Tiling with unique vertex corona

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Tilings

Definition

A collection $\mathcal{T}$ of (convex) polytopes in $\mathbb{R}^d$ is called *(locally finite) tiling* if

- union of all polytopes from $\mathcal{T}$ is $\mathbb{R}^d$;
- they do not intersect in internal points;
- every ball intersects only finite number of polytopes from $\mathcal{T}$.

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Definition

A tiling is called *face-to-face* or *normal* if intersection of any two tiles is face of both.

Periodic and aperiodic tilings

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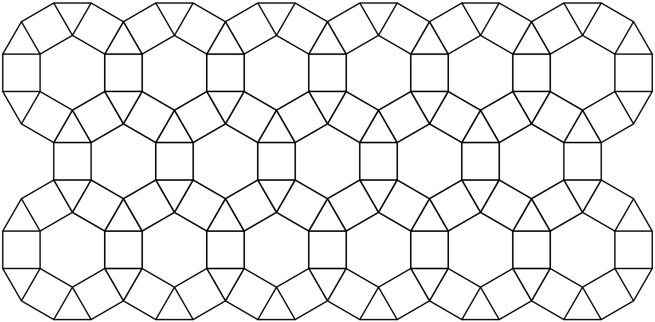

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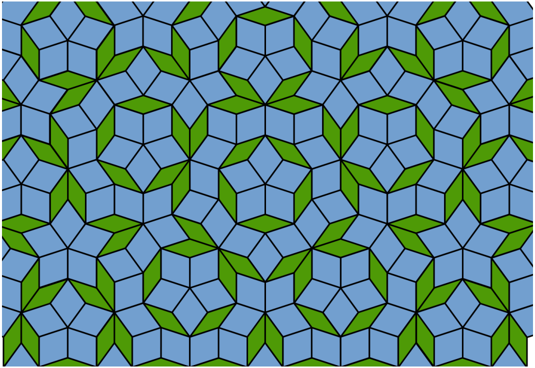

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In Euclidean space crystallographic tiling is the same as periodic tiling.

But this definition can be used in Hyperbolic space $\mathbb{H}^d$ too.

Coronas of a tile

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Definition

The *k-th corona* of P is the collection of all tiles of $\mathcal{T}$ that can be reached from P be at most k steps across facets of the tiling.

Local Theorem

Theorem (Generalized Local Theorem by N. Dolbilin and M. Shtogrin)

A tiling of $\mathbb{R}^d$ (or $\mathbb{H}^d$) is crystallographic iff for some k the following conditions hold.

- *For the number $N(k)$ of k-coronas we have: $N(k) = N(k+1)$ and this number is finite.*
- *For every i the symmetry groups $S_i(k)$ and $S_i(k+1)$ of k- and $(k+1)$-coronas of the i-th type coincides.*

Vertex corona

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Definition

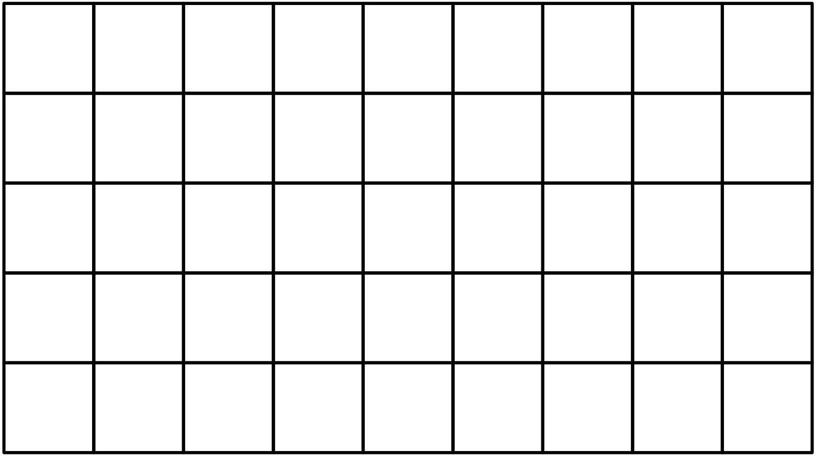
A tiling $\mathcal{T}$ is said to be a *unique vertex corona tiling* if all its vertex coronas are congruent. This means not only collections of polytopes are the same but also that they arranged at correspondent vertices in the same way.

