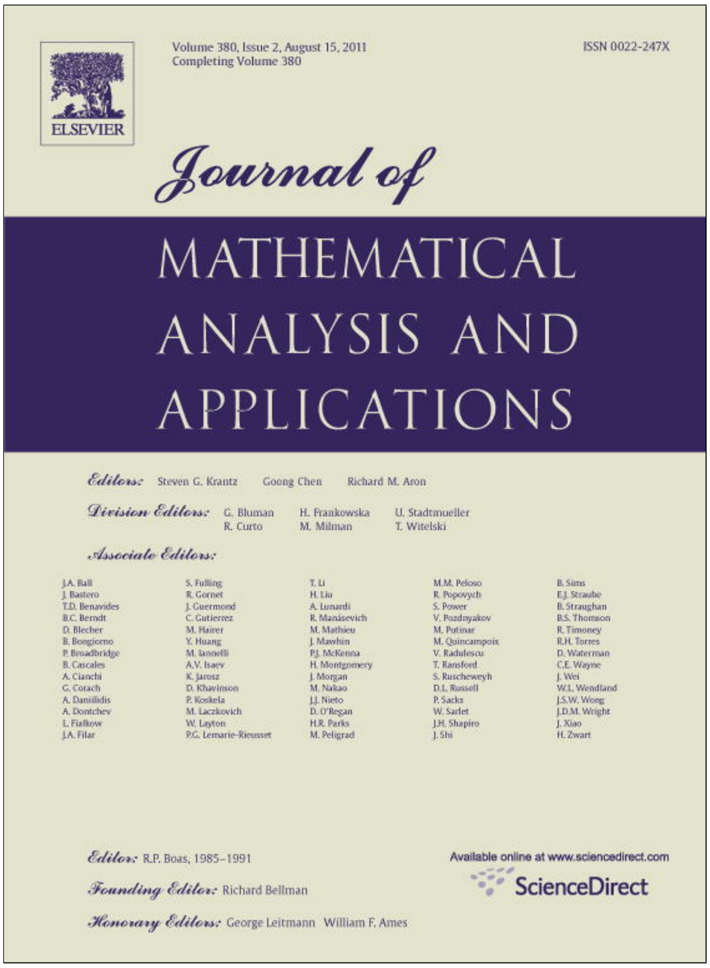
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Darboux transformation and explicit solutions for two integrable equations

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ABSTRACT

A new N-fold Darboux transformation for two integrable equations is constructed with the help of a gauge transformation for the spectral problem proposed by Qiao [Z.J. Qiao, Phys. Lett. A 192 (1994) 316–322]. By the Darboux transformation, explicit soliton and multi-soliton solutions for the two equations are obtained. In particular, soliton and complexiton solutions are shown through some figures.

1. Introduction

The investigation of the explicit solutions of integrable nonlinear evolution equations (NLEEs) plays an important role in the study of nonlinear physical phenomena. It is well known that there are many approaches to obtain explicit solutions of NLEEs, such as the Inverse Scattering Transform (IST) [1], the Hirota bilinear technique [2], the Bäcklund and Darboux transformation [3–7], the algebra-geometric approach [8–10] and so on [11–19]. Among them, the Darboux transformation (DT) is a powerful approach to get explicit soliton and multi-soliton solutions of integrable NLEEs. The key is to expose a kind of covariant properties that the corresponding spectral problems possess. There have been many tricks to do this for getting explicit solutions to various soliton equations including the Korteweg–de Vries (KdV) equation [3,4], the Kadomtsev–Petviashvili (KP) equation [3], the Davey–Stewartson (DS) equation [3], the sine-Gordon (SG) equation [3], the nonlinear Schrödinger (NLS) equation [3–5], the Boussinesq equation [6], the Nizhnik–Novikov–Veselov (NNV) equation [7], and so on [11–19].

In 1984, Neugebauer and Meinel developed a systematic method to construct an explicit determinant formula for an N-fold DT of the AKNS hierarchy [11]. The N-fold DT formula includes all N-soliton solutions of the AKNS hierarchy in a unified form. Moreover, the solutions of each equation in the AKNS hierarchy are reduced to solving a linear algebraic system, which is easy to generate various solutions by symbolic computation in the computer. Many multi-solitons of integrable NLEEs are obtained through this systematic method [11–19].

In this paper, we consider the spectral problem proposed by Qiao in 1994 [20]

$$\phi_x = U\phi, \qquad U = \begin{pmatrix} \lambda - \frac{1}{2}uv & u \\ \lambda v & -\lambda + \frac{1}{2}uv \end{pmatrix}, \tag{1.1}$$

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where $\phi = (\phi_1, \phi_2)^T$, u, v are two potentials, and λ is a spectral parameter. Eq. (1.1) is equivalent to the following form [20]

$$L \cdot \phi \equiv \begin{pmatrix} \partial + \frac{1}{2}uv & -u \\ v\partial + \frac{1}{2}uv^2 & -\partial - \frac{1}{2}uv \end{pmatrix} \cdot \phi = \lambda \phi. \tag{1.2}$$

A simple calculation leads to

$$L_{*w}(\xi) \triangleq \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} L(w + \varepsilon \xi) = \begin{pmatrix} \frac{u\xi_2 + v\xi_1}{2} & -\xi_1 \\ \frac{v(u\xi_2 + v\xi_1)}{2} & \frac{u\xi_2 - v\xi_1}{2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \xi_2 & 0 \end{pmatrix} L, \tag{1.3}$$

where $w = (u, v)^T$, $\xi = (\xi_1, \xi_2)^T$, and $L_{*w}(\xi)$ is an injective homomorphism.

The spectral problem (1.2) has the following recursion relation [20,21]

$$\mathcal{L} \cdot \nabla \lambda = \lambda \nabla \lambda, \tag{1.4}$$

where

$$\mathcal{L} = \frac{1}{2} \begin{pmatrix} -\partial - \partial v \partial^{-1} u + uv^2 \partial^{-1} u + uv & \partial v \partial^{-1} v - uv^2 \partial^{-1} v \\ \partial u \partial^{-1} u + u^2 v \partial^{-1} u & \partial - \partial u \partial^{-1} v - u^2 v \partial^{-1} v + uv \end{pmatrix} \tag{1.5}$$

and

$$\nabla \lambda \equiv \begin{pmatrix} \frac{\delta \lambda}{\delta u} \\ \frac{\delta \lambda}{\delta v} \end{pmatrix} = \begin{pmatrix} \phi_2^2 - v\phi_1\phi_2 \\ -\phi_1^2 - u\phi_1\phi_2 \end{pmatrix}. \tag{1.6}$$

The operator

$$J = -2 \begin{pmatrix} -u\partial^{-1} u & -1 + u\partial^{-1} v \\ 1 + v\partial^{-1} u & -v\partial^{-1} v \end{pmatrix} \tag{1.7}$$

yields the following integrable hierarchy [20,21]

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_m} = J\mathcal{L}^{m+1} \cdot \begin{pmatrix} v \\ u \end{pmatrix}, \quad m = 0, 1, 2, \ldots. \tag{1.8}$$

The first system in the hierarchy (1.8) is trivial. The second and third systems are

$$u_{t_1} = \frac{1}{2}u_{xx} + \frac{1}{2}u^2 v_x - \frac{1}{4}u^3 v^2, \qquad v_{t_1} = -\frac{1}{2}v_{xx} + \frac{1}{2}v^2 u_x + \frac{1}{4}v^3 u^2, \tag{1.9}$$

and

$$u_{t_2} = \frac{1}{4}u_{xxx} + \frac{3}{4}uu_x v_x - \frac{3}{8}u^2 v^2 u_x, \qquad v_{t_2} = \frac{1}{4}v_{xxx} - \frac{3}{4}vu_x v_x - \frac{3}{8}u^2 v^2 v_x. \tag{1.10}$$

Eqs. (1.9) and (1.10) are not special cases of the AKNS hierarchy because they have more high order nonlinear terms than the second and third members in the AKNS hierarchy. But, (1.9) is indeed gauge-equivalent to the Gerdjikov–Ivanov equation through some complex variable transformation [22–24], which has physical applications in nonlinear optics. However, (1.10) is a new integrable equation proposed in Qiao' work [20]. In our paper, we focus on Eqs. (1.9) and (1.10) and study their explicit solutions.

In Ref. [20], under a constraint between the potentials and the eigenfunctions, the spectral problem (1.1) was nonlinearized as a completely integrable finite-dimensional system in the Liouville sense. The Lax representations and the involutive solutions of the hierarchy (1.8) were also presented.

The aim of the present paper is to construct an N-fold DT of the spectral problem (1.1). As an application of the N-fold DT, multi-soliton and complexiton solutions of the two integrable equations (1.9) and (1.10) are obtained. In our results, the soliton and complexiton solutions are expressed in terms of exponential functions and combinations of trigonometric and exponential functions. All of solutions are real. One of potentials u, v could have a smooth soliton while the other has singularity. This means that we may have found an example of a 2-component integrable system having no smooth solitons and complexitons.

Our paper is organized as follows. In Section 2, an N-fold DT for the integrable equations (1.9) and (1.10) is constructed. In Section 3, their multi-solitons are derived from the N-fold DT. In Section 4, new complexiton solutions of the integrable equations (1.9) and (1.10) are given. The paper is concluded by summarizing the results in Section 5.

