$.

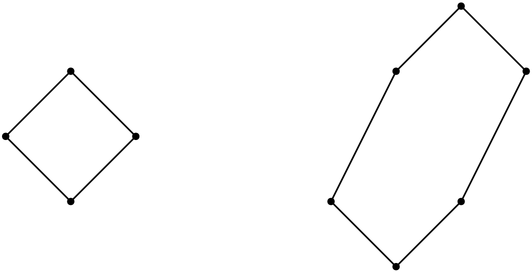
Let P be a five-dimensional parallelohedron.

- If P can be extended, then its extension has **combinatorics** of one of the 110244 Voronoi parallelohedra in $\mathbb{R}^5$;
- In the five-dimensional case, the **global combinatorics** of a Voronoi parallelohedron guarantees the geometric part of the Voronoi conjecture.
- **Local combinatorics** can be used to show that P can be extended.

COMBINATORICS OF EXTENDED PARALLELOHEDRA

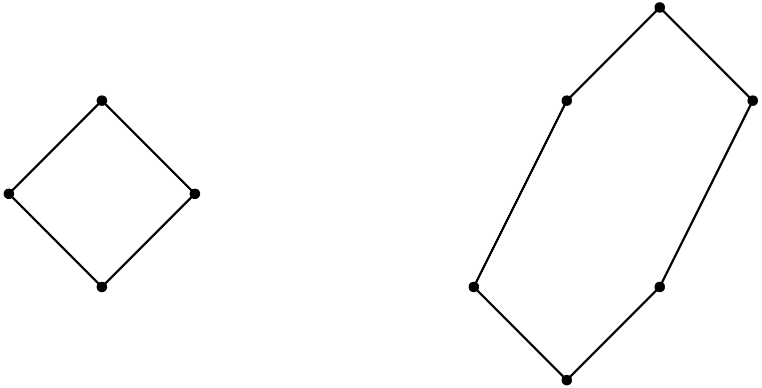
Theorem (G., Magazinov)

Let P be a d-dimensional parallohedron. If I is a free direction for P and the projection of P along I satisfies the Voronoi conjecture, then $P + I$ has the combinatorics of a Voronoi parallelohedron.

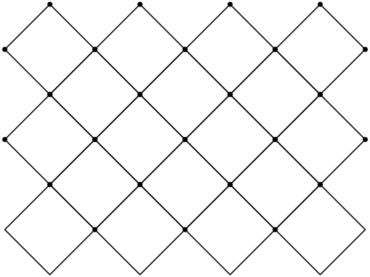


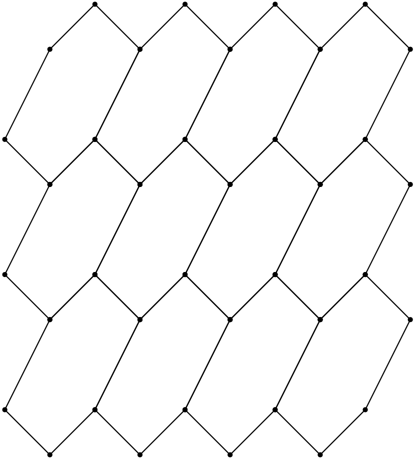
A parallelohedron and its extension

Combinatorics of extended parallelohedra

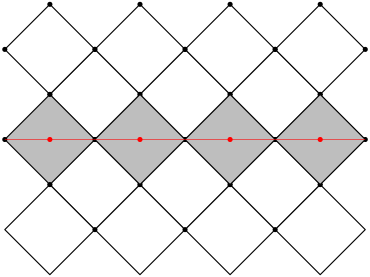


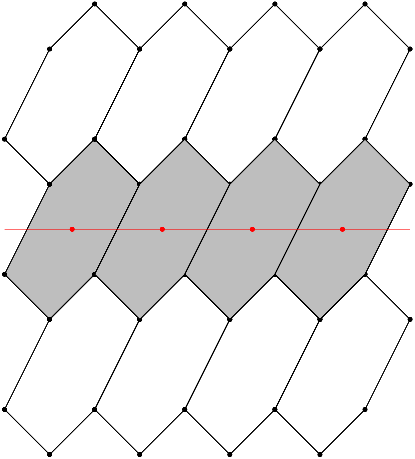
Combinatorics of extended parallelohedra