Unique vertex corona and periodicity

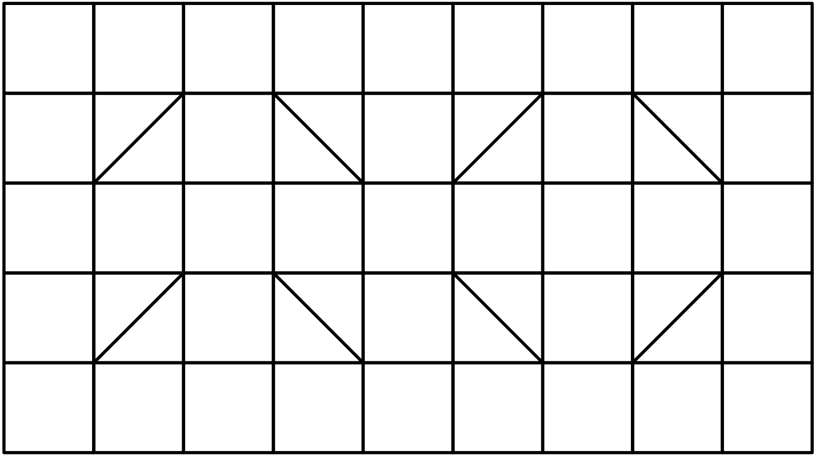
Question

Is it true that unique vertex corona of a tiling $\mathcal{T}$ implies that $\mathcal{T}$ is periodic (crystallographic)?

Unique vertex corona and periodicity



Unique vertex corona and periodicity



Idea of face-to-face classification: topological structure

Lemma

If every vertex corona contains n polygons and k_i of them are i-gons then

$$\sum \frac{k_i}{i} = \frac{n-2}{2}.$$

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And only finitely many global topological structures of the whole tiling.

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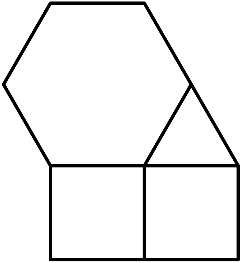
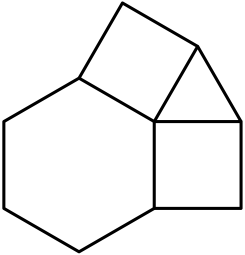
Two theoretical local structures are





Idea of face-to-face classification: topological structure

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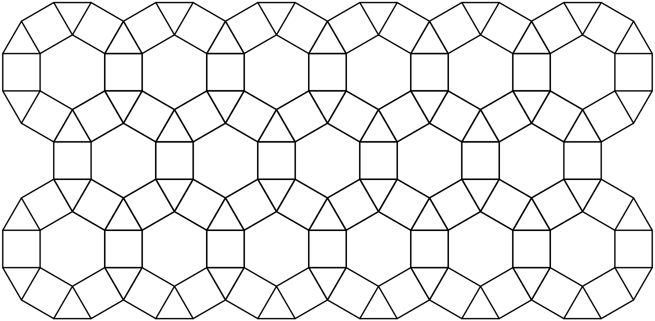
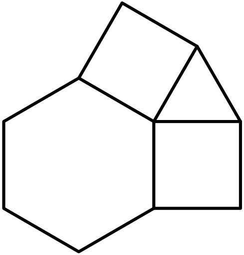
And the only possible topological structure is the following:



Idea of face-to-face classification: segments and angles

- We mark segments that are equal with the same color.



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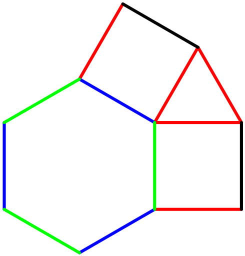

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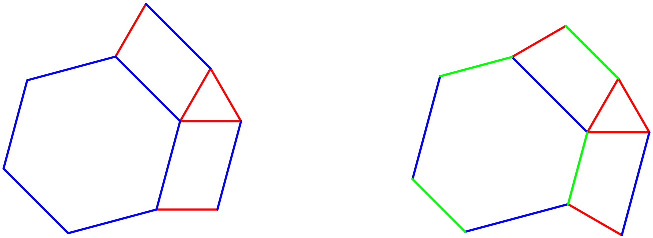

In this particular case there are two possibilities.