2. Darboux transformation

In order to construct a DT of the integrable equations (1.9) and (1.10), let us first recover their Lax representations from [20]:

The Lax representation of (1.9) is composed of the spectral problem (1.1) and the following auxiliary problem

$$\phi_{t_1} = V^{(2)}\phi, \qquad V^{(2)} = \begin{pmatrix} M_{11}^{(1)} & u\lambda + \frac{1}{2}u_x \\ v\lambda^2 - \frac{1}{2}v_x\lambda & -M_{11}^{(1)} \end{pmatrix}, \tag{2.1}$$

where $M_{11}^{(1)} = \lambda^2 - \frac{1}{2}uv\lambda + \frac{1}{8}(-u^2v^2 - 2vu_x + 2uv_x)$.

The Lax representation of (1.10) is given by the spectral problem (1.1) and the following auxiliary problem

$$\phi_{t_2} = V^{(3)}\phi, \qquad V^{(3)} = \begin{pmatrix} M_{11}^{(2)} & M_{12}^{(2)} \\ M_{21}^{(2)} & -M_{11}^{(2)} \end{pmatrix}, \tag{2.2}$$

where

$$\begin{aligned} M_{11}^{(2)} &= \lambda^3 - \frac{1}{2}uv\lambda^2 - \frac{1}{8}\left(u^2v^2 + 2vu_x - 2uv_x\right)\lambda + \frac{1}{16}\left(u^3v^3 + 2u_xv_x - 2vu_{xx} - 2uv_{xx}\right), \\ M_{12}^{(2)} &= u\lambda^2 + \frac{1}{2}u_x\lambda + \frac{1}{4}\left(uvu_x - \frac{1}{2}u\left(u^2v^2 + 2vu_x - 2uv_x\right) + u_{xx}\right), \\ M_{21}^{(2)} &= v\lambda^3 - \frac{1}{2}v_x\lambda^2 + \frac{1}{4}\left(-uvv_x - \frac{1}{2}v\left(u^2v^2 + 2vu_x - 2uv_x\right) + v_{xx}\right)\lambda. \end{aligned}$$

In fact, a direct calculation shows that the zero curvature equation $U_{t_\kappa} - V_x^{(\kappa+1)} + [U, V^{(\kappa+1)}] = 0$ ($\kappa = 1, 2$), implies the nonlinear equations (1.9) ($\kappa = 1$) and (1.10) ($\kappa = 2$), respectively. Therefore, (1.9) and (1.10) are integrable in the sense of Lax pair.

Based on the Lax pairs ((1.1), (2.1)) and ((1.1), (2.2)) for the integrable equations (1.9) and (1.10), let us consider the following gauge transformation:

$$\bar{\phi} = T\phi, \tag{2.3}$$

where T is a nonsingular matrix and $\bar{\phi}$ admits the form of the Lax pairs ((1.1), (2.1)) and ((1.1), (2.2))

$$\bar{\phi}_x = \overline{U}\bar{\phi}, \qquad \overline{U} = (T_x + TU)T^{-1}, \tag{2.4}$$

$$\bar{\phi}_{t_1} = \overline{V}^{(2)}\bar{\phi}, \qquad \overline{V}^{(2)} = \left(T_{t_1} + TV^{(2)}\right)T^{-1}, \tag{2.5}$$

$$\bar{\phi}_{t_2} = \overline{V}^{(3)}\bar{\phi}, \qquad \overline{V}^{(3)} = \left(T_{t_2} + TV^{(3)}\right)T^{-1}. \tag{2.6}$$

Differentiating ((2.4), (2.5)) or ((2.4), (2.6)) yields

$$\overline{U}_{t_\kappa} - \overline{V}_x^{(\kappa+1)} + \left[\overline{U}, \overline{V}^{(\kappa+1)}\right] = T\left(U_{t_\kappa} - V_x^{(\kappa+1)} + \left[U, V^{(\kappa+1)}\right]\right)T^{-1} \quad (\kappa = 1, 2). \tag{2.7}$$

In order to make the integrable equations (1.9) and (1.10) invariant under the transformation (2.3), it is crucial to find an appropriate matrix T such that $\overline{U}$, $\overline{V}^{(\kappa+1)}$ have the same forms as $U, V^{(\kappa+1)}$. The old potentials u and v in $U, V^{(\kappa+1)}$ are mapped into the new potentials $\bar{u}$ and $\bar{v}$ in $\overline{U}, \overline{V}^{(\kappa+1)}$ simultaneously.

Suppose that the Darboux matrix T in (2.3) is in the form of

$$T = T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \tag{2.8}$$

where

$$A(\lambda) = \lambda^N + \sum_{k=0}^{N-1} A_k\lambda^k, \qquad B(\lambda) = \sum_{k=0}^{N-1} B_k\lambda^k, \tag{2.9}$$

$$C(\lambda) = \sum_{k=0}^{N-1} C_k\lambda^{k+1}, \qquad D(\lambda) = \lambda^N + \sum_{k=0}^{N-1} D_k\lambda^k, \tag{2.10}$$

A_k, B_k, C_k and D_k ($0 \leqslant k \leqslant N-1$) are functions of x and t.

Let $\phi(\lambda_j) = (\varphi_1(\lambda_j), \varphi_2(\lambda_j))^T$, $\psi(\lambda_j) = (\psi_1(\lambda_j), \psi_2(\lambda_j))^T$ be two basic solutions of (1.1) and (2.1) (or (2.2)). In the following, we discuss ((2.4), (2.5)) and ((2.4), (2.6)), separately.

For ((2.4), (2.5)), there exist constants r_j $(1 \leqslant j \leqslant 2N)$ such that

$$\big(A(\lambda_j)\varphi_1(\lambda_j) + B(\lambda_j)\varphi_2(\lambda_j)\big) - r_j\big(A(\lambda_j)\psi_1(\lambda_j) + B(\lambda_j)\psi_2(\lambda_j)\big) = 0, \tag{2.11}$$

$$\big(C(\lambda_j)\varphi_1(\lambda_j) + D(\lambda_j)\varphi_2(\lambda_j)\big) - r_j\big(C(\lambda_j)\psi_1(\lambda_j) + D(\lambda_j)\psi_2(\lambda_j)\big) = 0. \tag{2.12}$$

Furthermore, (2.11) and (2.12) can be rewritten as a linear algebraic system

$$A(\lambda_j) + \delta_j B(\lambda_j) = 0, \qquad C(\lambda_j) + \delta_j D(\lambda_j) = 0, \tag{2.13}$$

or

$$\sum_{k=0}^{N-1}(A_k + \delta_j B_k)\lambda_j^k = -\lambda_j^N, \qquad \sum_{k=0}^{N-1}(C_k\lambda_j + \delta_j D_k)\lambda_j^k = -\delta_j\lambda_j^N, \tag{2.14}$$

with

$$\delta_j = \frac{\varphi_2(\lambda_j) - r_j\psi_2(\lambda_j)}{\varphi_1(\lambda_j) - r_j\psi_1(\lambda_j)}, \quad 1 \leqslant j \leqslant 2N. \tag{2.15}$$

If one takes care of a proper selection of constants λ_j, r_j ($\lambda_k \neq \lambda_j$ as $k \neq j$), then the determinant of coefficients for (2.14) is non-zero. Therefore, A_k, B_k, C_k and D_k $(0 \leqslant k \leqslant N-1)$ are uniquely determined by (2.14).

Eq. (2.8) shows that $\det T(\lambda)$ is a $2N$th-order polynomial in λ, and

$$\det T(\lambda_j) = A(\lambda_j)D(\lambda_j) - B(\lambda_j)C(\lambda_j). \tag{2.16}$$

On the other hand, from (2.13) we have

$$A(\lambda_j) = -\delta_j B(\lambda_j), \qquad C(\lambda_j) = -\delta_j D(\lambda_j). \tag{2.17}$$

Therefore, we obtain

$$\det T(\lambda_j) = 0, \tag{2.18}$$

which implies that λ_j $(1 \leqslant j \leqslant 2N)$ are $2N$ roots of the $2N$th degree polynomial $\det T(\lambda)$, namely,

$$\det T(\lambda) = \prod_{j=1}^{2N}(\lambda - \lambda_j). \tag{2.19}$$

Using the above facts, we are able to prove the following proposition.

Proposition 1. *The matrix $\overline{U}$ given by* (2.4) *has the same form as U:*

$$\overline{U} = \begin{pmatrix} \lambda - \frac{1}{2}\bar{u}\bar{v} & \bar{u} \\ \lambda\bar{v} & -\lambda + \frac{1}{2}\bar{u}\bar{v} \end{pmatrix}$$

where

$$\bar{u} = u - 2B_{N-1}, \qquad \bar{v} = v + 2C_{N-1}, \tag{2.20}$$

and

$$\begin{aligned}
A_{N-m,x} &= \frac{1}{2}(2\bar{u}C_{N-m-1} - 2vB_{N-m-1} + uvA_{N-m} - \bar{u}\bar{v}A_{N-m}),\\
B_{N-m,x} &= \frac{1}{2}(4B_{N-m-1} - 2uA_{N-m} + 2\bar{u}D_{N-m} - uvB_{N-m} - \bar{u}\bar{v}B_{N-m}),\\
C_{N-m,x} &= \frac{1}{2}(-4C_{N-m-1} - 2vD_{N-m} + uvC_{N-m} + 2\bar{v}A_{N-m} + \bar{u}\bar{v}C_{N-m}),\\
D_{N-m,x} &= \frac{1}{2}(-2uC_{N-m-1} - uvD_{N-m} + 2\bar{v}B_{N-m-1} + \bar{u}\bar{v}D_{N-m}) \quad (m = 1, 2, \ldots, N),
\end{aligned} \tag{2.21}$$

with $B_{-1} = C_{-1} = 0$.

