Measuring weighted cut-and-project sets

based on a joint work with Dirk Frettlöh

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Bounded distance equivalence

Definition

A set $X \subset \mathbb{R}^d$ is called a **Delone set** if it is uniformly discrete and relatively dense.

Definition

Two (Delone) sets X and Y are called **bounded distance equivalent** (or **b.d.e.**) if there is bijection $f : X \longrightarrow Y$ such that

$$\sup_{x \in X} ||f(x) - x|| < \infty.$$

Laczkovich criterion

Theorem (Laczkovich, 1992)

A Delone set X is b.d.e. to a lattice of density α if and only if there is a positive constant C such that for every bounded, measurable set $S \subset \mathbb{R}^d$ the inequality

$$\left| \#(X \cap S) - \alpha \lambda_d(S) \right| \leq C \cdot p_1(S)$$

holds.
Here λ_d is the d-dimensional Lebesgue measure, and $p_1(S)$ is the Lebesgue measure of the 1-neighborhood of the boundary of S.

CPS a.k.a Quasicrystals

Definition

A **cut-and-project set** (or **CPS**) Λ is given by a collection of maps and spaces:

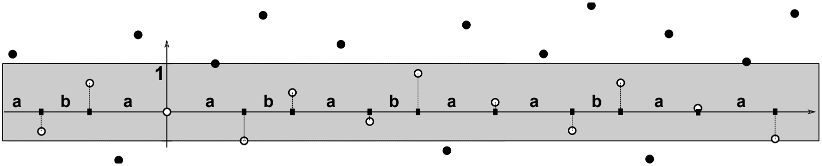
$$\begin{array}{ccccc} G & \xleftarrow{\pi_1} & G \times H & \xrightarrow{\pi_2} & H \\ \cup & & \cup & & \cup \\ \Lambda & & \Gamma & & W \end{array}$$

where in general G and H are locally compact abelian groups. Furthermore, Γ is a lattice in $G \times H$, W is a relatively compact set in H, and π_1 and π_2 are projections to G and to H respectively, such that $\pi_1|_\Gamma$ is one-to-one, and $\pi_2(\Gamma)$ is dense in W. Then

$$\Lambda = \{\pi_1(x) \,|\, x \in \Gamma,\ \pi_2(x) \in W\}$$

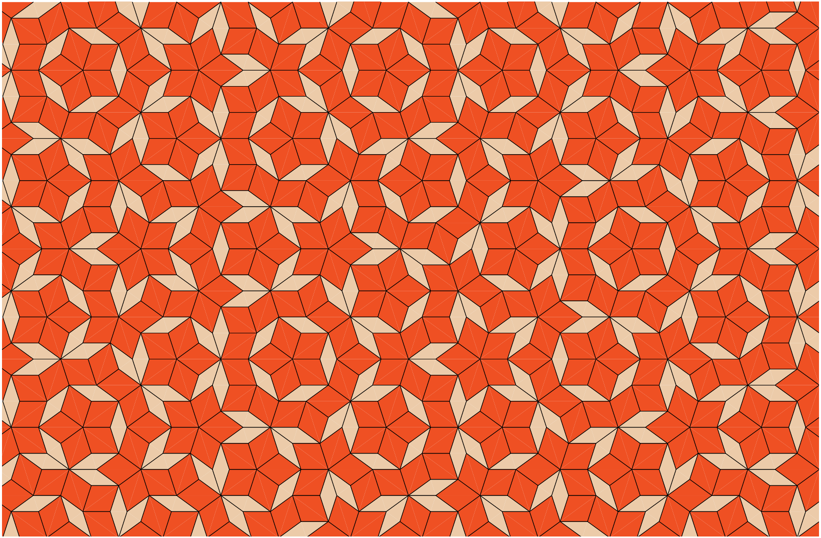
is called a CPS.

Examples



Fibonacci sequence is a CPS with $G = \mathbb{R}$ (**direct space**, horizontal) and $H = \mathbb{R}$ (**internal space**, vertical).
The lattice Γ is spanned by the vectors $(1, 1)$ and $(\tau, -\tau^{-1})$.
Here $\tau = \frac{\sqrt{5}+1}{2}$.
Both projections π_1 and π_2 are orthogonal projections on the corresponding spaces.
The **window** $W = \left[-\tau^{-1}, 1\right)$.

Examples



CPS equivalent to a lattice

Theorem (Kesten, 1966)

A $\mathbb{R} \times \mathbb{R}$ CPS Λ with a window $W = [a, b]$ is b.d.e. to a lattice if and only if there is a vector $\mathbf{e} \in \Gamma$ such that $\pi_2(\mathbf{e}) = b - a$.

