Enumeration of five-dimensional Dirichlet-Voronoi parallelohedra

based on a joint work with Mathieu Dutour-Sikirić, Achill Schürmann, and Clara Waldmann

Alexey Garber

The University of Texas Rio Grande Valley

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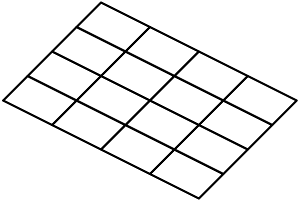
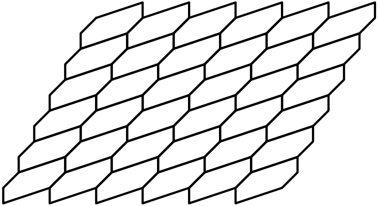
Parallelohedra

Definition

Convex d-dimensional polytope P is called a **parallelohedron** if $\mathbb{R}^d$ can be (face-to-face) tiled into parallel copies of P.





Two types of two-dimensional parallelohedra

Three-dimensional parallelohedra

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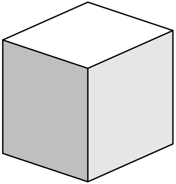

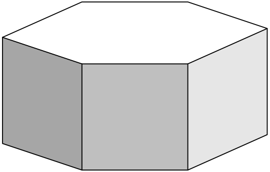

Parallelepiped and hexagonal prism with centrally symmetric base.

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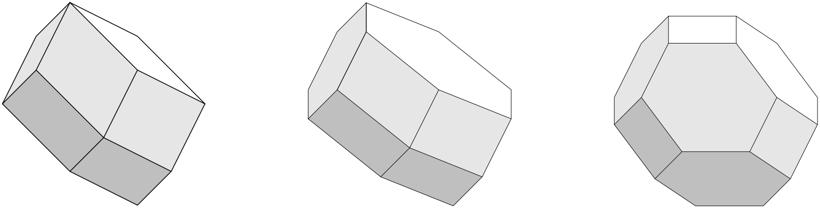
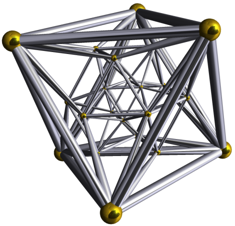


Rhombic dodecahedron, elongated dodecahedron, and truncated octahedron

Four-dimensional parallelohedra

There are 52 four-dimensional parallelohedra. They were found by Delone (1929) and Shtogrin (1973) in proportion 51 to 1.



24-cell is the “first” example of non-zonotopal parallelohedron.

Five-dimensional parallelohedra

In 1976 Baranovskii and Ryshkov found 221 primitive five-dimensional parallelohedra.

In 2000 Engel reported that he found 103769 combinatorially different five-dimensional parallelohedra.

But it appears, that he used subordination symbols to distinguish them, and not all combinatorially different parallelohedra were listed.

In 2002 Engel and Grishukhin used Engel's list to found the last 222nd primitive parallelohedron in $\mathbb{R}^5$.

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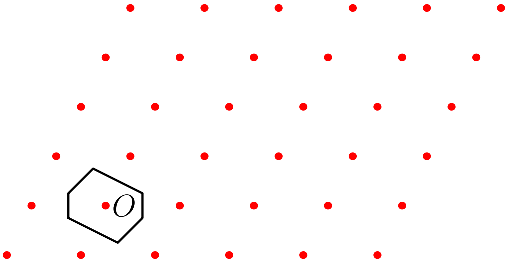
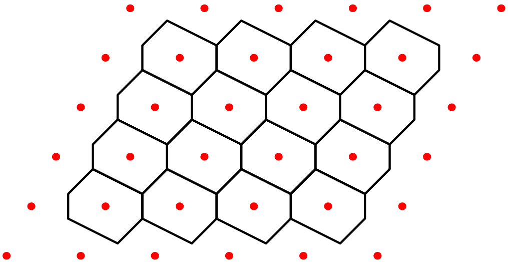


Dirichlet-Voronoi polytopes

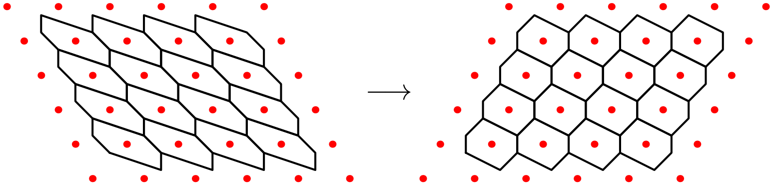
- For arbitrary lattice Λ we can construct a parallelohedron associated with it.
- Take a polytope consists of points that are closer to a point O of Λ than to any other lattice point.
- This is the **Dirichlet-Voronoi polytope** of the lattice, and it is a parallelohedron.