Proof. Let $T^{-1} = T^*/\det T$ and

$$(T_x + TU)T^* = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix}. \tag{2.22}$$

It is easy to see that $f_{11}(\lambda)$, $f_{22}(\lambda)$ and $f_{21}(\lambda)$ are three $(2N+1)$th degree polynomials in λ, while $f_{12}(\lambda)$ is a $2N$th degree polynomial. From (1.1) and (2.15), we find

$$\delta_{jx} = \lambda_j v + 2\left(-\lambda_j + \frac{1}{2}uv\right)\delta_j - u\delta_j^2. \tag{2.23}$$

Through a direct calculation, we know that all λ_j $(1 \leqslant j \leqslant 2N)$ are roots of $f_{ni}(\lambda)$ $(n, i = 1, 2)$. Therefore, (2.22) reads as

$$(T_x + TU)T^* = (\det T)P(\lambda), \tag{2.24}$$

where

$$P(\lambda) = \begin{pmatrix} p_{11}^{(1)}\lambda + p_{11}^{(0)} & p_{12}^{(0)} \\ p_{21}^{(1)}\lambda + p_{21}^{(0)} & p_{22}^{(1)}\lambda + p_{22}^{(0)} \end{pmatrix}, \tag{2.25}$$

and $p_{ni}^{(s)}$ $(n, i = 1, 2,\ s = 0, 1)$ are independent of λ. So, we obtain

$$(T_x + TU) = P(\lambda)T. \tag{2.26}$$

Comparing the coefficients of λ^{N+1} and λ^N on both sides of (2.26) leads to

$$\lambda^{N+1}:\quad p_{11}^{(1)} = -p_{22}^{(1)} = 1, \qquad p_{21}^{(1)} = v + 2C_{N-1} = \bar{v}, \tag{2.27}$$

$$\lambda^N:\quad p_{12}^{(0)} = u - 2B_{N-1} = \bar{u}, \tag{2.28}$$

$$\begin{aligned} p_{11}^{(0)} = -p_{22}^{(0)} &= \frac{1}{2}(4B_{N-1}C_{N-1} - 2uC_{N-1} + 2vB_{N-1} - uv) \\ &= -\frac{1}{2}(u - 2B_{N-1})(v + 2C_{N-1}) = -\frac{1}{2}\bar{u}\bar{v}, \end{aligned} \tag{2.29}$$

$$\begin{aligned} p_{21}^{(0)} &= 2C_{N-2} + 2B_{N-1}C_{N-1}^2 - uC_{N-1}^2 + vB_{N-1}C_{N-1} + vD_{N-1} - uvC_{N-1} \\ &\quad - (v + 2C_{N-1})A_{N-1} + C_{N-1,x}. \end{aligned} \tag{2.30}$$

Substituting (2.20) and (2.21) into (2.30) produces

$$p_{21}^{(0)} = 0.$$

From (1.1) and (2.25), we see $\overline{U} = P(\lambda)$. The proof is thus complete. □

Proposition 2. *The matrix $\overline{V}^{(2)}$ given by* (2.5) *has the same form as $V^{(2)}$, where original potentials u and v are mapped into new potentials $\bar{u}$ and $\bar{v}$ through same transformation* (2.21)*.*

Proof. Let $T^{-1} = T^*/\det T(\lambda)$ and

$$(T_{t_1} + TV)T^* = \begin{pmatrix} g_{11}(\lambda) & g_{12}(\lambda) \\ g_{21}(\lambda) & g_{22}(\lambda) \end{pmatrix}. \tag{2.31}$$

One can easily see that $g_{11}(\lambda)$, $g_{22}(\lambda)$, $g_{21}(\lambda)$ are three $(2N+2)$th degree polynomials and $g_{12}(\lambda)$ is a $(2N+1)$th degree polynomial in λ. When $\lambda = \lambda_j$ $(1 \leqslant j \leqslant 2N)$, using (2.1) and (2.15), we obtain a Riccati equation

$$\delta_{jt_1} = v\lambda_j^2 - \frac{1}{2}v_x\lambda_j - 2\left(\lambda_j^2 - \frac{1}{2}uv\lambda_j + \frac{1}{8}\left(-u^2v^2 - 2vu_x + 2uv_x\right)\right)\delta_j - \left(u\lambda_j + \frac{1}{2}u_x\right)\delta_j^2. \tag{2.32}$$

Obviously, all λ_j $(1 \leqslant j \leqslant 2N)$ are roots of $g_{ni}(\lambda)$ $(n, i = 1, 2)$. Furthermore, we have

$$T_{t_1} + TV = Q(\lambda)T, \tag{2.33}$$

where

$$Q(\lambda) = \begin{pmatrix} q_{11}^{(2)}\lambda^2 + q_{11}^{(1)}\lambda + q_{11}^{(0)} & q_{12}^{(1)}\lambda + q_{12}^{(0)} \\ q_{21}^{(1)}\lambda^2 + q_{21}^{(0)}\lambda & q_{22}^{(2)}\lambda^2 + q_{22}^{(1)}\lambda + q_{22}^{(0)} \end{pmatrix},$$

and $q_{ni}^{(s)}$ $(n, i = 1, 2,\ s = 0, 1, 2)$ are independent of λ. So, we obtain

$$T_{t_1} + TV = Q(\lambda)T. \tag{2.34}$$

Comparing the coefficients of $\lambda^{N+2}, \lambda^{N+1}$ and λ^N on both sides of Eq. (2.34) leads to

$$\lambda^{N+2}\colon\ q_{11}^{(2)} = -q_{22}^{(2)} = 1, \qquad q_{21}^{(1)} = v + 2C_{N-1} = \bar{v}, \tag{2.35}$$

$$\lambda^{N+1}\colon\ q_{12}^{(1)} = u - 2B_{N-1} = \bar{u}, \tag{2.36}$$

$$q_{11}^{(1)} = -q_{22}^{(1)} = \frac{1}{2}(4B_{N-1}C_{N-1} - 2uC_{N-1} + 2vB_{N-1} - uv) = -\frac{1}{2}\bar{u}\bar{v}, \tag{2.37}$$

$$\begin{aligned} q_{21}^{(0)} = \frac{1}{2}\big(&4C_{N-2} - 4A_{N-1}C_{N-1} + 4B_{N-1}C_{N-1}^2 - 2uC_{N-1}^2 - 2vA_{N-1} \\ &+ 2vB_{N-1}C_{N-1} + 2vD_{N-1} - 2uvC_{N-1} - v_x\big), \end{aligned} \tag{2.38}$$

$$\begin{aligned} \lambda^N\colon\ q_{12}^{(0)} = \frac{1}{2}\big(&-4B_{N-1} - 4B_{N-1}^2C_{N-1} + 4B_{N-1}D_{N-1} + 2uA_{N-1} + 2uB_{N-1}C_{N-1} \\ &- 2uD_{N-1} - 2vB_{N-1}^2 + 2uvB_{N-1} + u_x\big), \end{aligned} \tag{2.39}$$

$$\begin{aligned} q_{11}^{(0)} = -q_{22}^{(0)} = \frac{1}{8}\big(&16B_{N-1}C_{N-2} + 16B_{N-2}C_{N-1} - 16A_{N-1}B_{N-1}C_{N-1} + 16B_{N-1}^2C_{N-1}^2 - 16B_{N-1}C_{N-1}D_{N-1} \\ &- 8uC_{N-2} - 8uB_{N-1}C_{N-1}^2 + 8uC_{N-1}D_{N-1} + 8vB_{N-2} - 8vA_{N-1}B_{N-1} + 8vB_{N-1}C_{N-1} \\ &- 8uvB_{N-1}C_{N-1} - u^2 - 4u_xC_{N-1}v^2 - 2vu_x - 4v_xB_{N-1} + 2uv_x\big). \end{aligned} \tag{2.40}$$

After substituting (2.20) and (2.21) into (2.38), (2.39) and (2.40), we arrive at

$$q_{21}^{(0)} = \frac{1}{2}\bar{v}_x, \qquad q_{12}^{(1)} = \frac{1}{2}\bar{u}_x, \qquad q_{11}^{(0)} = -q_{22}^{(0)} = \frac{1}{8}\big(-\bar{u}^2\bar{v}^2 - 2\bar{v}\bar{u}_x + 2\bar{u}\bar{v}_x\big).$$

This completes the proof. □

On the other hand, starting from ((2.4), (2.6)) and using the way similar to the proof of Propositions 1 and 2, we are able to obtain the following results.

Proposition 3. *The matrix $\overline{V}^{(3)}$ given by* (2.6) *has the same form as $V^{(3)}$, where original potentials u and v are mapped into new potentials $\bar{u}$ and $\bar{v}$ through the same transformation* (2.20)*.*

Propositions 1–3 show that the transformation (2.20) mapped the Lax pairs ((1.1), (2.1)) and ((1.1), (2.2)) into another set of Lax pairs ((2.4), (2.5)) and ((2.4), (2.6)) with the same format, respectively. Therefore, both of the Lax pairs yield the same Eqs. (1.9) and (1.10). So, the transformation (2.20) is a DT of the integrable equations (1.9) and (1.10). From the above three propositions, we have the following theorem.