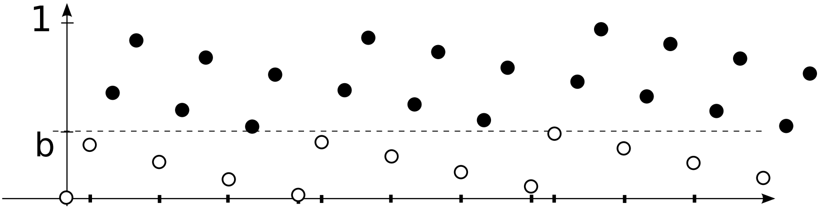
Theorem (Duneau and Oguey, 1990)

Let Λ be a $\mathbb{R}^d \times \mathbb{R}^n$ CPS. If the window of Λ is a π_2-projection of fundamental domain of n-sublattice of Γ, then Λ is b.d.e. to a d-lattice.

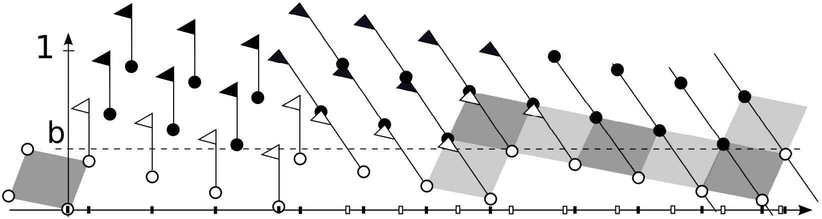
Theorem (Grepstad and Lev, 2015, as reformulated by Haynes and Koivusalo, 2015)

Let Λ be a $\mathbb{R} \times \mathbb{R}^n$ CPS with a measurable window W. Λ is b.d.e. to a lattice if and only if W is $\pi_2(\Gamma)$-equidecomposable to a π_2-projection of fundamental domain of n-sublattice of Γ.

Proof of "if" part



Proof of "if" part



From sets to measures

Definition

For a set $X \subset \mathbb{R}^d$ we can construct the associated **Dirac comb** measure

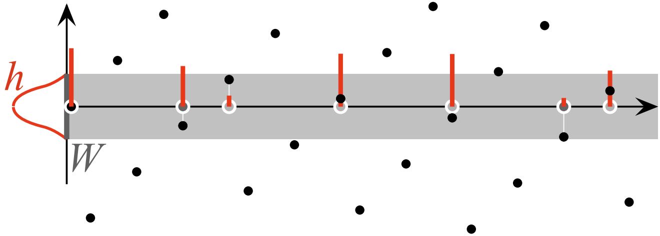
$$\mu_X = \sum_{x \in X} \delta_x.$$

Definition

Two measures μ and ν on $\mathbb{R}$ are called b.d.e. if there is a constant C such that for all intervals $[a, b]$

$$|\mu([a, b]) - \nu([a, b])| < C.$$

Weighted CPS



Definition

Weighted CPS Λ_h is the **weighted** Dirac comb

$$\sum_{x\in\Lambda} h(x^\star)\delta_x,$$

where $x^\star = \pi_2(\pi_1^{-1}(x))$ and h is a function on W.

Main result

Let α be a positive irrational number with certain "mild" restriction on its continued fraction expansion.

Theorem

Let Λ be a CPS with lattice $\Gamma = \mathbb{Z}^2$, $G = \binom{1}{\alpha}\mathbb{R}$ and $H = G^\perp$, window $W = [a, b] \subset H$, and $h \in C(H)$ with support in W. Also let π_1 and π_2 be orthogonal projections on G and H respectively.

1. *If h is piecewise linear, or*
2. *if h is twice differentiable on W, and h'' is uniformly bounded on W,*

then the weighted CPS Λ_h is bounded distance equivalent to $m\lambda$, where λ is the standard Lebesgue measure and $m = \int_a^b h(t)dt$.

Idea of the proof

Idea: use bounded remainder sets (BRS).

Definition

Let $\alpha > 0$ be an irrational number. A measurable set $S \subset [0,1)^2$ is called a **BRS** for the continuous irrational rotation with slope α and starting point $\mathbf{x} = (x_1, x_2)$ if the error

$$\int_0^T \chi_S(\{x_1 + t\}, \{x_2 + \alpha t\})dt - T\lambda(S)$$

is uniformly bounded for all $T > 0$.

IDEA OF THE PROOF

Theorem (Grepstad and Larcher, 2016)

For **almost all** *irrational α and for every $\mathbf{x} \in [0, 1)^2$ every polygon in $[0, 1)^2$ without edge slope α and every convex region with twice differentiable boundary of positive curvature is a BRS with slope α and starting point $\mathbf{x}$.*

The rest:

- Transform BRS language into CPS language and vice versa.
- Show that piecewise linear or twice differentiable functions in weighted CPS lead to BRS.

WEIGHTED VS NON-WEIGHTED

Non-weighted CPS:

- Can be treated as weighted with weight function equal to the indicator function of the window.
- If W is an interval, then Λ is b.d.e. to a lattice only for countably many lengths of W.

Weighted CPS:

- Any interval can produce many weighted CPS b.d.e. to lattices.

Thank you!