Voronoi conjecture for parallelohedra

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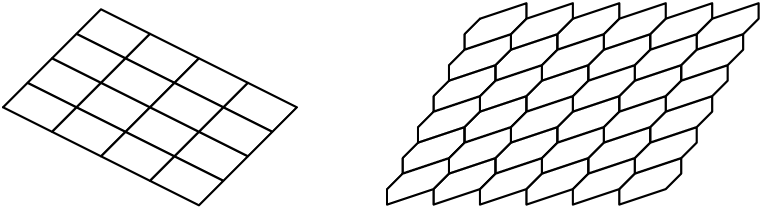
Parallelohedra

Definition

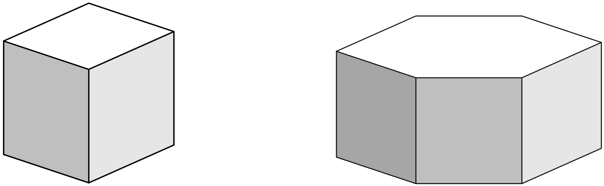
Convex d-dimensional polytope P is called a **parallelohedron** if $\mathbb{R}^d$ can be (face-to-face) tiled into parallel copies of P.



Two types of two-dimensional parallelohedra

Three-dimensional parallelohedra

In 1885 Russian crystallographer Fedorov listed all types of three-dimensional parallelohedra.



Parallelepiped and hexagonal prism with centrally symmetric base.

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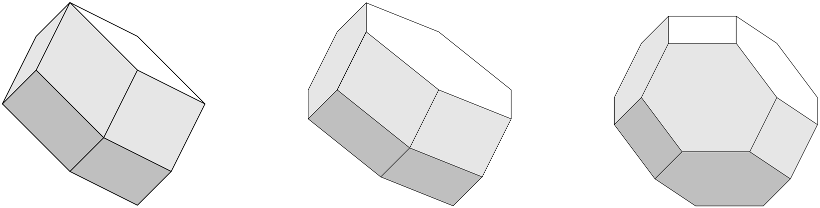
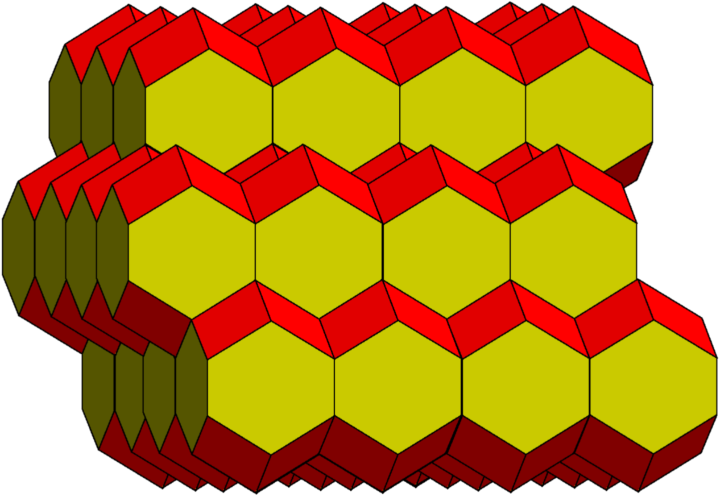


Rhombic dodecahedron, elongated dodecahedron, and truncated octahedron

Tiling by elongated dodecahedra (from Wikipedia)



Minkowski-Venkov conditions

Theorem (Minkowski, 1897; Venkov, 1954; and McMullen, 1980)

P is a d-dimensional parallelohedron iff it satisfies the following conditions:

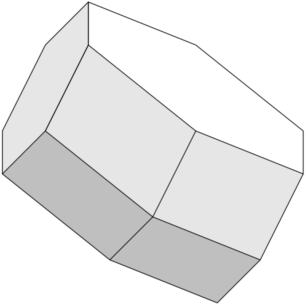
1. *P is centrally symmetric;*
2. *Any facet of P is centrally symmetric;*
3. *Projection of P along any its $(d-2)$-dimensional face is parallelogram or centrally symmetric hexagon.*

Particularly, if P tiles $\mathbb{R}^d$ in a non-face-to-face way, then it satisfies Minlowski-Venkov conditions, and hence tiles $\mathbb{R}^d$ in a face-to-face way as well.

Belts of parallelohedra

Definition

The set of facets parallel to a given $(d-2)$-face is called **belt**. These facets are projected onto sides of a parallelogram or a hexagon.