Theorem 1. *Under the DT* (2.20)*, the integrable equations* (1.9) *and* (1.10) *admit the following solutions*

$$\bar{u} = u - 2\frac{\Delta_{B_{N-1}}}{\Delta_{N-1}}, \qquad \bar{v} = v + 2\frac{\Omega_{C_{N-1}}}{\Omega_{N-1}}, \tag{2.41}$$

where Δ_{N-1} is the determinant of the coefficients for the first linear algebraic system of (2.14):

$$\Delta_{N-1} = \begin{vmatrix} 1 & \delta_1 & \lambda_1 & \delta_1\lambda_1 & \cdots & \lambda_1^{N-1} & \delta_1\lambda_1^{N-1} \\ 1 & \delta_2 & \lambda_2 & \delta_2\lambda_2 & \cdots & \lambda_2^{N-1} & \delta_2\lambda_2^{N-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \delta_{2N-1} & \lambda_{2N-1} & \delta_{2N-1}\lambda_{2N-1} & \cdots & \lambda_{2N-1}^{N-1} & \delta_{2N-1}\lambda_{2N-1}^{N-1} \\ 1 & \delta_{2N} & \lambda_{2N} & \delta_{2N}\lambda_{2N} & \cdots & \lambda_{2N}^{N-1} & \delta_{2N}\lambda_{2N}^{N-1} \end{vmatrix},$$

Ω_{N-1} is the determinant of the coefficients for the second linear algebraic system of (2.14):

$$\Omega_{N-1} = \begin{vmatrix} \lambda_1 & \delta_1 & \lambda_1^2 & \delta_1\lambda_1 & \cdots & \lambda_1^N & \delta_1\lambda_1^{N-1} \\ \lambda_2 & \delta_2 & \lambda_2^2 & \delta_2\lambda_2 & \cdots & \lambda_2^N & \delta_2\lambda_2^{N-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_{2N-1} & \delta_{2N-1} & \lambda_{2N-1}^2 & \delta_{2N-1}\lambda_{2N-1} & \cdots & \lambda_{2N-1}^N & \delta_{2N-1}\lambda_{2N-1}^{N-1} \\ \lambda_{2N} & \delta_{2N} & \lambda_{2N}^2 & \delta_{2N}\lambda_{2N} & \cdots & \lambda_{2N}^N & \delta_{2N}\lambda_{2N}^{N-1} \end{vmatrix},$$

$\Delta_{B_{N-1}}$ is produced from Δ_{N-1} by replacing its $2N$th column with $(-\lambda_1^N, \ldots, -\lambda_{2N}^N)^T$, $\Omega_{C_{N-1}}$ is produced from Ω_{N-1} by replacing its $(2N-1)$th column with $(-\delta_1\lambda_1^N, \ldots, -\delta_{2N}\lambda_{2N}^N)^T$, δ_j $(1 \leqslant j \leqslant 2N)$ are given by (2.15) and λ_j $(1 \leqslant j \leqslant 2N)$ are spectral parameters.

In the next two sections, we will construct the soliton and complexiton solutions of the integrable equations (1.9) and (1.10) according to Theorem 1.

3. Solitons

Substituting trivial solutions $u = v = 0$ into the Lax pairs (1.1) and (2.1), we choose the following basic solutions

$$\phi(\lambda_j) = \begin{pmatrix} \exp(\lambda_j(x+\lambda_j t)) \\ 0 \end{pmatrix}, \qquad \psi(\lambda_j) = \begin{pmatrix} 0 \\ \exp(-\lambda_j(x+\lambda_j t)) \end{pmatrix}.$$

According to (2.15), we have

$$\delta_j^{(1)} = -r_j \exp\bigl(-2\lambda_j(x+\lambda_j t)\bigr), \quad 1 \leqslant j \leqslant 2N. \tag{3.1}$$

Substituting trivial solutions $u = v = 0$ into the Lax pairs (1.1) and (2.2), we choose the following basic solutions

$$\phi(\lambda_j) = \begin{pmatrix} \exp(\lambda_j(x+\lambda_j^2 t)) \\ 0 \end{pmatrix}, \qquad \psi(\lambda_j) = \begin{pmatrix} 0 \\ \exp(-\lambda_j(x+\lambda_j^2 t)) \end{pmatrix}.$$

According to (2.15), we have

$$\delta_j^{(2)} = -r_j \exp\bigl(-2\lambda_j\bigl(x+\lambda_j^2 t\bigr)\bigr), \quad 1 \leqslant j \leqslant 2N. \tag{3.2}$$

For simplicity, let us discuss soliton solutions of the integrable equations (1.9) and (1.10) in the special case of $N = 1$. Solving the linear algebraic system (2.14) with $\lambda = \lambda_j$ $(j = 1, 2)$ leads to

$$B_0 = \frac{\Delta_{B_0}}{\Delta_0}, \qquad C_0 = \frac{\Omega_{C_0}}{\Omega_0}, \tag{3.3}$$

where

$$\Delta_0 = \begin{vmatrix} 1 & \delta_1^{(\kappa)} \\ 1 & \delta_2^{(\kappa)} \end{vmatrix}, \qquad \Omega_0 = \begin{vmatrix} \lambda_1 & \delta_1^{(\kappa)} \\ \lambda_2 & \delta_2^{(\kappa)} \end{vmatrix}, \qquad \Delta_{B_0} = \begin{vmatrix} 1 & -\lambda_1 \\ 1 & -\lambda_2 \end{vmatrix}, \qquad \Omega_{C_0} = \begin{vmatrix} -\delta_1^{(\kappa)}\lambda_1 & \delta_1^{(\kappa)} \\ -\delta_2^{(\kappa)}\lambda_2 & \delta_2^{(\kappa)} \end{vmatrix}.$$

According to (2.41), we obtain a solution of the integrable equations (1.9) and (1.10)

$$\bar{u} = u - 2\frac{\Delta_{B_0}}{\Delta_0} = u - 2\frac{\lambda_1 - \lambda_2}{\delta_2^{(\kappa)} - \delta_1^{(\kappa)}}, \tag{3.4}$$

$$\bar{v} = v + 2\frac{\Omega_{C_0}}{\Omega_0} = v + 2\frac{(\lambda_2 - \lambda_1)\delta_1^{(\kappa)}\delta_2^{(\kappa)}}{\lambda_1\delta_2^{(\kappa)} - \lambda_2\delta_1^{(\kappa)}} \quad (\kappa = 1, 2), \tag{3.5}$$

where $\delta_j^{(\kappa)}$ $(\kappa = 1, 2;\ j = 1, 2)$ are given by (3.1) or (3.2). Parameters r_j, λ_j $(j = 1, 2)$ are properly chosen so that the denominators in (3.4) and (3.5) are non-zero.

As $\kappa = 1$, Eqs. (3.4) and (3.5) give explicit solutions of the integrable equation (1.9):

$$\begin{aligned} \bar{u} &= -\frac{2(\lambda_1 - \lambda_2)}{r_1\exp(-2\lambda_1(x+\lambda_1 t)) - r_2\exp(-2\lambda_2(x+\lambda_2 t))}, \\ \bar{v} &= \frac{2r_1r_2(\lambda_2 - \lambda_1)\exp(-2(\lambda_1+\lambda_2)x - 2(\lambda_1^2+\lambda_2^2)t)}{-r_2\lambda_1\exp(-2\lambda_2(x+\lambda_2 t)) + r_1\lambda_2\exp(-2\lambda_1(x+\lambda_1 t))}. \end{aligned} \tag{3.6}$$

In order to obtain one-soliton solution of the integrable equation (1.9), the following two especial cases are discussed under the condition $\lambda_2 = -\lambda_1$.

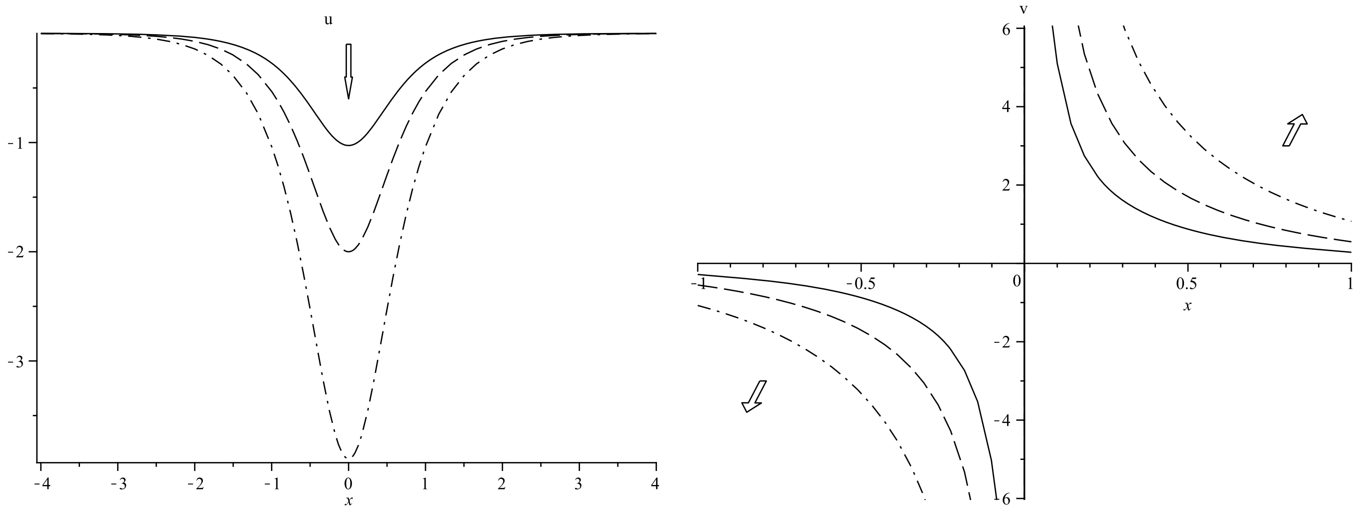

Fig. 1. One-soliton solution (3.7) with $r_1 = -r_2 = \lambda_1 = -\lambda_2 = 1$, the solid line (—) is wave elevation at $t = -\frac{1}{3}$, the evenly dashed line (– – –) is for wave elevation at $t = 0$, and unevenly dashed line (– · – · –) is for wave elevation at $t = \frac{1}{3}$.