Voronoi conjecture

Conjecture (Voronoi, 1909)

Every parallelohedron is affine equivalent to Dirichlet-Voronoi polytope of some lattice Λ.



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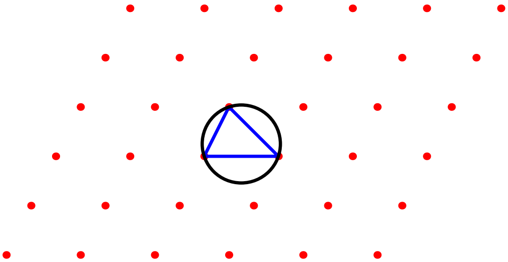
- Magazinov in 2014 for Minkowski sums of parallelohedra and segments
- G., Gavrilyuk, and Magazinov in 2015 for parallehedra with $H_1(P_\pi, \mathbb{Q})$ generated by half-belts

Delone tiling

Delone tiling is the tiling with "empty spheres".

A polytope P is in the Delone tiling $\mathrm{Del}(\Lambda)$ iff it is inscribed in an empty sphere.



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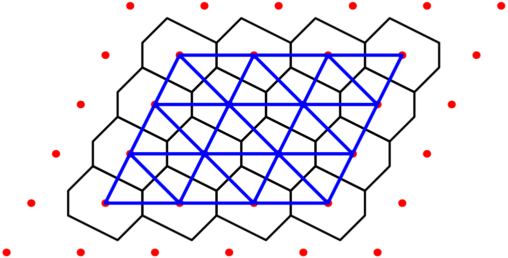


The Delone tiling is dual to the Voronoi tiling.

From lattices to PQF

An affine transformation can take a lattice to $\mathbb{Z}^d$, but it changes the metric from $\mathbf{x}^t\mathbf{x}$ to $\mathbf{x}^t Q\mathbf{x}$ for some positive definite quadratic form Q.

Task

Find all combinatorially different Delone tilings of $\mathbb{Z}^d$.

Definition

The Delone tiling $\mathrm{Del}(\mathbb{Z}^d, Q)$ of the lattice $\mathbb{Z}^d$ with respect to PQF Q is the tiling of $\mathbb{Z}^d$ with empty ellipsoids determined by Q (spheres in the metric $\mathbf{x}^t Q\mathbf{x}$).

Secondary cones

Let $\mathcal{S}^d \subset \mathbb{R}^{\frac{d(d+1)}{2}}$ denotes the cone of all PQF.

Definition

The **secondary cone** of a Delone tiling $\mathcal{D}$ is the set of all PQFs Q with Delone tiling equal to $\mathcal{D}$.

$$\mathrm{SC}(\mathcal{D}) = \left\{ Q \in \mathcal{S}^d | \mathcal{D} = \mathrm{Del}(\mathbb{Z}^d, Q) \right\}$$

Theorem (Voronoi, 1909)

$\mathrm{SC}(\mathcal{D})$ is a convex polyhedron in $\mathcal{S}^d$.

Secondary cones II

Theorem (Voronoi, 1909)

The set of closures all secondary cones gives a face-to-face tiling of the closure of $\mathcal{S}^d$ (that is the cone of positive semidefinite quadratic forms).

- Full-dimensional secondary cones correspond to Delone triangulations
- One-dimensional secondary cones are called **extreme rays**

Lemma

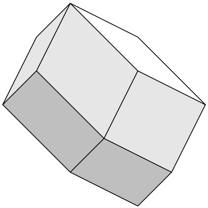
Two Delone tilings $\mathcal{D}$ and $\mathcal{D}'$ are affinely equivalent iff there is a matrix $\mathcal{A} \in GL_d(\mathbb{Z})$ such that

$$\mathcal{A}(\mathrm{SC}(\mathcal{D})) = \mathrm{SC}(\mathcal{D}').$$

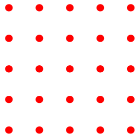
Octahedra and tetrahedra

Do you remember the Egon's picture with octahedra and tetrahedra?