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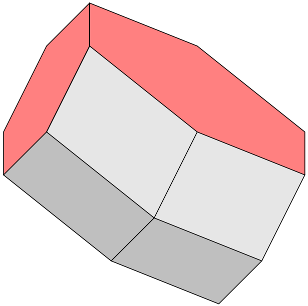

Belts of parallelohedra

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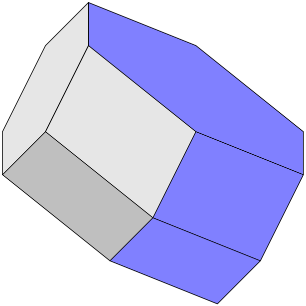
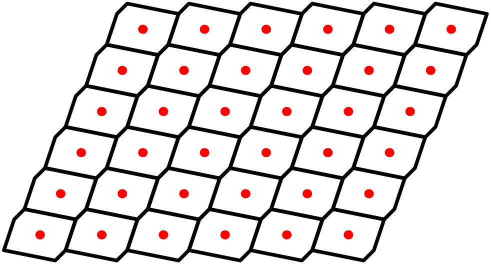


Parallelohedra to Lattices

Let $\mathcal{T}_P$ be the unique face-to-face tiling of $\mathbb{R}^d$ into parallel copies of P. Then centers of tiles forms a lattice Λ_P.

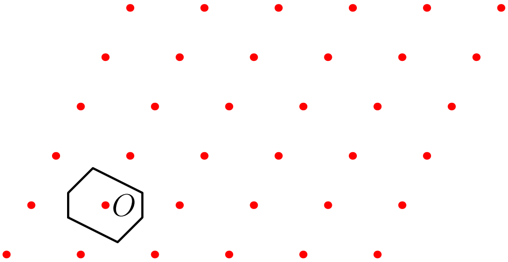


Lattices to Paralleohedra

- Let Λ be an arbitrary d-dimensional and let O be a point of Λ.

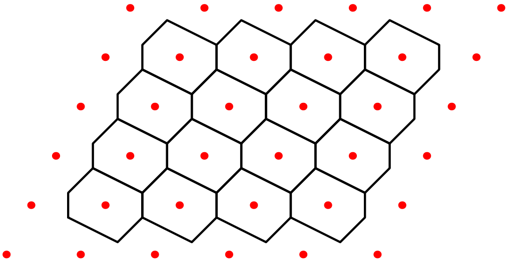
Lattices to Paralleohedra

- Let Λ be an arbitrary d-dimensional and let O be a point of Λ.
- Construct the polytope consisting of points that are closer to O than to any other point of Λ (Dirichlet-Voronoi polytope of Λ).



Lattices to Paralleohedra

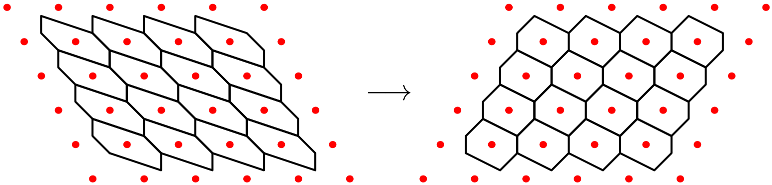
- Let Λ be an arbitrary d-dimensional and let O be a point of Λ.
- Construct the polytope consisting of points that are closer to O than to any other point of Λ (Dirichlet-Voronoi polytope of Λ).
- Then DV_Λ is a parallelohedron and points of Λ are centers of corresponding tiles.



Voronoi conjecture

Conjecture (G.Voronoi, 1909)

Every parallelohedron is affine equivalent to Dirichlet-Voronoi polytope of some lattice Λ.



Voronoi conjecture in $\mathbb{R}^2$

- Each parallelogram can be transformed into rectangle and all rectangles are Voronoi polygons.
- Each centrally-symmetric hexagon can be transformed into one inscribed in a circle. This transformation is **unique** modulo isometry and/or homothety. Similarly, all centrally-symmetric hexagons inscribed in circles are Voronoi polygons.

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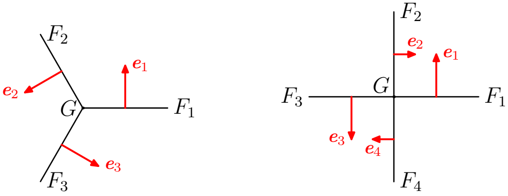
For a given parallelohedron, how one can check whether it satisfies the Voronoi conjecture?
Answer (not the only answer): use canonical scaling.

Canonical scaling

Definition

A (positive) real-valued function $n(F)$ defined on set of all facets of the parallelohedral tiling is called a **canonical scaling**, if it satisfies the following conditions for facets F_i that contain arbitrary $(d-2)$-face G:



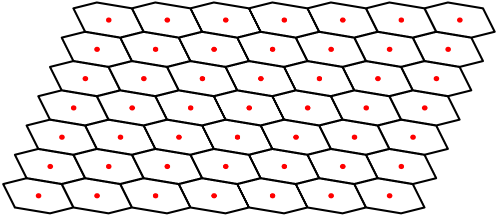
$$\sum \pm n(F_i)\mathbf{e}_i = \mathbf{0}$$

Constructing canonical scaling

How to construct a canonical scaling for a given tiling?