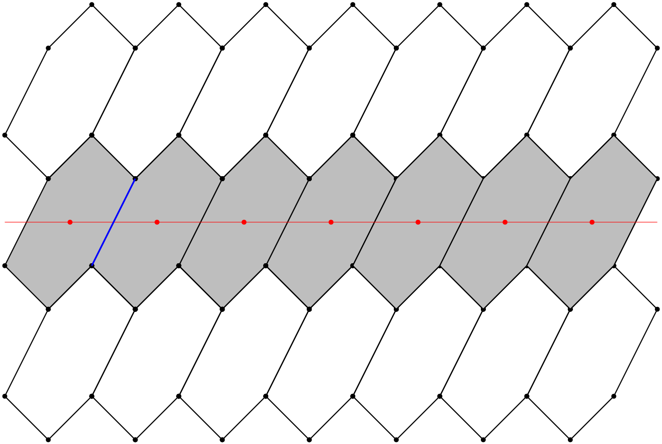


Combinatorics of extended parallelohedra



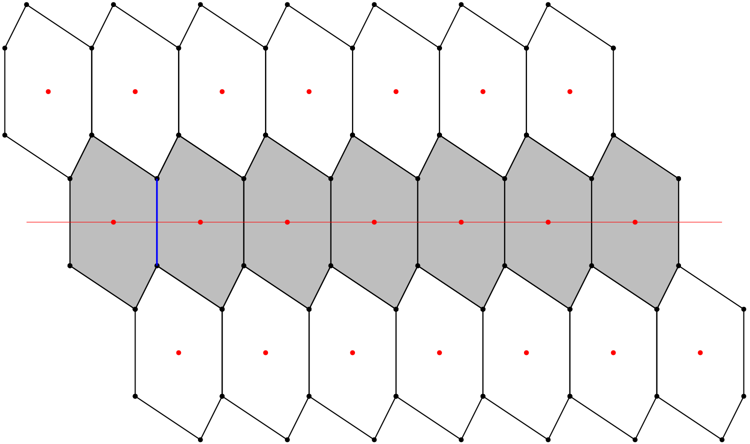


Combinatorics of extended parallelohedra



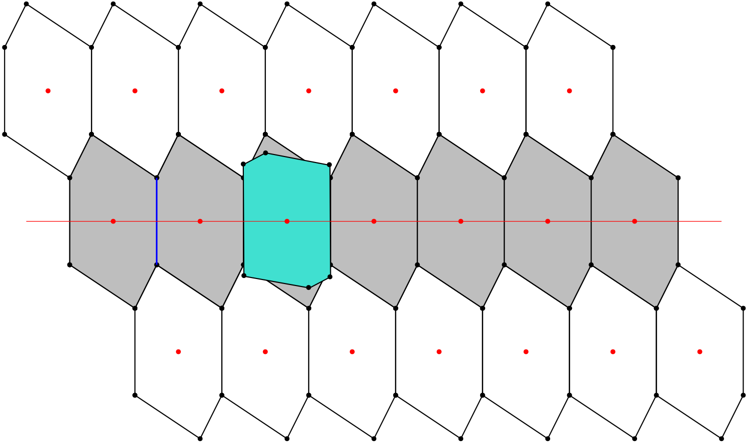
$P + I$ can be extended along blue direction and form layers along red sublattice

Combinatorics of extended parallelohedra



We can transform blue direction into orthogonal to the red sublattice and transform the section into the Voronoi polytope

Combinatorics of extended parallelohedra



If the blue edge is long enough, then the combinatorial equivalence is "natural"

Proof of the Voronoi conjecture in $\mathbb{R}^5$

Let P be a five-dimensional parallelohedron.