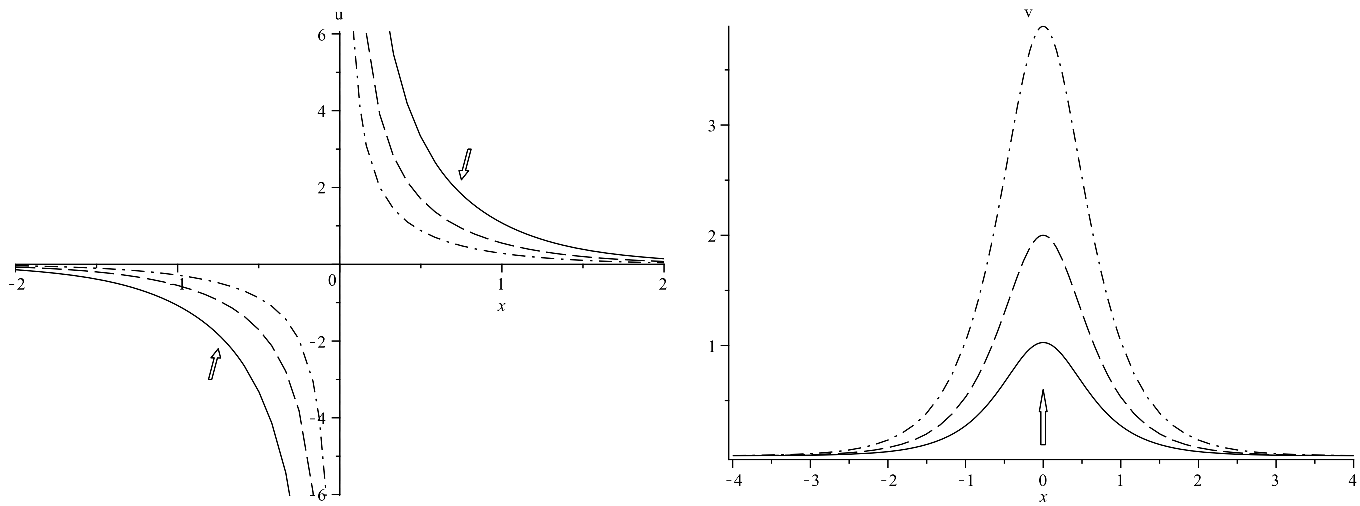

Fig. 2. One-soliton solution (3.8) with $r_1 = r_2 = \lambda_1 = -\lambda_2 = 1$, the solid line (—) is wave elevation at $t = -\frac{1}{3}$, the evenly dashed line (– – –) is for wave elevation at $t = 0$ and unevenly dashed line (– · – · –) is for wave elevation at $t = \frac{1}{3}$.

Case 1. When $r_2 = -r_1$, (3.6) becomes one-soliton solution

$$\bar{u} = -\frac{2\lambda_1}{r_1}\exp\bigl(2\lambda_1^2 t\bigr)\operatorname{sech}(2\lambda_1 x), \qquad \bar{v} = 2r_1\exp\bigl(-2\lambda_1^2 t\bigr)\operatorname{csch}(2\lambda_1 x), \tag{3.7}$$

and their graphs are shown in Fig. 1.

Case 2. When $r_2 = r_1$, (3.6) becomes one-soliton solution

$$\bar{u} = \frac{2\lambda_1}{r_1}\exp\bigl(2\lambda_1^2 t\bigr)\operatorname{csch}(2\lambda_1 x), \qquad \bar{v} = 2r_1\exp\bigl(-2\lambda_1^2 t\bigr)\operatorname{sech}(2\lambda_1 x), \tag{3.8}$$

and their graphs are shown in Fig. 2.

For $\kappa = 2$, (3.4) and (3.5) give explicit solutions of the integrable equation (1.10):

$$\begin{aligned}
\bar{u} &= -\frac{2(\lambda_1 - \lambda_2)}{r_1\exp(-2\lambda_1(x + \lambda_1^2 t)) - r_2\exp(-2\lambda_2(x + \lambda_2^2 t))}, \\
\bar{v} &= \frac{2r_1 r_2(\lambda_2 - \lambda_1)\exp(-2(\lambda_1 + \lambda_2)x - 2(\lambda_1^3 + \lambda_2^3)t)}{-r_2\lambda_1\exp(-2\lambda_2(x + \lambda_2^2 t)) + r_1\lambda_2\exp(-2\lambda_1(x + \lambda_1^2 t))}.
\end{aligned} \tag{3.9}$$

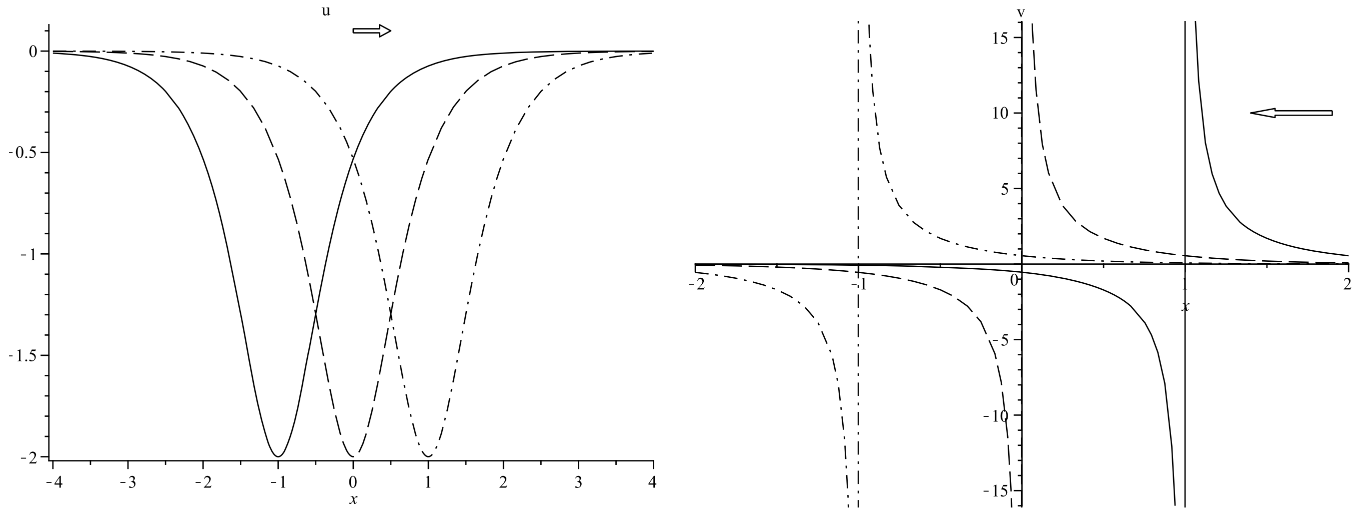

Fig. 3. One-soliton solution (3.10) with $r_1 = -r_2 = \lambda_1 = -\lambda_2 = 1$, the solid line (—) is wave elevation at $t = -1$, the evenly dashed line (− − −) is for wave elevation at $t = 0$, and unevenly dashed line (− · − · −) is for wave elevation at $t = 1$.

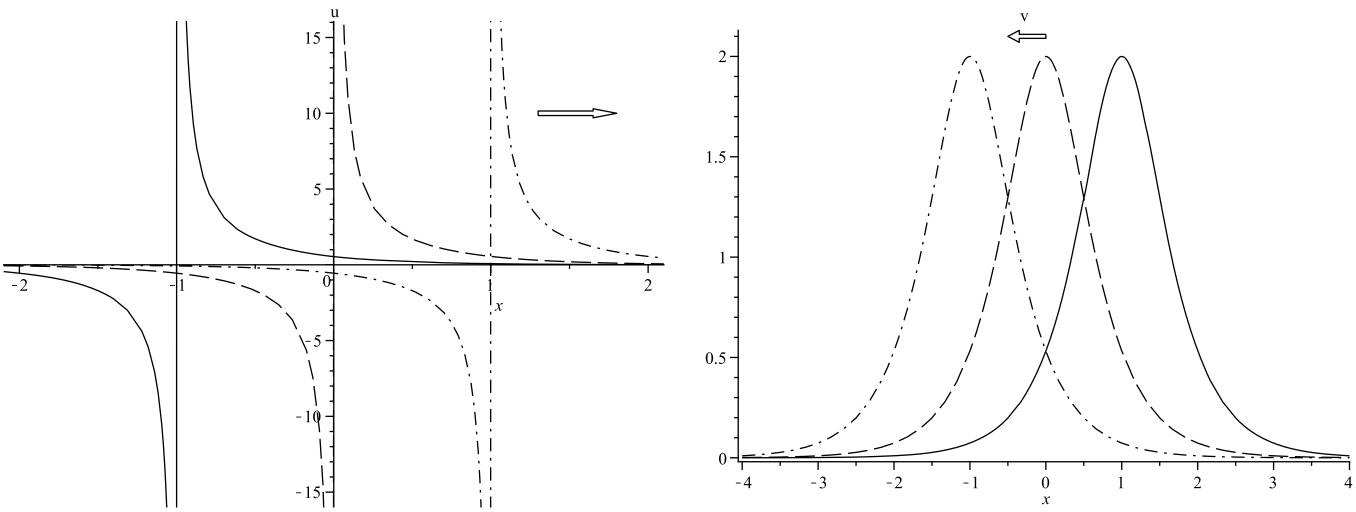

Fig. 4. One-soliton solution (3.11) with $r_1 = r_2 = \lambda_1 = -\lambda_2 = 1$, the solid line (—) is wave elevation at $t = -1$, the evenly dashed line (− − −) is for wave elevation at $t = 0$, and unevenly dashed line (− · − · −) is for wave elevation at $t = 1$.