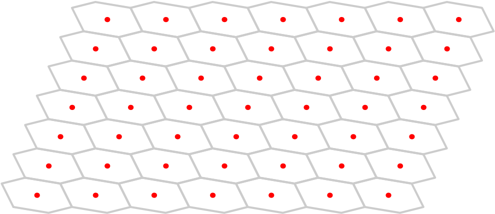
- If two facets F_1 and F_2 of the tiling have a common $(d-2)$-face from 6-belt, then the value of canonical scaling on F_1 uniquely defines the value on F_2 and vice versa.
- If facets F_1 and F_2 have a common $(d-2)$-face from 4-belt then the only condition is that if these facets are opposite then values of canonical scaling on F_1 and F_2 are equal.
- If facets F_1 and F_2 are opposite in one parallelohedron then values of canonical scaling on F_1 and F_2 are equal.

Voronoi's generatrix



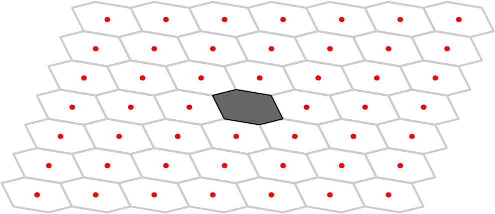
Consider we have a canonical scaling defined on the tiling with copies of P.

Voronoi's generatrix



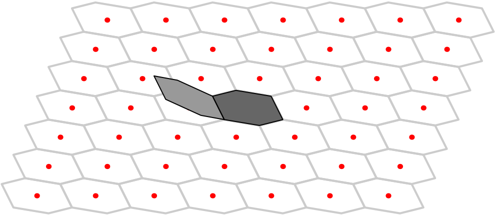
We will construct a piecewise linear generatrix function $\mathcal{G} : \mathbb{R}^d \longrightarrow \mathbb{R}$.

Voronoi's generatrix



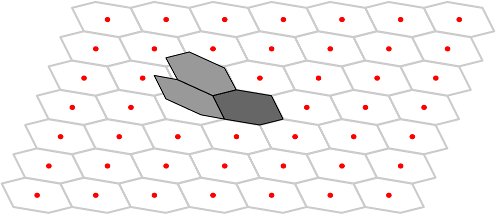
Step 1: Put $\mathcal{G}$ equal to 0 on one of the tiles.

Voronoi's generatrix



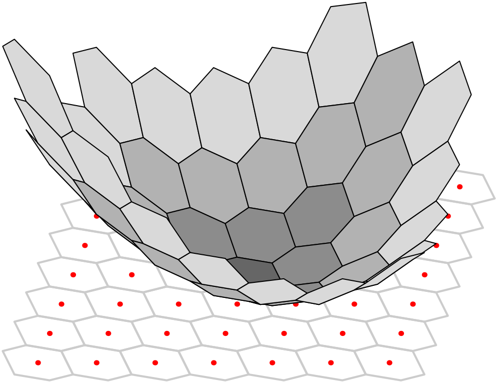
Step 2: When we pass through one facet of the tiling the gradient of $\mathcal{G}$ changes accordingly to the canonical scaling.

Voronoi's generatrix



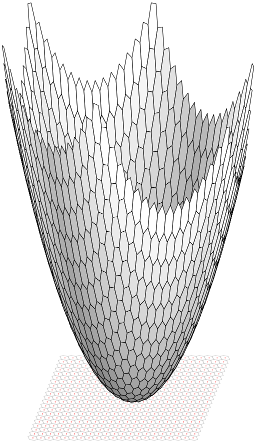
Step 2: Namely, if we pass a facet F with the normal vector $\mathbf{e}$, then we add the vector $n(F)\mathbf{e}$ to the gradient.

Voronoi's generatrix



We obtain the graph of the generatrix function $\mathcal{G}$.

Voronoi's generatrix II



Properties of generatrix

- The graph of generatrix $\mathcal{G}$ looks like a "piecewise linear" paraboloid.
- And actually there is a paraboloid $y = \mathbf{x}^T Q \mathbf{x}$ for some positive definite quadratic form Q tangent to generatrix in the centers of its shells.
- Moreover, if we consider an affine transformation $\mathcal{A}$ of this paraboloid into paraboloid $y = \mathbf{x}^T \mathbf{x}$ then the tiling by copies of P will transform into the Voronoi tiling for some lattice.

So to prove the Voronoi conjecture it is sufficient and, to some extent, necessary to construct a canonical scaling on the tiling by copies of P.

Primitive parallelohedra

Definition

A d-dimensional parallelohedron P is called **primitive**, if every vertex of the corresponding tiling belongs to exactly $d+1$ copies of P.