- If P can be extended, then its extension has **combinatorics** of one of 110244 Voronoi parallelohedra in $\mathbb{R}^5$;
- In the five-dimensional case, **global combinatorics** of a Voronoi parallelohedron guarantees the geometric part of the Voronoi conjecture.
- **Local combinatorics** can be used to show that P can be extended.

PROOF OF THE VORONOI CONJECTURE IN $\mathbb{R}^5$

Let P be a five-dimensional parallelohedron.

- If P can be extended, then its extension has **combinatorics** of one of 110244 Voronoi parallelohedra in $\mathbb{R}^5$; **Done!**
- In the five-dimensional case, **global combinatorics** of a Voronoi parallelohedron guarantees the geometric part of the Voronoi conjecture. **Done!**
- **Local combinatorics** can be used to show that P can be extended. **Careful analysis of dual 3- and 4-cells.**

Dual cell approach: $\mathbb{R}^5$ and dual 3-cells

Let P be a 5-dimensional paralleohedron.

Dual 3-cells of P can be:

- tetrahedra, octahedra, pyramids;
- triangular prisms;
- parallelepipeds.

What information do they carry?

- If all dual 3-cells are tetrahedra, octahedra, or pyramids, then P satisfies the Voronoi conjecture (Ordine's case).
- If there is a two-dimensional face F of P such that $\mathcal{D}(F)$ is a parallelepiped, then every edge of F is a free direction for P.

CUBIC DUAL CELLS AND FREE DIRECTIONS

Def: **6-belt** of P is a set of 6 facets parallel to one face of codimension 2.

Lemma (Grishukhin, Magazinov)

A direction I is free for P if and only if every 6-belt of P has at least one facet parallel to I.

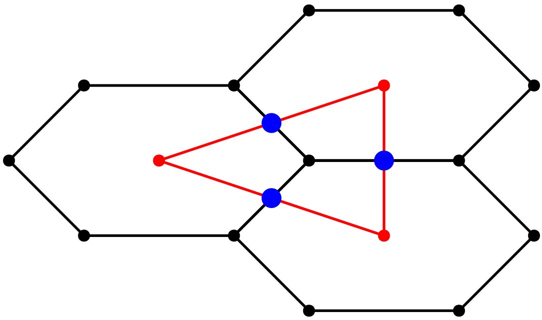
Lemma

If F is a face with cubical dual cell, then every edge of F is a free direction.

Tool: the space of **half-lattice points** $\Lambda_{1/2} = (\frac{1}{2}\Lambda)/\Lambda$.

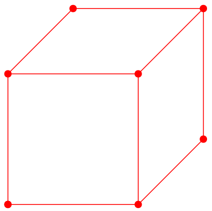
- The half-lattice points serve as symmetry points for the tiling into copies of P and for the dual cell complex;
- They split into 32 classes and can be identified with $\mathbb{F}_2^5$.

Half-lattice points for 6-belts



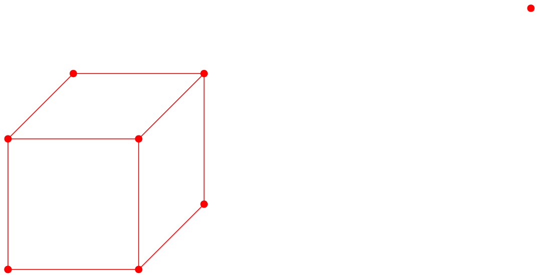
For every 6-belt, the **centers** of facets parallel to this belt give a two-dimensional subspace of $\mathbb{F}_2^5$ (except the origin).

Half-lattice points for edges of F



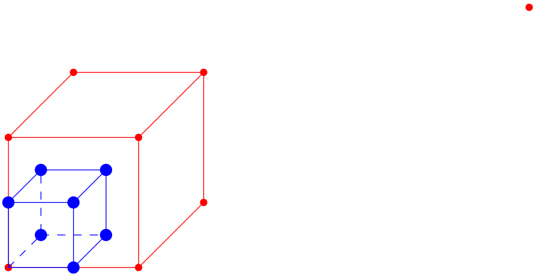
The dual cell of F is a combinatorial cube.