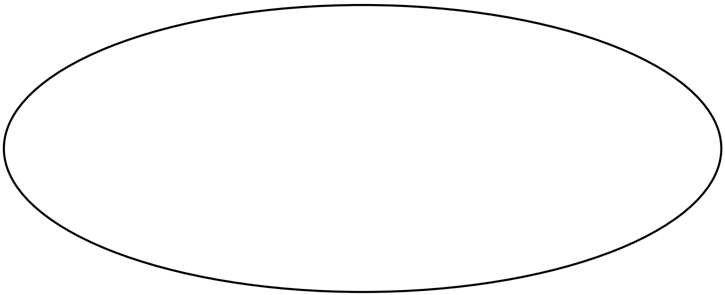
It is a (combinatorial) Delone decomposition, but not a Delone triangulation.



Secondary cones in dimension 2

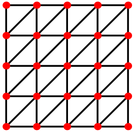


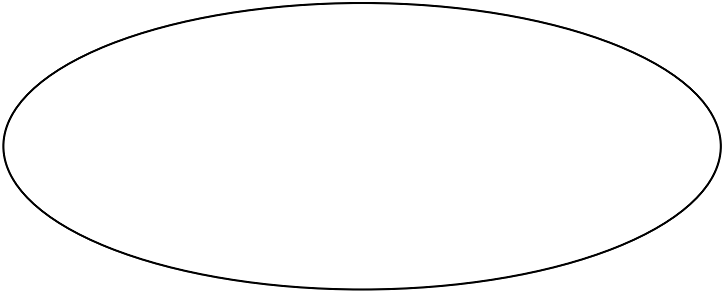
Any PQF $Q = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ can be represented by a point in a cone over open disc.



Secondary cones in dimension 2

We will find the secondary cone of Delone triangulation on the right.





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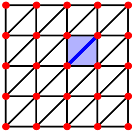
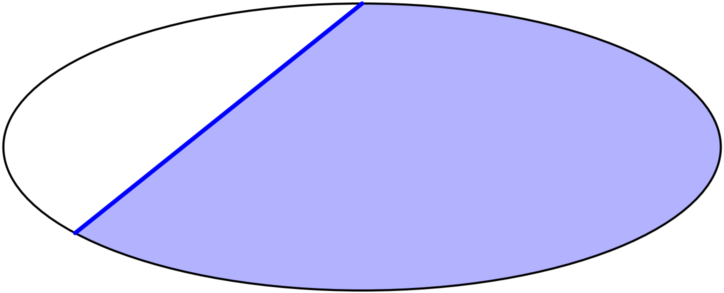
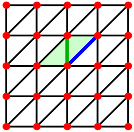


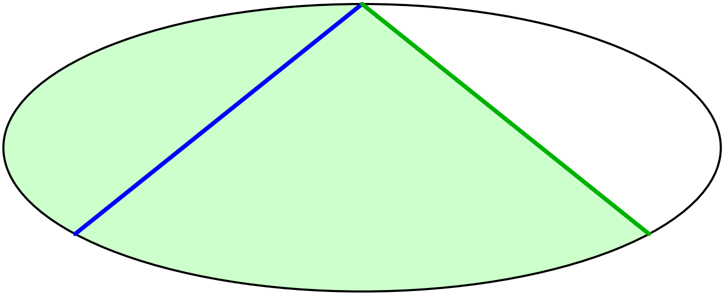
Each pair of adjacent triangles defines one linear inequality for secondary cone. For **blue** pair the inequality is $b < 0$.



Secondary cones in dimension 2

The **green** pair of triangles gives us inequality $b + c > 0$.





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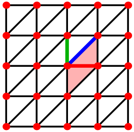
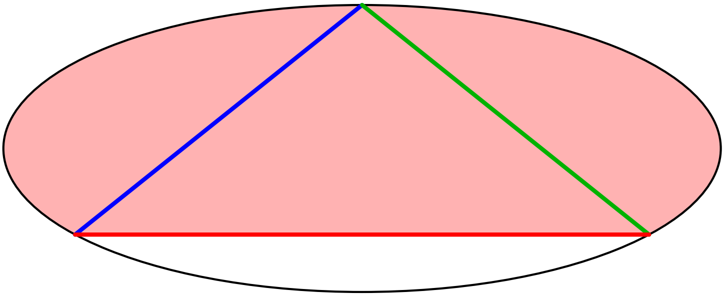
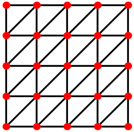


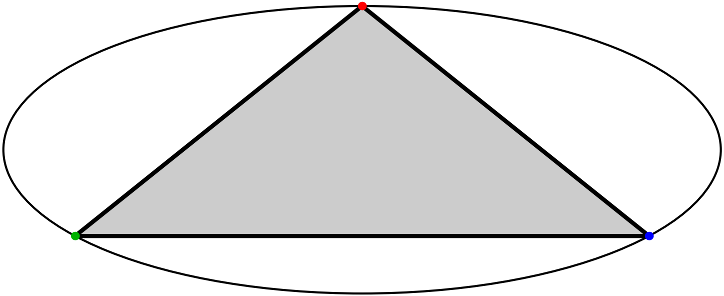
The **red** pair gives us inequality $a + b > 0$.