Primitive parallelohedra appear exactly as dual to Delone triangulations (not arbitrary Delone decompositions).

Theorem (Voronoi, 1909)

The Voronoi conjecture is true for primitive parallelohedra.

Primitive parallelohedra II

Definition

A d-dimensional parallelohedron P is called **k-primitive** if every k-face of the corresponding tiling belongs to exactly $d+1-k$ copies of P.

Theorem (Zhitomirskii, 1929)

The Voronoi conjecture is true for $(d-2)$-primitive d-dimensional parallelohedra. Or the same, it is true for parallelohedra without belts of length 4.

Dual cells

Definition

The **dual cell** of a face F of given parallelohedral tiling is the set of all centers of parallelohedra that share F.
If F is $(d-k)$-dimensional then the corresponding cell is called ***k*-cell**.

The set of all dual cells of the tiling with corresponding incidence relation determines a structure of a cell complex.

Conjecture (Dimension conjecture)

The dimension of a dual k-cell is equal to k.

The dimension conjecture is necessary for the Voronoi conjecture.

Dual 3-cells and 4-dimensional parallelohedra

Lemma (Delone, 1929)

There are five types of three-dimensional dual cells: tetrahedron, octahedron, quadrangular pyramid, triangular prism and cube.

Theorem (Delone, 1929)

The Voronoi conjecture is true for four-dimensional parallelohedra.

Theorem (Ordine, 2005)

The Voronoi conjecture is true for parallelohedra without cubical or prismatic dual 3-cells.

Gain function instead of canonical scaling

We know how canonical scaling should change when we pass from one facet to a neighbor facet across a primitive $(d-2)$-face of F.

Definition

We will call the multiple of canonical scaling that we achieve by passing across F the **gain function** g on F.

For any generic curve γ on surface of P that do not cross non-primitive $(d-2)$-faces we can define the value $g(\gamma)$.

Lemma

The Voronoi conjecture is true for P iff for any generic cycle $g(\gamma) = 1$.

Properties of the gain function

Definition

Consider a manifold P_δ that is a surface of parallelohedron P with deleted closed non-primitive $(d-2)$-faces. We will call this manifold the **δ-surface** of P.

The gain function is well defined on any cycle on P_δ.

Lemma (G., Gavrilyuk, Magazinov)

The gain function gives us a homomorphism

$$g : \pi_1(P_\delta) \longrightarrow \mathbb{R}^+$$

and the Voronoi conjecture is true for P iff this homomorphism is trivial.

Improvements of the main lemma

- Values of a canonical scaling should be equal on opposite facets of P. So we can consider a **π-surface** P_π of P obtained from P_δ by gluing its opposite points.
- Any **half-belt cycle** which starts at the center of a facet and end at the center of the opposite facet crossing only three parallel primitive $(d-2)$-faces will be mapped to 1 by g.
- The group $\mathbb{R}^+$ is commutative, so we can factorize the fundamental group $\pi_1(P_\pi)$ by the commutator and get the group of one-dimensional homologies over $\mathbb{Z}$ instead of the fundamental group.
- We can eliminate the torsion part of the group $H_1(P_\pi, \mathbb{Z})$ since there is no torsion in the group $\mathbb{R}^+$.

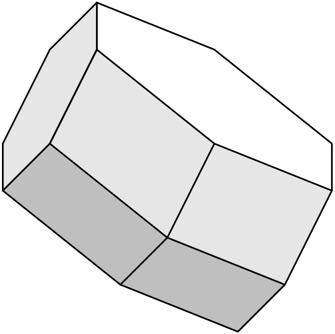
Finally we get the group $H_1(P_\pi, \mathbb{Q})$ instead of the initial fundamental group $\pi_1(P_\delta)$.

Global combinatorics for the Voronoi conjecture

Theorem (G., Gavrilyuk, Magazinov, 2015)

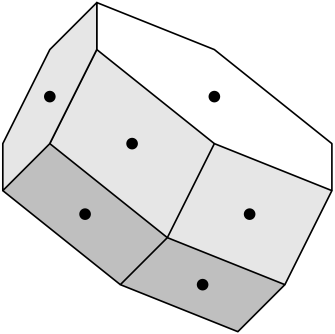
If the group of one-dimensional homologies $H_1(P_\pi, \mathbb{Q})$ of the π-surface of a parallelohedron P is generated by the half-belt cycles then the Voronoi conjecture is true for P.

How one can apply this theorem?



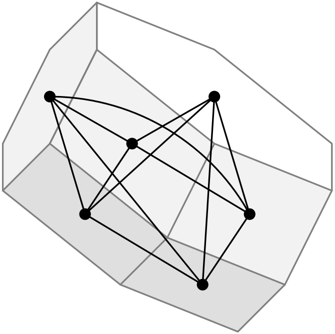
We start from a parallelohedron P.