Half-lattice points for edges of F



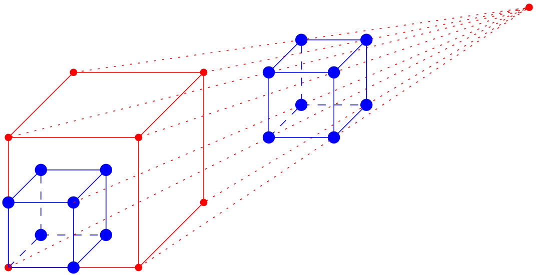
If e is an edge of F, then the dual cell of e contains an additional point.

Half-lattice points for edges of F



The dual cell $\mathcal{D}(e)$ defines 7 non-trivial half-integer classes within the cube.

Half-lattice points for edges of F



And 8 more classes with the additional vertex. In total there are 15 classes that give a four-dimensional subspace of $\mathbb{F}_2^5$.

$2 + 4 > 5$

- The two- and four-dimensional subspaces of $\mathbb{F}_2^5$ intersect non-trivially;
- The intersection class is such that the symmetry in its representative
  - preserves one of the facets from the belt, and
  - swaps two copies of P that contain e.
- This is possible only if e is parallel to the facet which is preserved.

Dual cell approach: $\mathbb{R}^5$ and dual 3-cells

Let P be a 5-dimensional paralleohedron.

Dual 3-cells of P can be:

- ~~tetrahedra, octahedra, pyramids;~~
- **triangular prisms**;
- ~~parallelepipeds.~~

Suppose F is a face of P with prismatic dual cell.

Lemma

Either F is a triangle or some edge of F is a free direction for P.

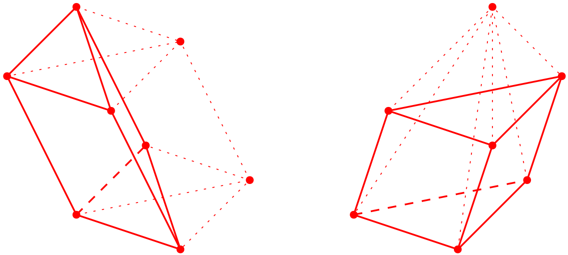
What if F is not a triangle?

We consider the space of parity classes $\Lambda_p = \Lambda/2\Lambda$. This space is also isomorphic to $\mathbb{F}_2^5$ and has 32 representatives. It serves as the source of possible additional points in all dual cells.

- F is a two-dimensional face of P;
- The dual cell of F spans a three-dimensional affine subspace π_F of Λ_p;
- Each edge of F has an additional point in its dual cell, and since F is an n-gon for $n > 3$, at least two edges give points in one translation of π_F;
- After that we can "fix" the vertices of $\mathcal{D}(F)$ among the parity classes and consider all cases for additional vertices exhaustively.

DUAL 4-CELLS FOR THE EDGES OF F

The same exhaustive search gives possible 4-cells of the edges of F. There are two possible types that don't give a free direction right away.



Prism over tetrahedron or pyramid over triangular prism

It remains to consider four cases for different combinations of these dual cells among the edges of F.

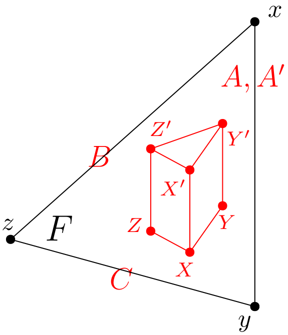
Cases for dual 4-cells

- Prism-Prism-Prism
- Prism-Prism-Pyramid
- Prism-Pyramid-Pyramid
- Pyramid-Pyramid-Pyramid

