Belt diameter of some class of space filling zonotopes

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Polytopes with centrally symmetric facets

Consider a family $\mathcal{P}_C$ of all $d$-dimensional polytopes with centrally symmetric facets.

Theorem (A.D. Alexandrov anf G.C. Shephard)

*Every polytope from $\mathcal{P}_C$ is centrally symmetric itself.*

Let $P$ be any polytope from $\mathcal{P}_C$. 
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For any $(d - 2)$-dimensional face $F$ of $P$ we can define a *belt* $\mathcal{B}_P(F)$ as a set of all facets of $P$ parallel to $F$. We can construct it by moving from one facet to another across $(d - 2)$-faces parallel to $F$. 

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Belt diameter of zonotopes
Constructing belts
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![Diagram of a belt in a zonotope]

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Belt distance and belt diameter

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*Belt distance* between two facets of $P$ is the minimal number of belts that we need to travel from one facets to another.
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*Belt diameter* of $P$ is the maximal belt distance between its facets.
Why this is interesting?

Conjecture (G. Voronoi, 1909)

*Every convex polytope that tiles space with translation copies is affinely equivalent to Dirichlet-Voronoi polytope of some lattice.*
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One of the most popular (and one the most successful for now) approaches to prove the Voronoi conjecture in parallelohedra theory uses a canonical scaling function.

And values of canonical scaling can be uniquely defined on facets in one belt of length equal to 6.
Questions

Problem
What is the maximal belt diameter of $d$-dimensional polytope?

It is easy to see that answer for $d = 3$ is 2. We can add some restriction for maximal length of any belt or we can restrict ourselves only to certain subfamily of polytopes with centrally symmetric facets. Or both.
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The situation is much more interesting for those polytopes who has belts of length 4 or 6. It was proved that conditions to have centrally symmetric facets and belts of length at most 6 are necessary (H. Minkowski, 1897) and sufficient (B. Venkov, 1954) for convex polytope to be a parallelohedron, i.e. to tile \(\mathbb{R}^d\) with parallel copies.
Small belt lengths

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There are 5 three-dimensional parallelohedra, 52 four-dimensional and dozens of thousands of five-dimensional. And maximal belt diameter is unknown for all cases except $\mathbb{R}^3$. 
Zonotopes

Definition

A polytope is called a *zonotope* if it is a projection of cube. Or equivalently it is a Minkowski sum of finite number of segments. In that case we will denote this polytope as $Z(U)$ where $U$ is the correspondent vector set.
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The equivalent property is the following.

Theorem (P. McMullen, 1980)

For $d > 3$ a $d$-dimensional polytope is a zonotope if and only if all $(d - 2)$-faces of $P$ are centrally symmetric.
Questions about belt diameter of zonotopes

Problem

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**Claim**

If $k \geq d$ or if we do not restrict the maximal value of belt length then the answer is $d - 1$ and it is sharp.
An idea how to get an upper bound

**Definition**

Two sets $E$ and $F$ of $d - 1$ vectors in $\mathbb{R}^d$, each are called conjugated if
\[
\dim(E \cup f_i) = \dim(F \cup e_j) = d.
\]
Correspondent zonotope $Z(E \cup F)$ is called symmetric.

**Theorem (A.G.)**

The maximal belt diameter for zonotope is also achieved on some symmetric zonotope of smaller or equal dimension.
A new basis for symmetric zonotopes

Lemma

For space-filling zonotopes we can find an upper bound only for zonotopes with set of zone vectors

$$V = \left( \begin{array}{c|c} E_{d-1} & A \\ \hline 0 \ldots 0 & 1 \ldots 1 \end{array} \right),$$

where $A$ is a 0/1-matrix with at least half of zeros in each row.
Results on belt diameter of zonotopes

Theorem (A.G.)

*Belt diameter of $d$-dimensional space-filling zonotope is not greater than*

\[
\left\lceil \log_2 \left( \frac{4}{5} d \right) \right\rceil.
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Results on belt diameter of zonotopes

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Using similar technique for arbitrary belt length we can prove

Claim

Belt diameter of $d$-dimensional zonotope with belt length at most $k$ does not exceed

$$\left\lceil \log_1 + \frac{1}{k-2} \right\rceil d.$$
\textbf{Definition}

If $U$ is a subset of the vector set $E(d) = \{e_i - e_j\}_{i,j=1}^{d+1}$ where $e_i$ is the standard basis in $\mathbb{R}^{d+1}$ then zonotope $Z(U)$ is called a \textit{\Pi-zonotope}.
Π-zonotopes and their diameters

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**Theorem (A.G.)**

*Belt diameter of $d$-dimensional Π-zonotope is not greater than 2 if $d \leq 6$ and 3 if $d > 6$. These bounds are sharp in any dimension.*
THANK YOU!