Some examples of single corona tilings

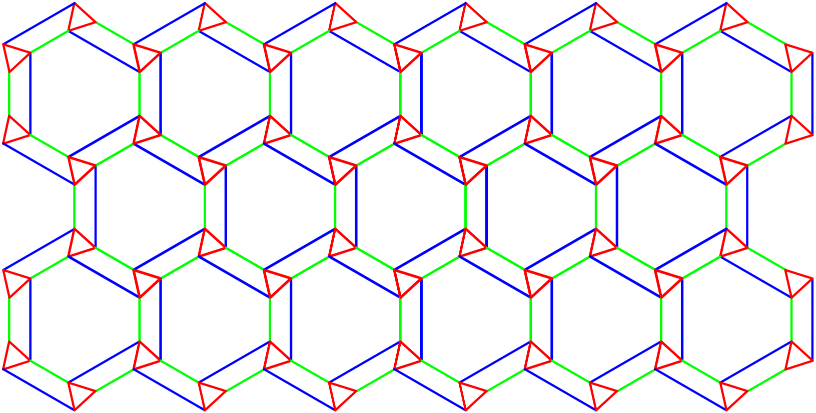

Figure : Tiling with regular triangle, hexagon and trapezoid.

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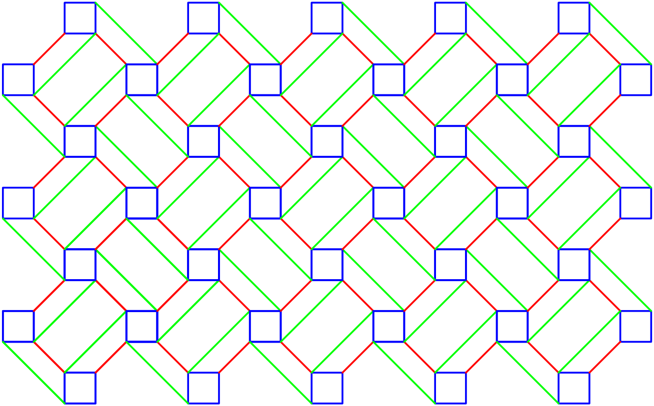

Figure : Tiling with two different rectangles and trapezoid.

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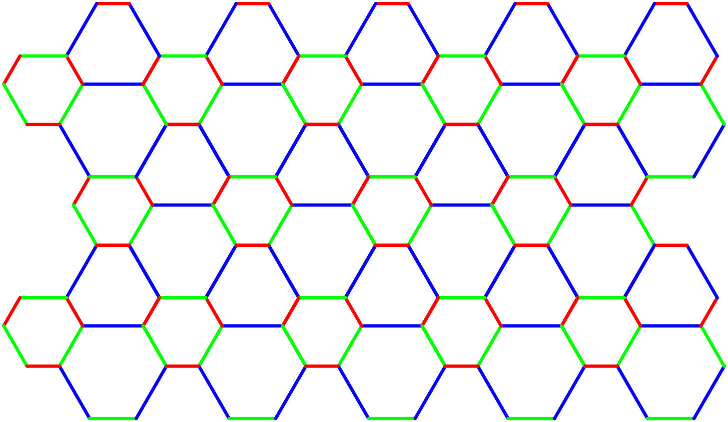

Figure : Tiling with three hexagons.

Example of non-periodic tiling

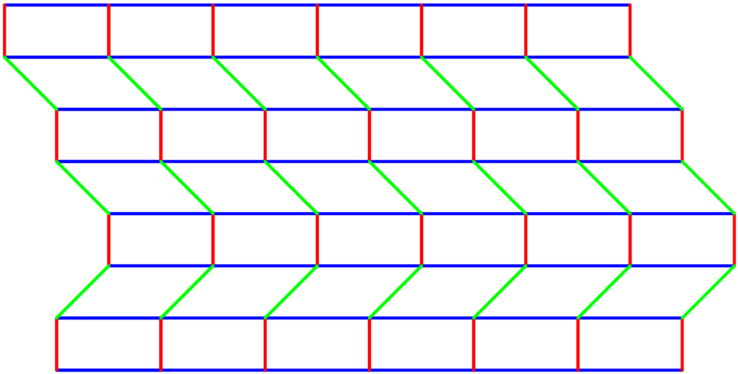

Figure : Tiling with one-dimensional translation group.