Cases for dual 4-cells

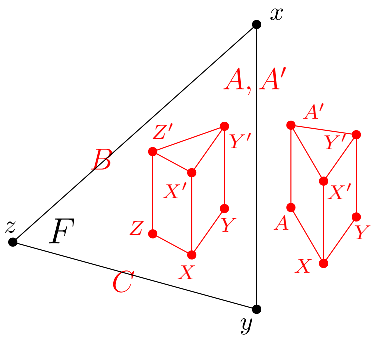
- Prism-Prism-Prism
  - P is a direct product of two parallelohedra.
- Prism-Prism-Pyramid
  - We can look on the parity classes more carefully.
- **Prism-Pyramid-Pyramid**
  - Again, we can look on the parity classes more carefully.
- Pyramid-Pyramid-Pyramid
  - A modification of Ordine's approach to the Voronoi generatrix construction works.

Prism-Pyramid-Pyramid case



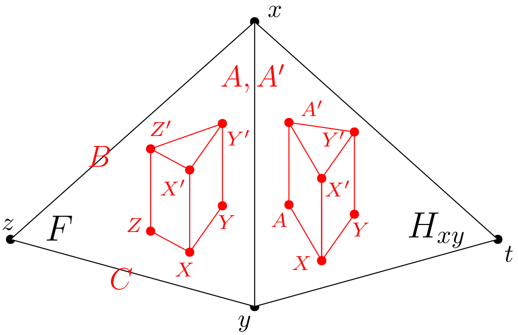
We start from the face $F = xyz$ with the prismatic dual cell $XYZX'Y'Z'$.

Prism-Pyramid-Pyramid case



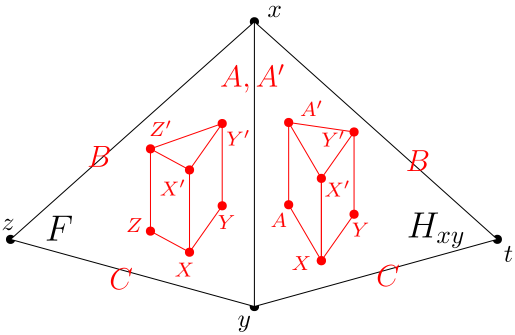
The dual cell of xy is the prism $AXYZA'X'Y'Z'$ and $AXYA'X'Y'$ is another prismatic dual 3-cell.

Prism-Pyramid-Pyramid case



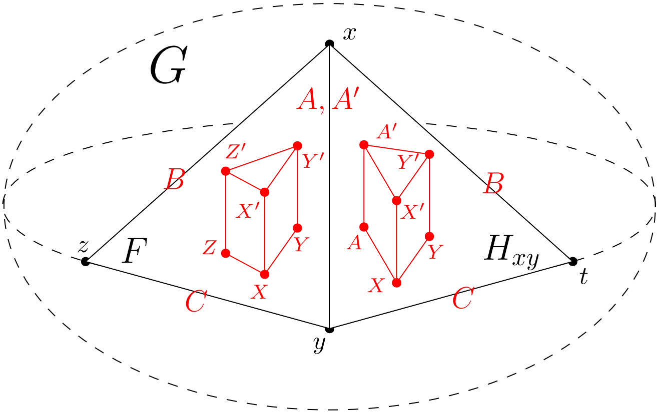
The corresponding two-dimensional face of P must be a triangle or there is a free direction.

Prism-Pyramid-Pyramid case



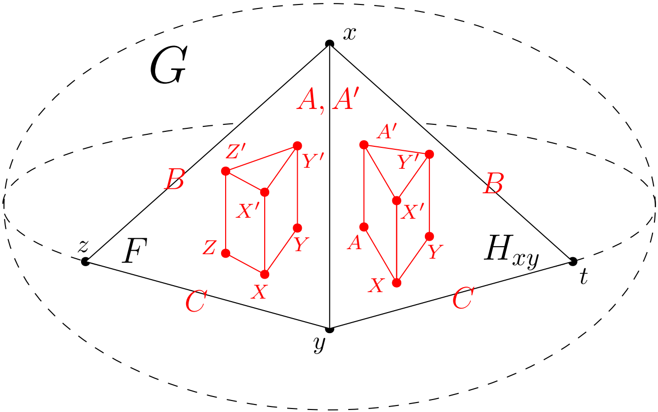
Exhaustively checking all parity classes we can identify the dual cells of the edges *xt* and *yt*.