Secondary cones in dimension 2

The secondary cone is a cone over triangle with vertices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.





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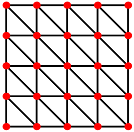
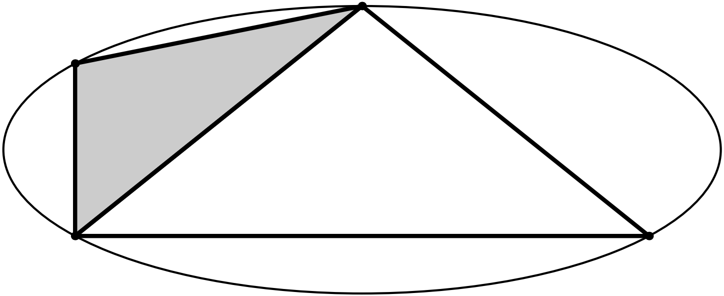


Similarly we can construct secondary cones for other triangulations.



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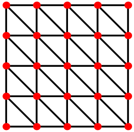
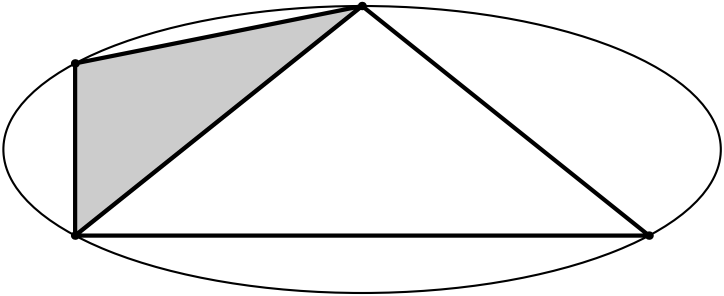


Triangulations corresponding to adjacent secondary cones differ by a (bi-stellar) flip.



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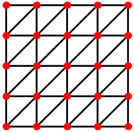

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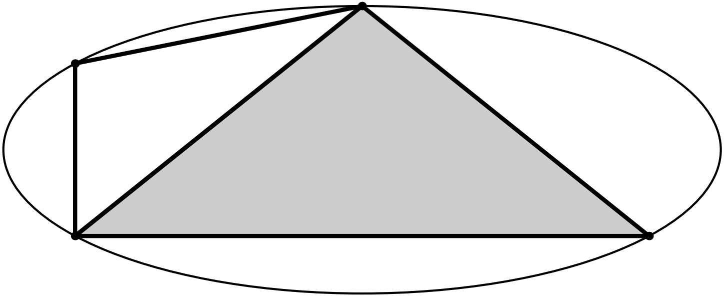

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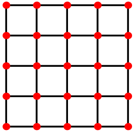
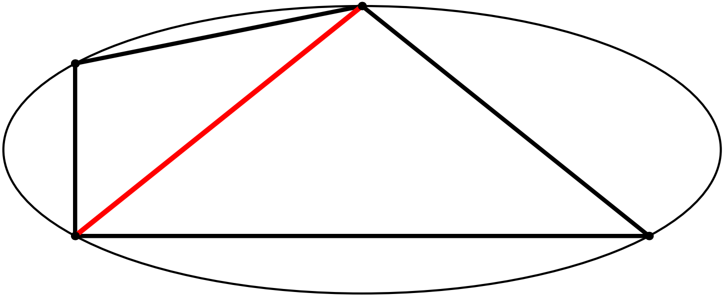


Cones of smaller dimensions are secondary cones of non-generic Delone decompositions.



ALGORITHM

- We start from all the secondary cones of Delone triangulations. These are the cones of codimension 0.
- Compute all facets of each cone and pick those which are non-equivalent. These are the cones of codimension 1.
- Repeat until we get different extreme rays.

To check $GL_d(\mathbb{Z})$-equivalence of secondary cones we use `isom` by Bernd Souvignier.

Results

Theorem (Dutour-Sikirić, G., Schürmann, Waldmann, 2015+)

There are 110244 *affine types of lattice Delone subdivisions in dimension* 5*.*

Three independent implementations: `Haskell` code, `polyhedral` package of `GAP`, and C++ code `scc` v.2.0 (secondary cone cruiser).

Additionally, all these classes generate combinatorially different Dirichlet-Voronoi parallelohedra.

THANK YOU!