Idea of non face-to-face classification

Lemma

Assume every vertex corona of A contains n polygons one of which contains vertex on its side. And k_i of them that has A as a vertex are i-gons. Then

$$\sum \frac{k_i}{i} = \frac{n-2}{2}.$$

Properties of two-dimensional tilings

Claim

If a two-dimensional tiling $\mathcal{T}$ has a unique vertex corona then it could be non-periodic.

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But $\mathcal{T}$ always has at least one-dimensional translation group $G_{\mathcal{T}}$.

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But $\mathcal{T}$ always has at least one-dimensional translation group $G_\mathcal{T}$.

Claim

If $G_\mathcal{T}$ is one-dimensional then we will need to use corona $C_\mathcal{T}$ and its reflected image (rotations are not enough).

Further questions about tilings in arbitrary dimensions

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What is the minimal dimension of translation group $G_{\mathcal{T}}$ a tiling with unique vertex corona can have?

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Question

Can a tiling with unique "non-reflected" vertex corona be non-periodic?

Estimates for face-to-face tilings

Theorem (D. Frettlöh, A.G.)

There are d-dimensional face-to-face tilings with unique vertex corona with translation group of dimension at most $\frac{d}{2}$.

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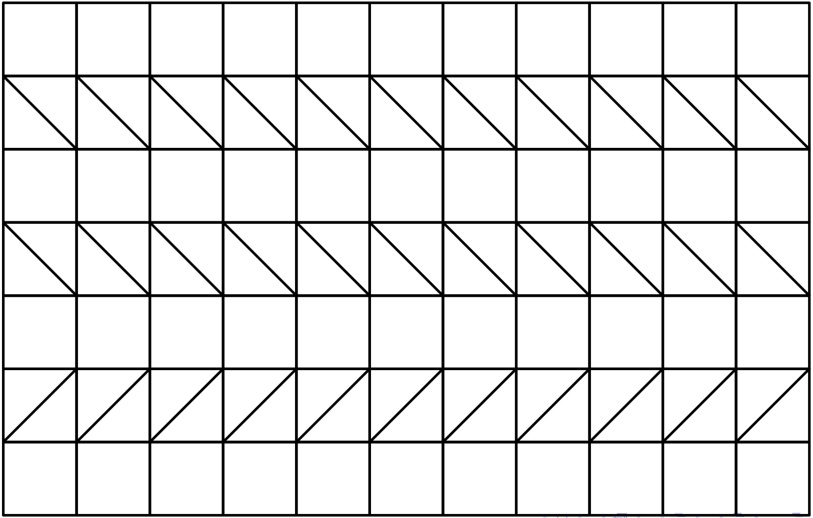
Theorem (D. Frettlöh, A.G.)

There are d-dimensional face-to-face tilings with unique vertex corona with translation group of dimension at most $\frac{d}{2}$.
But for now we need to use both reflections of corona.

Aperiodic non face-to-face tiling



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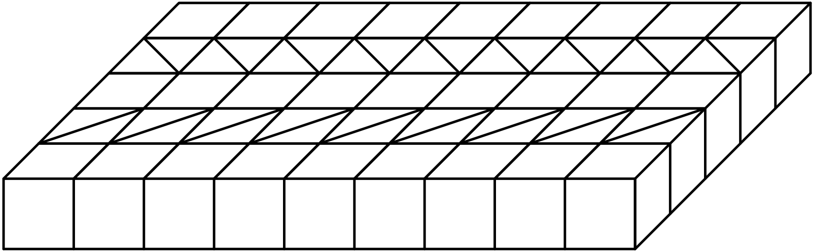