How one can apply this theorem?



Then put a vertex of the graph G for every pair of opposite facets of P.

How one can apply this theorem?



Draw edges of G between pairs of facets with a common primitive $(d-2)$-face.

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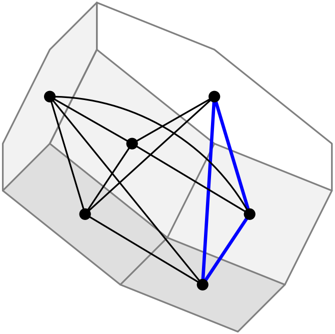
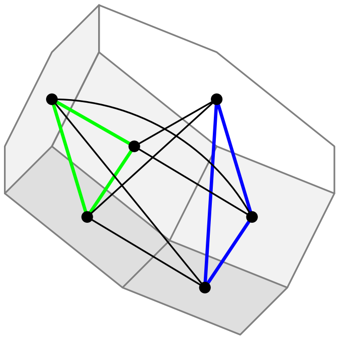


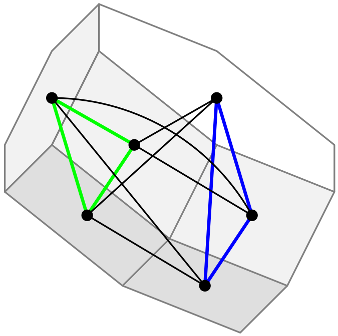
List all "basic" cycles γ that has gain function 1 for sure. These are half-belt cycles.

How one can apply this theorem?



List all "basic" cycles γ that has gain function 1 for sure. These are half-belt cycles. And trivially contractible cycles around $(d-3)$-faces.

How one can apply this theorem?



Check that the basic cycles generate all cycles of the graph G.

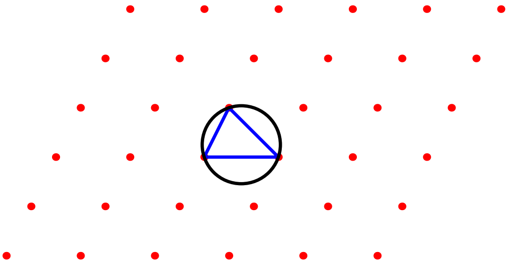
How many parallelohedra satisfy GGM condition?

- All 5 parallelohedra in $\mathbb{R}^3$. The full list was obtained by Fedorov (1885).
- All 52 parallelohedra in $\mathbb{R}^4$. The full list was obtained by Delone (1929) with a correction by Stogrin (1974).
- All 110244 **Voronoi** parallelohedra in $\mathbb{R}^5$ (Preprint of Dutour-Sikirić, G., and Magazinov).

DELONE TILING

Delone tiling is the tiling with "empty spheres".

A polytope P is in the Delone tiling $\mathrm{Del}(\Lambda)$ iff it is inscribed in an empty sphere.



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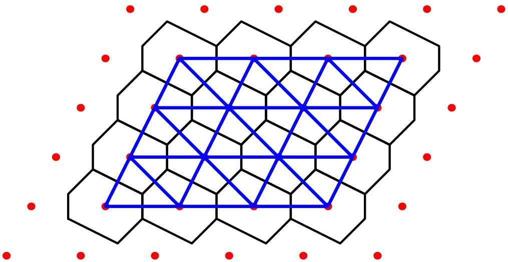


The Delone tiling is dual to the Voronoi tiling.

From lattices to PQF

An affine transformation can take a lattice to $\mathbb{Z}^d$, but it changes metrics from $\mathbf{x}^t\mathbf{x}$ to $\mathbf{x}^t Q\mathbf{x}$ for some positive definite quadratic form Q.

Task

Find all combinatorially different Delone tilings of $\mathbb{Z}^d$.

Definition

The Delone tiling $\mathrm{Del}(\mathbb{Z}^d, Q)$ of the lattice $\mathbb{Z}^d$ with respect to PQF Q is the tiling of $\mathbb{Z}^d$ with empty ellipsoids determined by Q (spheres in the metric $\mathbf{x}^t Q\mathbf{x}$).

Secondary cones

Let $\mathcal{S}^d \subset \mathbb{R}^{\frac{d(d+1)}{2}}$ denotes the cone of all PQF.

Definition

The **secondary cone** of a Delone tiling $\mathcal{D}$ is the set of all PQFs Q with Delone tiling equal to $\mathcal{D}$.

$$\mathrm{SC}(\mathcal{D}) = \left\{ Q \in \mathcal{S}^d | \mathcal{D} = \mathrm{Del}(\mathbb{Z}^d, Q) \right\}$$

Theorem (Voronoi, 1909)

$\mathrm{SC}(\mathcal{D})$ *is a convex polyhedron in* $\mathcal{S}^d$.