In order to obtain one-soliton solution of the integrable equation (1.10), the following two especial cases are discussed under the condition $\lambda_2 = -\lambda_1$.

Case 1. When $r_2 = -r_1$, (3.9) becomes one-soliton solution

$$\bar{u} = -\frac{2\lambda_1}{r_1}\operatorname{sech}\big(2\lambda_1\big(x + \lambda_1^2 t\big)\big), \qquad \bar{v} = 2r_1 \operatorname{csch}\big(2\lambda_1\big(x + \lambda_1^2 t\big)\big), \tag{3.10}$$

and their graphs are shown in Fig. 3.

Case 2. When $r_2 = r_1$, (3.9) becomes one-soliton solution

$$\bar{u} = \frac{2\lambda_1}{r_1}\operatorname{csch}\big(2\lambda_1\big(x + \lambda_1^2 t\big)\big), \qquad \bar{v} = 2r_1 \operatorname{sech}\big(2\lambda_1\big(x + \lambda_1^2 t\big)\big), \tag{3.11}$$

and their graphs are shown in Fig. 4.

4. Complexiton solutions

On the basis of Theorem 1, we are able to generate the complexiton solutions for the integrable equations (1.9) and (1.10) by using the method proposed in [17]. Substituting trivial solutions $u = v = 0$ and complex parameters

$$\lambda_{4j-3}=\alpha_j+i\beta_j:=\lambda_{1j}^{(\kappa)},\qquad \lambda_{4j-2}=-(\alpha_j+i\beta_j):=\lambda_{2j}^{(\kappa)},$$
$$\lambda_{4j-1}=\alpha_j-i\beta_j:=\overline{\lambda_{1j}^{(\kappa)}},\qquad \lambda_{4j}=-\alpha_j+i\beta_j:=\overline{\lambda_{2j}^{(\kappa)}} \tag{4.1}$$

$(j=1,2,\dots,N;\ \kappa=1)$ into Lax pairs (1.1) and (2.1), we have their basic solutions:

$$\begin{pmatrix}\varphi_1^{(1)}(\lambda_{4j-3})\\ \varphi_2^{(1)}(\lambda_{4j-3})\end{pmatrix}=\begin{pmatrix}\exp(\eta_j^+)(\cos\xi_j^+ + i\sin\xi_j^+)\\ \exp(-\eta_j^+)(\cos\xi_j^+ - i\sin\xi_j^+)\end{pmatrix}, \tag{4.2}$$

$$\begin{pmatrix}\varphi_1^{(1)}(\lambda_{4j-2})\\ \varphi_2^{(1)}(\lambda_{4j-2})\end{pmatrix}=\begin{pmatrix}\exp(\eta_j^-)(\cos\xi_j^- + i\sin\xi_j^-)\\ \exp(-\eta_j^-)(\cos\xi_j^- - i\sin\xi_j^-)\end{pmatrix}, \tag{4.3}$$

$$\begin{pmatrix}\varphi_1^{(1)}(\lambda_{4j-1})\\ \varphi_2^{(1)}(\lambda_{4j-1})\end{pmatrix}=\begin{pmatrix}\exp(\eta_j^+)(\cos\xi_j^+ - i\sin\xi_j^+)\\ \exp(-\eta_j^+)(\cos\xi_j^+ + i\sin\xi_j^+)\end{pmatrix}, \tag{4.4}$$

$$\begin{pmatrix}\varphi_1^{(1)}(\lambda_{4j})\\ \varphi_2^{(1)}(\lambda_{4j})\end{pmatrix}=\begin{pmatrix}\exp(\eta_j^-)(\cos\xi_j^- - i\sin\xi_j^-)\\ \exp(-\eta_j^-)(\cos\xi_j^- + i\sin\xi_j^-)\end{pmatrix}, \tag{4.5}$$

where

$$\eta_j^+=\alpha_j x+\left(\alpha_j^2-\beta_j^2\right)t,\qquad \xi_j^+=\beta_j x+2\alpha_j\beta_j t,$$
$$\eta_j^-=-\alpha_j x+\left(\alpha_j^2-\beta_j^2\right)t,\qquad \xi_j^-=-\beta_j x+2\alpha_j\beta_j t,$$

α_j, β_j are arbitrarily real constants, and $\overline{\lambda_j^{(1)}}$ and $\overline{\lambda_j^{(2)}}$ are conjugates of $\lambda_j^{(1)}$ and $\lambda_j^{(2)}$.

According to Eq. (2.15), Eqs. (4.2)–(4.5), choosing $r_j=0$, $j=1,2,\dots,N$, we obtain

$$\delta_{4j-3}^{(1)}=\frac{\varphi_2^{(1)}(\lambda_{4j-3})}{\varphi_1^{(1)}(\lambda_{4j-3})}=\exp\left(-2\eta_j^+\right)\left(\cos\xi_j^+ - i\sin\xi_j^+\right)^2:=\delta_{1j}^{(1)}, \tag{4.6}$$

$$\delta_{4j-2}^{(1)}=\frac{\varphi_2^{(1)}(\lambda_{4j-2})}{\varphi_1^{(1)}(\lambda_{4j-2})}=\exp\left(-2\eta_j^-\right)\left(\cos\xi_j^- - i\sin\xi_j^-\right)^2:=\delta_{2j}^{(1)}, \tag{4.7}$$

$$\delta_{4j-1}^{(1)}=\frac{\varphi_2^{(1)}(\lambda_{4j-1})}{\varphi_1^{(1)}(\lambda_{4j-1})}=\exp\left(-2\eta_j^+\right)\left(\cos\xi_j^+ + i\sin\xi_j^+\right)^2:=\overline{\delta^{(1)}}_{1j}, \tag{4.8}$$

$$\delta_{4j}^{(1)}=\frac{\varphi_2^{(1)}(\lambda_{4j})}{\varphi_1^{(1)}(\lambda_{4j})}=\exp\left(-2\eta_j^-\right)\left(\cos\xi_j^- + i\sin\xi_j^-\right)^2:=\overline{\delta^{(1)}}_{2j}. \tag{4.9}$$

Substituting trivial solutions $u=v=0$ and complex parameters (4.1) for $\kappa=2$ into the Lax pairs (1.1) and (2.2), we have their basic solutions:

$$\begin{pmatrix}\varphi_1^{(2)}(\lambda_{4j-3})\\ \varphi_2^{(2)}(\lambda_{4j-3})\end{pmatrix}=\begin{pmatrix}\exp(\eta_j)(\cos\xi_j + i\sin\xi_j)\\ \exp(-\eta_j)(\cos\xi_j - i\sin\xi_j)\end{pmatrix}, \tag{4.10}$$

$$\begin{pmatrix}\varphi_1^{(2)}(\lambda_{4j-2})\\ \varphi_2^{(2)}(\lambda_{4j-2})\end{pmatrix}=\begin{pmatrix}\exp(-\eta_j)(\cos\xi_j - i\sin\xi_j)\\ \exp(\eta_j)(\cos\xi_j + i\sin\xi_j)\end{pmatrix}, \tag{4.11}$$

$$\begin{pmatrix}\varphi_1^{(2)}(\lambda_{4j-1})\\ \varphi_2^{(2)}(\lambda_{4j-1})\end{pmatrix}=\begin{pmatrix}\exp(\eta_j)(\cos\xi_j - i\sin\xi_j)\\ \exp(-\eta_j)(\cos\xi_j + i\sin\xi_j)\end{pmatrix}, \tag{4.12}$$

$$\begin{pmatrix}\varphi_1^{(2)}(\lambda_{4j})\\ \varphi_2^{(2)}(\lambda_{4j})\end{pmatrix}=\begin{pmatrix}\exp(-\eta_j)(\cos\xi_j + i\sin\xi_j)\\ \exp(\eta_j)(\cos\xi_j - i\sin\xi_j)\end{pmatrix}, \tag{4.13}$$

where

$$\eta_j=\alpha_j x+\left(\alpha_j^3-3\alpha\beta_j^2\right)t,\qquad \xi_j=\beta_j x+\left(3\alpha_j^2\beta_j-\beta_j^3\right)t.$$

According to (2.15), (4.10)–(4.13), choosing $r_j=0$, $j=1,2,\dots,N$, we have

$$\delta_{4j-3}^{(2)}=\frac{\varphi_2^{(2)}(\lambda_{4j-3})}{\varphi_1^{(2)}(\lambda_{4j-3})}=\exp(-2\eta_j)(\cos\xi_j - i\sin\xi_j)^2:=\delta_{1j}^{(2)}, \tag{4.14}$$

$$\delta_{4j-2}^{(2)} = \frac{\varphi_2^{(2)}(\lambda_{4j-2})}{\varphi_1^{(2)}(\lambda_{4j-2})} = \exp(2\eta_j)(\cos\xi_j + i\sin\xi_j)^2 := \delta_{2j}^{(2)}, \tag{4.15}$$

$$\delta_{4j-1}^{(2)} = \frac{\varphi_2^{(2)}(\lambda_{4j-1})}{\varphi_1^{(2)}(\lambda_{4j-1})} = \exp(-2\eta_j)(\cos\xi_j + i\sin\xi_j)^2 := \overline{\delta^{(2)}}_{1j}, \tag{4.16}$$

$$\delta_{4j}^{(2)} = \frac{\varphi_2^{(2)}(\lambda_{4j})}{\varphi_1^{(2)}(\lambda_{4j})} = \exp(2\eta_j)(\cos\xi_j - i\sin\xi_j)^2 := \overline{\delta^{(2)}}_{2j}. \tag{4.17}$$