Prism-Pyramid-Pyramid case



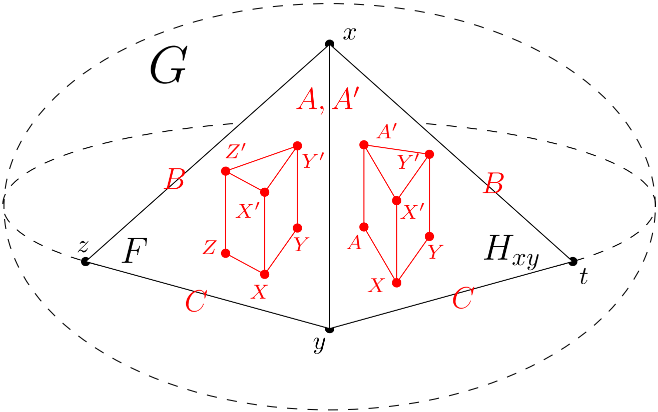
Now we look on the 3-dimensional face G with dual cell $XYY'X'$. Triangles xyz and xyt are two faces of G.

Prism-Pyramid-Pyramid case



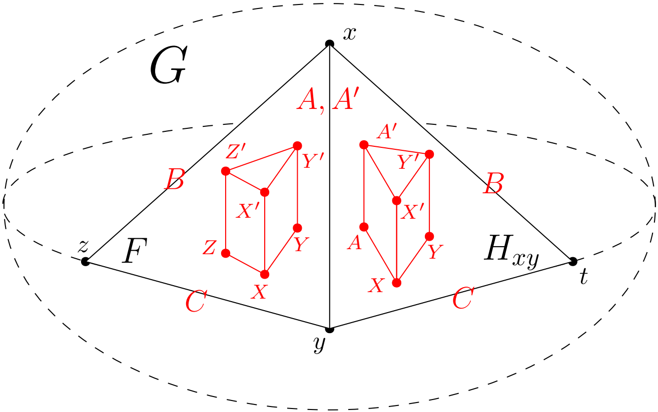
The two-dimensional face of G with pyramidal dual cell $BXYY'X'$ contains edges xz and xt.

PRISM-PYRAMID-PYRAMID CASE



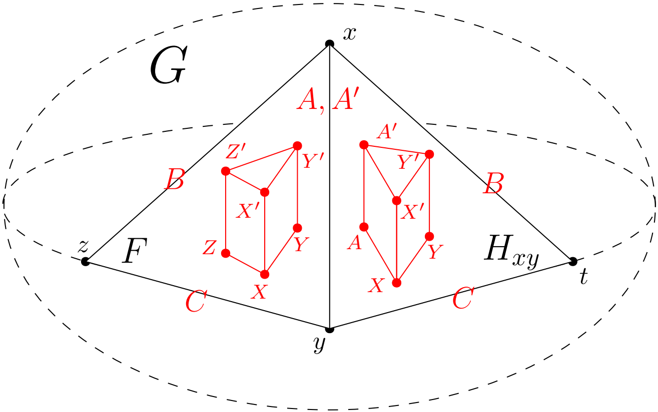
Similarly, the two-dimensional face of G with pyramidal dual cell $CXYY'X'$ contains edges yz and yt.

Prism-Pyramid-Pyramid case



These two faces both contain z and t and this is possible only if G is the tetrahedron $xyzt$.

Prism-Pyramid-Pyramid case



The face G is not centrally symmetric, but it has to be because its dual cell is $XYY'X'$.

What about $\mathbb{R}^6$?

Challenges in six-dimensional case.

- There is a significant jump in the number of parallelohedra. Baburin and Engel (2013) reported about half a billion different Delone triangulations (primitive parallelohedra) in $\mathbb{R}^6$.

- The classification of dual 4-cells is not known and dual 3-cells might be not enough.

THANK YOU!