Secondary cones II

Theorem (Voronoi, 1909)

The set of closures all secondary cones gives a face-to-face tiling of the closure of $\mathcal{S}^d$ (that is the cone of positive semidefinite quadratic forms).

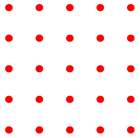
- Full-dimensional secondary cones correspond to Delone triangulations
- One-dimensional secondary cones are called **extreme rays**

Lemma

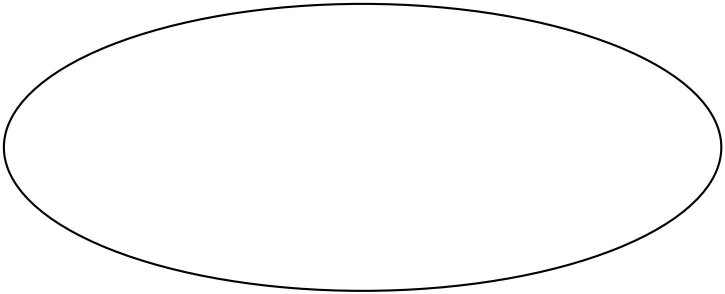
Two Delone tilings $\mathcal{D}$ and $\mathcal{D}'$ are affinely equivalent iff there is a matrix $\mathcal{A} \in GL_d(\mathbb{Z})$ such that

$$\mathcal{A}(\mathrm{SC}(\mathcal{D})) = \mathrm{SC}(\mathcal{D}').$$

Secondary cones in dimension 2

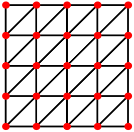


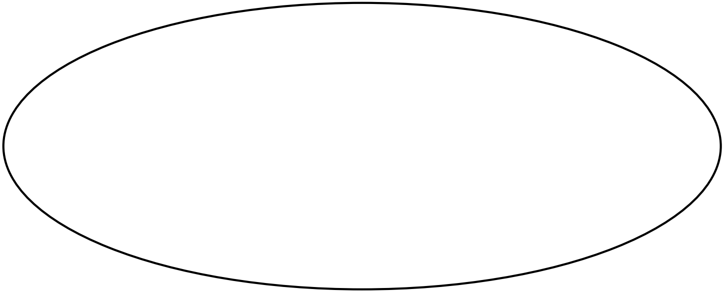
Any PQF $Q = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ can be represented by a point in a cone over open disc.



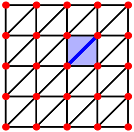
SECONDARY CONES IN DIMENSION 2

We will find the secondary cone of Delone triangulation on the right.

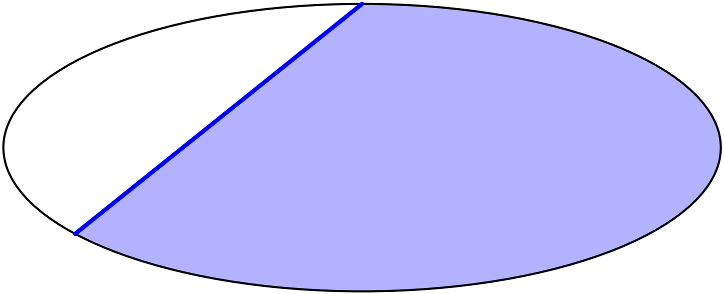




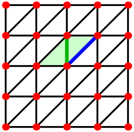
Secondary cones in dimension 2



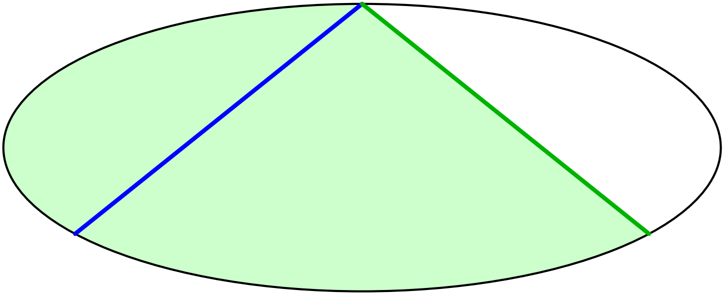
Each pair of adjacent triangles defines one linear inequality for secondary cone. For **blue** pair the inequality is $b < 0$.



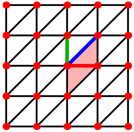
Secondary cones in dimension 2



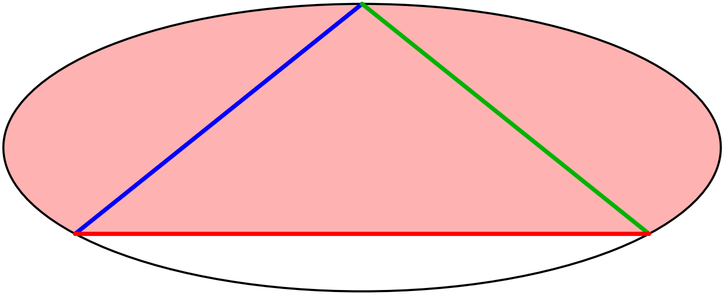
The **green** pair of triangles gives us inequality $b + c > 0$.