Substituting (4.1), (4.6)–(4.9), and (4.14)–(4.17) into (2.41) ($j = 1, 2, \ldots, N$), we obtain the following complexiton solutions of the integrable equations (1.9) and (1.10)

$$\bar{u} = -2\frac{\Delta_{B_{2N-1}}}{\Delta_{2N-1}}, \qquad \bar{v} = 2\frac{\Omega_{C_{2N-1}}}{\Omega_{2N-1}}, \tag{4.18}$$

where

$$\Delta_{2N-1} = \det\big(\sigma_{11}^{(\kappa)}, \sigma_{21}^{(\kappa)}, \overline{\sigma_{11}^{(\kappa)}}, \overline{\sigma_{21}^{(\kappa)}}, \ldots, \sigma_{1N}^{(\kappa)}, \sigma_{2N}^{(\kappa)}, \overline{\sigma_{1N}^{(\kappa)}}, \overline{\sigma_{2N}^{(\kappa)}}\big)^T,$$
$$\Omega_{2N-1} = \det\big(\rho_{11}^{(\kappa)}, \rho_{21}^{(\kappa)}, \overline{\rho_{11}^{(\kappa)}}, \overline{\rho_{21}^{(\kappa)}}, \ldots, \rho_{1N}^{(\kappa)}, \rho_{2N}^{(\kappa)}, \overline{\rho_{1N}^{(\kappa)}}, \overline{\rho_{2N}^{(\kappa)}}\big)^T,$$
$$\Delta_{B_{2N-1}} = \det\big(b_{11}^{(\kappa)}, b_{21}^{(\kappa)}, \overline{b_{11}^{(\kappa)}}, \overline{b_{21}^{(\kappa)}}, \ldots, b_{1N}^{(\kappa)}, b_{2N}^{(\kappa)}, \overline{b_{1N}^{(\kappa)}}, \overline{b_{2N}^{(\kappa)}}\big)^T,$$
$$\Omega_{C_{2N-1}} = \det\big(c_{11}^{(\kappa)}, c_{21}^{(\kappa)}, \overline{c_{11}^{(\kappa)}}, \overline{c_{21}^{(\kappa)}}, \ldots, c_{1N}^{(\kappa)}, c_{2N}^{(\kappa)}, \overline{c_{1N}^{(\kappa)}}, \overline{c_{2N}^{(\kappa)}}\big)^T,$$

with

$$\sigma_{lj}^{(k)} = \big(1, \delta_{lj}^{(\kappa)}, \lambda_{lj}^{(\kappa)}, \delta_{lj}^{(\kappa)}\lambda_{lj}^{(\kappa)}, \ldots, {\lambda_{lj}^{(\kappa)}}^{2N-2}, \delta_{lj}^{(\kappa)}{\lambda_{lj}^{(\kappa)}}^{2N-2}, {\lambda_{lj}^{(\kappa)}}^{2N-1}, \delta_{lj}^{(\kappa)}{\lambda_{lj}^{(\kappa)}}^{2N-1}\big), \tag{4.19}$$

$$\rho_{lj}^{(\kappa)} = \big(\lambda_{lj}^{(\kappa)}, \delta_{lj}^{(\kappa)}, {\lambda_{lj}^{(\kappa)}}^2, \delta_{lj}^{(\kappa)}\lambda_{lj}^{(\kappa)}, \ldots, {\lambda_{lj}^{(\kappa)}}^{2N-1}, \delta_{lj}^{(\kappa)}{\lambda_{lj}^{(\kappa)}}^{2N-2}, {\lambda_{lj}^{(\kappa)}}^{2N}, \delta_{lj}^{(\kappa)}{\lambda_{lj}^{(\kappa)}}^{2N-1}\big), \tag{4.20}$$

$$b_{lj}^{(\kappa)} = \big(1, \delta_{lj}^{(\kappa)}, \lambda_{lj}^{(\kappa)}, \delta_{lj}^{(\kappa)}\lambda_{lj}^{(\kappa)}, \ldots, {\lambda_{lj}^{(\kappa)}}^{2N-2}, \delta_{lj}^{(\kappa)}{\lambda_{lj}^{(\kappa)}}^{2N-2}, {\lambda_{lj}^{(\kappa)}}^{2N-1}, -{\lambda_{lj}^{(\kappa)}}^{2N}\big), \tag{4.21}$$

$$c_{lj}^{(\kappa)} = \big(\lambda_{lj}^{(\kappa)}, \delta_{lj}^{(\kappa)}, {\lambda_{lj}^{(\kappa)}}^2, \delta_{lj}^{(\kappa)}\lambda_{lj}^{(\kappa)}, \ldots, {\lambda_{lj}^{(\kappa)}}^{2N-1}, \delta_{lj}^{(\kappa)}{\lambda_{lj}^{(\kappa)}}^{2N-2}, -\delta_{lj}^{(\kappa)}{\lambda_{lj}^{(\kappa)}}^{2N}, \delta_{lj}^{(\kappa)}{\lambda_{lj}^{(\kappa)}}^{2N-1}\big) \tag{4.22}$$

where $\overline{\sigma_{lj}^{(\kappa)}}$, $\overline{\rho_{lj}^{(\kappa)}}$, $\overline{b_{lj}^{(\kappa)}}$, and $\overline{c_{lj}^{(\kappa)}}$ are the conjugates of $\sigma_{lj}^{(\kappa)}$, $\rho_{lj}^{(\kappa)}$, $b_{lj}^{(\kappa)}$ and $c_{lj}^{(\kappa)}$ $(l = 1, 2)$.

By properties of determinant and analytic function, we are able to arrive at the following solutions, which are called n-complexiton solution of the integrable equations (1.9) and (1.10)

$$\bar{u} = -2\frac{\Lambda_{B_{2N-1}}}{\Lambda_{2N-1}}, \qquad \bar{v} = 2\frac{\Gamma_{C_{2N-1}}}{\Gamma_{2N-1}}, \tag{4.23}$$

where

$$\Lambda_{2N-1} = \det\big(\operatorname{Re}\sigma_{11}^{(\kappa)}, \operatorname{Re}\sigma_{21}^{(\kappa)}, \operatorname{Im}\sigma_{11}^{(\kappa)}, \operatorname{Im}\sigma_{21}^{(\kappa)}, \ldots, \operatorname{Re}\sigma_{1N}^{(\kappa)}, \operatorname{Re}\sigma_{2N}^{(\kappa)}, \operatorname{Im}\sigma_{1N}^{(\kappa)}, \operatorname{Im}\sigma_{2N}^{(\kappa)}\big)^T,$$
$$\Gamma_{2N-1} = \det\big(\operatorname{Re}\rho_{11}^{(\kappa)}, \operatorname{Re}\rho_{21}^{(\kappa)}, \operatorname{Im}\rho_{11}^{(\kappa)}, \operatorname{Im}\rho_{21}^{(\kappa)}, \ldots, \operatorname{Re}\rho_{1N}^{(\kappa)}, \operatorname{Re}\rho_{2N}^{(\kappa)}, \operatorname{Im}\rho_{1N}^{(\kappa)}, \operatorname{Im}\rho_{2N}^{(\kappa)}\big)^T,$$
$$\Lambda_{B_{2N-1}} = \det\big(\operatorname{Re}b_{11}^{(\kappa)}, \operatorname{Re}b_{21}^{(\kappa)}, \operatorname{Im}b_{11}^{(\kappa)}, \operatorname{Im}b_{21}^{(\kappa)}, \ldots, \operatorname{Re}b_{1N}^{(\kappa)}, \operatorname{Re}b_{2N}^{(\kappa)}, \operatorname{Im}b_{1N}^{(\kappa)}, \operatorname{Im}b_{2N}^{(\kappa)}\big)^T,$$
$$\Gamma_{C_{2N-1}} = \det\big(\operatorname{Re}c_{11}^{(\kappa)}, \operatorname{Re}c_{21}^{(\kappa)}, \operatorname{Im}c_{11}^{(\kappa)}, \operatorname{Im}c_{21}^{(\kappa)}, \ldots, \operatorname{Re}c_{1N}^{(\kappa)}, \operatorname{Re}c_{2N}^{(\kappa)}, \operatorname{Im}c_{1N}^{(\kappa)}, \operatorname{Im}c_{2N}^{(\kappa)}\big)^T,$$

with $\sigma_{lj}^{(\kappa)}$, $\rho_{lj}^{(\kappa)}$, $b_{lj}^{(\kappa)}$ and $c_{lj}^{(\kappa)}$ given by (4.19)–(4.22).