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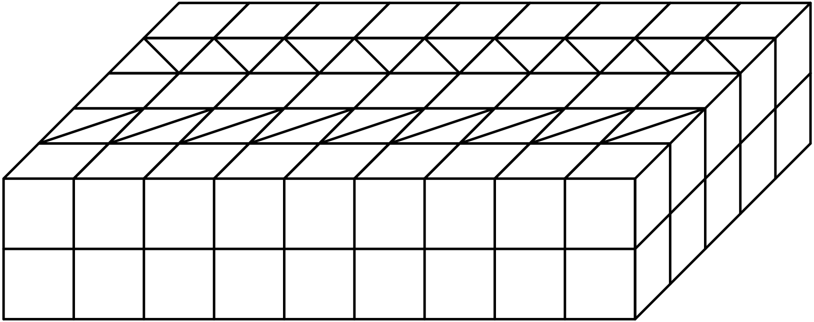

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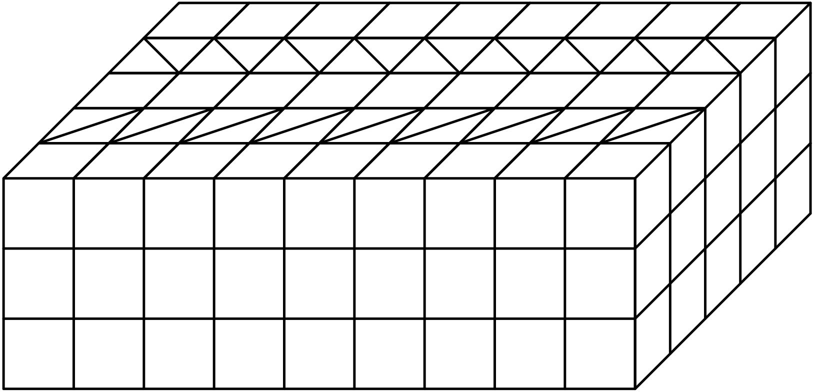
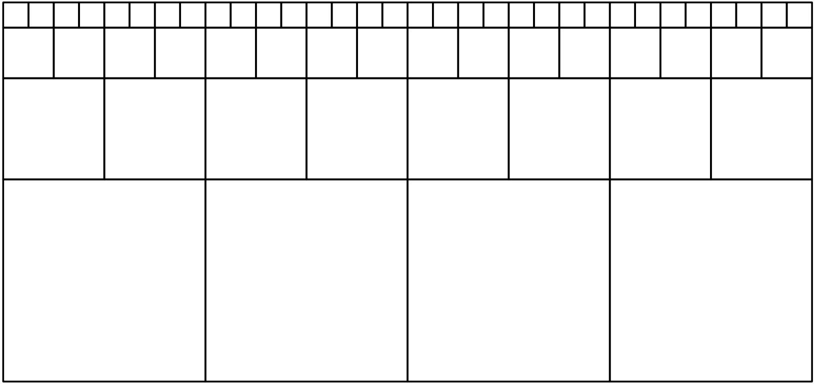
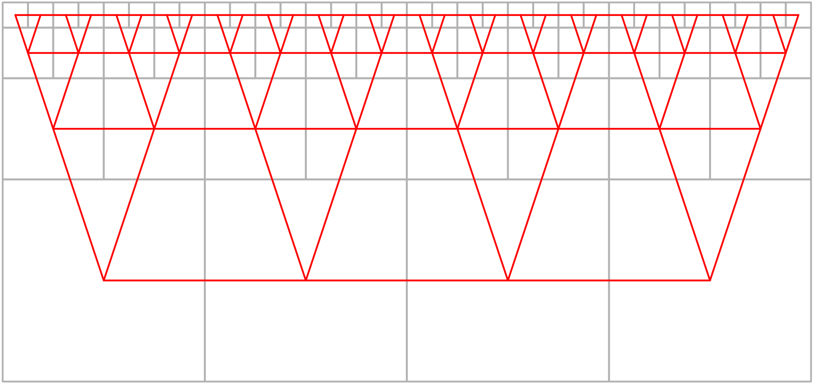


Böröczky tiling



Dual tiling is non-crystallographic



THANK YOU!