Secondary cones in dimension 2



The **red** pair gives us inequality $a + b > 0$.



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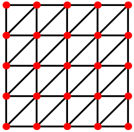
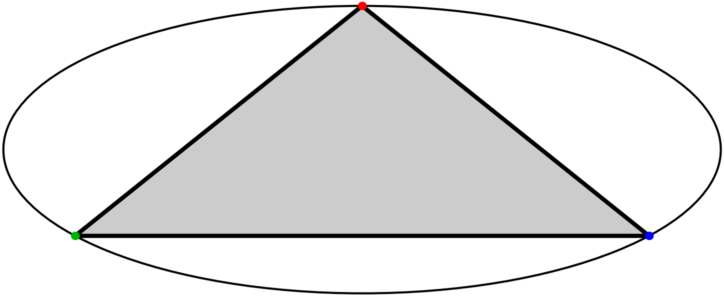


The secondary cone is a cone over triangle with vertices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.



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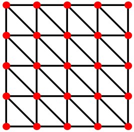
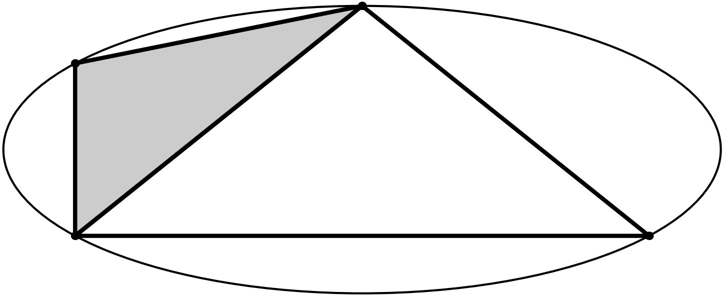
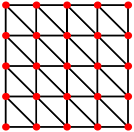


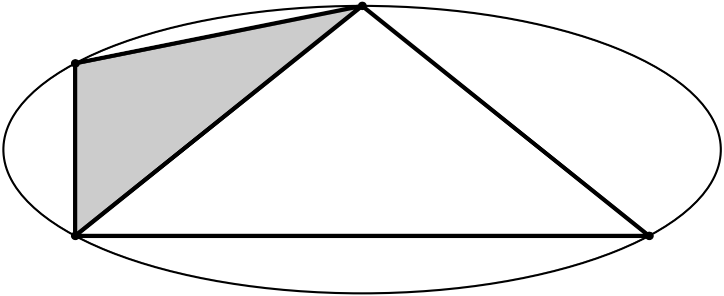
Similarly we can construct secondary cones for other triangulations.



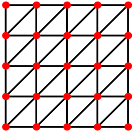
Secondary cones in dimension 2



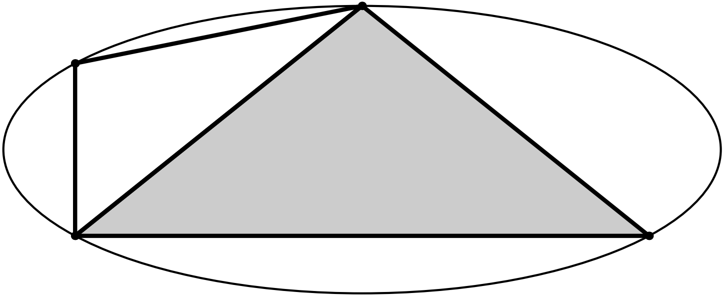
Triangulations corresponding to adjacent secondary cones differ by a (bi-stellar) flip.



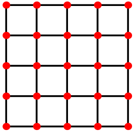
Secondary cones in dimension 2



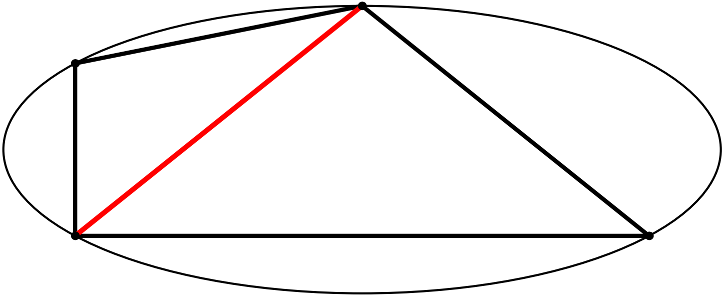
Triangulations corresponding to adjacent secondary cones differ by a (bi-stellar) flip.



Secondary cones in dimension 2



Cones of smaller dimensions are secondary cones of non-generic Delone decompositions.



Algorithm

- We start from one known Delone triangulation.
- Using all possible bistellar flips, we find secondary cones for all non-equivalent Delone triangulations. These are the cones of codimension 0.
- Compute all facets of each cone and pick those which are non-equivalent. These are the cones of codimension 1.
- Repeat until we get all non-equivalent extreme rays.

To check $GL_d(\mathbb{Z})$-equivalence of secondary cones we use `isom` by Bernd Souvignier for "central" rays.

Computations in $\mathbb{R}^5$

Theorem (Dutour-Sikirić, G., Schürmann, Waldmann, 2016)

There are 110244 *affine types of lattice Delone subdivisions in dimension* 5.

Three independent implementations: `Haskell` code, `polyhedral` package of `GAP`, and C++ code `scc` v.2.0 (secondary cone cruiser).

Additionally, all these classes generate combinatorially different Dirichlet-Voronoi parallelohedra.

Voronoi conjecture in small dimensions

- $\mathbb{R}^3$ is generally treated as a folklore.
- $\mathbb{R}^4$ was proved by Delone (1929).
- Some sources refer to Engel (1998) as a proof in $\mathbb{R}^5$.
  - Engel searched for new parallelohedra using possible extension and contraction.
  - Then the closure of corresponding secondary cones is searched for new types parallelohedra, and back to extension/contraction.
  - In the end, Engel concluded that since all parallelohedra he found satisfy the Voronoi conjecture, then it is true in $\mathbb{R}^5$.

Unfortunately, there is no justification that all parallelohedra (not only Voronoi) can be reached by this process.

Voronoi conjecture in $\mathbb{R}^5$

Theorem (G., Magazinov, 2019+)

The Voronoi conjecture is true in $\mathbb{R}^5$.

The proof uses

- classification of Voronoi parallelohedra in $\mathbb{R}^5$;
- dual 3-cells classification;
- extensions of parallelohedra;
- a lot of local combinatorics of parallelohedra tilings;
- and more

PROOF. FREE DIRECTION

Definition

Let I be a segment. If $P+I$ and P are both parallelohedra, then I is called a *free* direction for P.

Theorem (G., Magazinov)

Let P be a d-dimensional parallohedron. If I is a free direction for P and the projection of P along I satisfies the Voronoi conjecture, then $P+I$ has combinatorics of a Voronoi parallelohedron.

Corollary

If a 5-dimensional parallelohedron P has a free direction, then P satisfies the Voronoi conjecture.

Proof. Dual 3-cells

What are possible dual 3-cells of a five-dimensional parallelohedron P?

Proof. Dual 3-cells

What are possible dual 3-cells of a five-dimensional parallelohedron P?

- If all dual 3-cells are either tetrahedra, octahedra, or pyramids, then P satisfies the Voronoi conjecture (Ordine's theorem).
- If P has a cubical dual 3-cell, then it has a free direction, and hence satisfies the Voronoi conjecture (proof on the next slide).
- If two-dimensional face F of P has prismatic dual cell, then either an edge of F gives a free direction of P, or F is a triangle.

The main tool used is careful inspection of 32 parity classes of lattice points and all half-lattice points. Central symmetry in each half-lattice point preserves the tiling $\mathcal{T}(P)$, and lattice equivalent points must carry the same local combinatorics.

Proof. Cubic dual 3-cell

Lemma (Grishukhin, Magazinov)

A direction I is free for P if and only if every 6-belt of P has at least one facet parallel to I.

- The space of half-lattice points is isomorphic to five-dimensional space over $\mathbb{F}_2$.
- Let F have a cubical dual cell. An edge e of F has an additional point in its dual. Set of all midpoints between these nine points give a 4-dimensional subspace of the half-lattice space.
- The centers of facets of a 6-belt B give a two-dimensional subspace of the half-lattice space.
- 4- and 2-dimensional subspaces of 5-dimensional space intersect non-trivially, so there is a facet in B parallel to e.

Proof. Dual 4-cells

For a triangular face F of P with prismatic dual 3-cells, the edges may have only two types of dual 4-cells (or there is a free direction for P).

- Pyramid over triangular prism.
- Prism over tetrahedron.

In all four possible choices for dual cells of edges of F we were able to prove that either P has a free direction, or it admits a canonical scaling.

Again, using a lot of local combinatorics and in most cases exhaustively analyzing all 32 parity classes of lattice points.

What about $\mathbb{R}^6$?

Challenges in six-dimensional case.

- There is a significant jump in the number of parallelohedra. Baburin and Engel (2013) reported about more than half a billion of different Delone triangulations in $\mathbb{R}^6$.
- The classification of dual 4-cells in not known and dual 3-cells might be not enough.

THANK YOU!