For simplicity, let us discuss the special case of $N = 1$. A direct computation yields 1-complexiton solution of the integrable equations (1.9) and (1.10)

$$\bar{u} = -2\frac{\Lambda_{B_1}}{\Lambda_1}, \qquad \bar{v} = 2\frac{\Gamma_{C_1}}{\Gamma_1}, \tag{4.24}$$

where

$$\Lambda_1 = \det\big(\operatorname{Re}\sigma_{11}^{(\kappa)}, \operatorname{Re}\sigma_{21}^{(\kappa)}, \operatorname{Im}\sigma_{11}^{(\kappa)}, \operatorname{Im}\sigma_{21}^{(\kappa)}\big)^T, \qquad \Gamma_1 = \det\big(\operatorname{Re}\rho_{11}^{(\kappa)}, \operatorname{Re}\rho_{21}^{(\kappa)}, \operatorname{Im}\rho_{11}^{(\kappa)}, \operatorname{Im}\rho_{21}^{(\kappa)}\big)^T,$$
$$\Lambda_{B_1} = \det\big(\operatorname{Re}b_{11}^{(\kappa)}, \operatorname{Re}b_{21}^{(\kappa)}, \operatorname{Im}b_{11}^{(\kappa)}, \operatorname{Im}b_{21}^{(\kappa)}\big)^T, \qquad \Gamma_{C_1} = \det\big(\operatorname{Re}c_{11}^{(\kappa)}, \operatorname{Re}c_{21}^{(\kappa)}, \operatorname{Im}c_{11}^{(\kappa)}, \operatorname{Im}c_{21}^{(\kappa)}\big)^T,$$

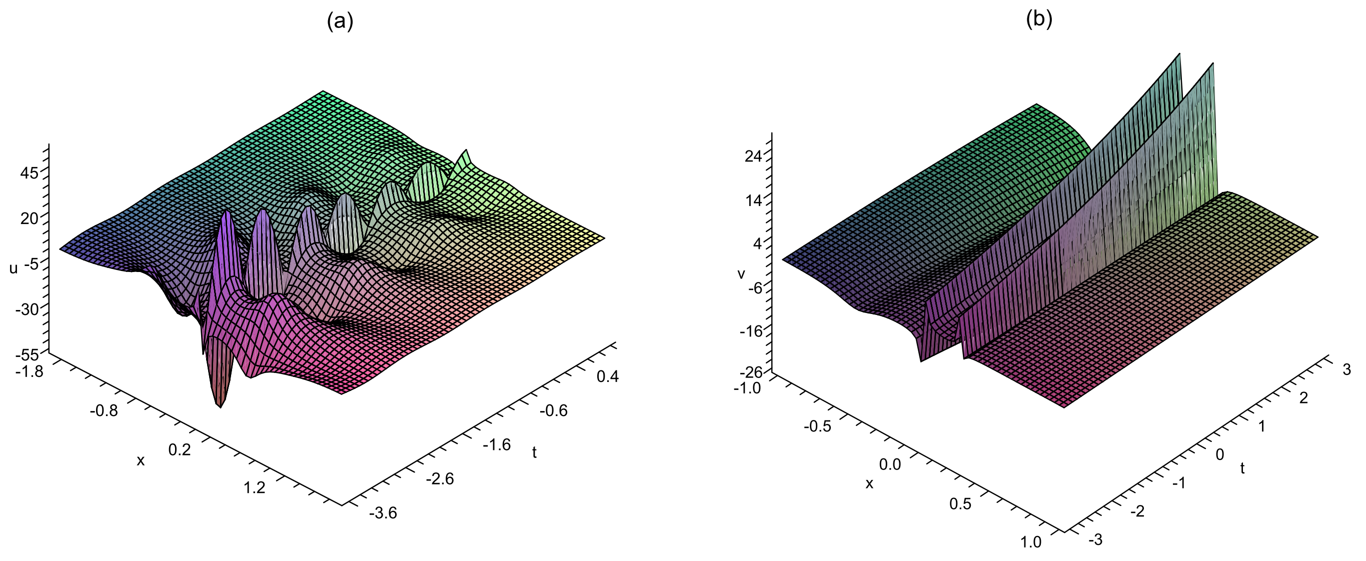

Fig. 5. The complexiton solution (4.25) of the nonlinear equation (1.9) with $\alpha_1 = 1$, $\beta_1 = -1.1$.

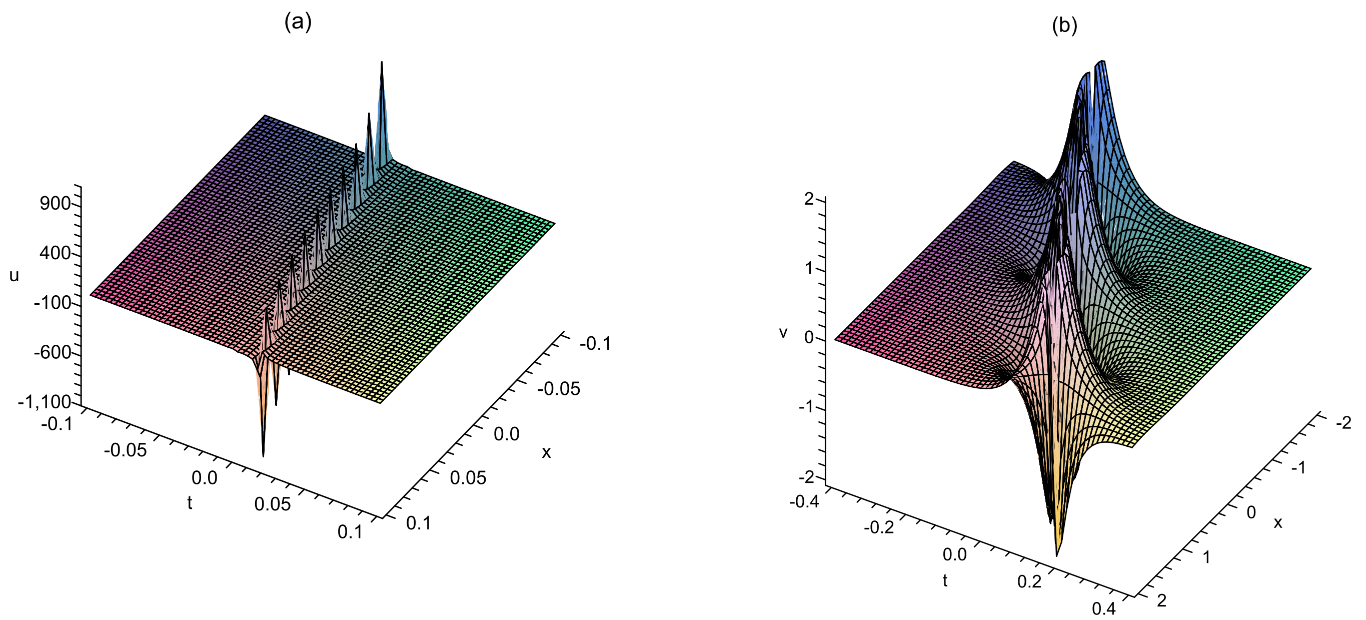

Fig. 6. The complexiton solution (4.26) of the nonlinear equation (1.10) with $\alpha_1 = 1$, $\beta_1 = -2$.

with

$$\sigma_{l1}^{(\kappa)} = \left(1, \delta_{l1}^{(\kappa)}, \lambda_{l1}^{(\kappa)}, \delta_{l1}^{(\kappa)}\lambda_{l1}^{(\kappa)}\right), \qquad \rho_{l1}^{(l)} = \left(\lambda_{l1}^{(\kappa)}, \delta_{l1}^{(\kappa)}, \lambda_{l1}^{(\kappa)2}, \delta_{l1}^{(\kappa)}\lambda_{l1}^{(\kappa)}\right),$$
$$c_{l1}^{(\kappa)} = \left(\lambda_{l1}^{(\kappa)}, \delta_{l1}^{(\kappa)}, -\delta_{l1}^{(\kappa)}\lambda_{l1}^{(\kappa)2}, \delta_{l1}^{(\kappa)}\lambda_{l1}^{(\kappa)}\right), \qquad b_{l1}^{(\kappa)} = \left(1, \delta_{l1}^{(\kappa)}, \lambda_{l1}^{(\kappa)}, -\lambda_{l1}^{(\kappa)2}\right)$$

$(l = 1, 2)$, $\delta_{11}^{(\kappa)}$, $\delta_{21}^{(\kappa)}$ are given by (4.6), (4.7), and $\lambda_{11}^{(\kappa)}$, $\lambda_{21}^{(\kappa)}$ are given by (4.1).

For $\kappa = 1$, Eq. (4.24) is a complexiton solution of the integrable equation (1.9):

$$\bar{u} = \frac{8\alpha_1\beta_1 \exp 2\eta_1^{+}[\beta_1 \cos 2\xi_1^{+} - \beta_1 \exp\zeta_1 \cos 2\xi_1^{-} + \alpha_1(\exp\zeta_1 \sin(-2\xi_1^{-}) + \sin 2\xi_1^{+})]}{\beta_1^2(1 + \exp 2\zeta_1) + 2\alpha_1^2 \exp\zeta_1 \cos(4\beta_1 x) - 2(\alpha_1^2 + \beta_1^2)\exp\zeta_1 \cos\varsigma_1},$$
$$\bar{v} = \frac{8\alpha_1\beta_1 \exp(-\eta_1^{-})[\exp\zeta_1 \sin 2\xi_1^{+} - \sin(-2\xi_1^{-})]}{\exp\zeta_1[2(\alpha_1^2 - \beta_1^2)\cos\varsigma_1 - 2\alpha_1^2\cos(4\beta_1 x)] + \beta_1[\beta_1 + \beta_1 \exp 2\zeta_1 - 4\alpha_1 \exp\zeta_1 \sin\varsigma_1]}, \tag{4.25}$$

where $\zeta_1 = 4\alpha_1 x$, $\varsigma_1 = 8\alpha_1\beta_1 t$. The graph of the complexiton solution (4.25) for the integrable equation (1.9) is shown in Fig. 5.

For $\kappa = 2$, Eq. (4.24) gives a complexiton solution of the integrable equation (1.10):

$$\bar{u} = \frac{8\alpha_1\beta_1 \exp 2\eta_1 [\beta_1(1-\exp 4\eta_1)\cos 2\xi_1 + \alpha_1(1+\exp 4\eta_1)\sin 2\xi_1]}{\beta_1^2 + \beta_1^2 \exp 8\eta_1 - 2(\alpha_1^2+\beta_1^2)\exp 4\eta_1 + 2\alpha_1^2 \exp 4\eta_1 \cos 4\xi_1},$$
$$\bar{v} = \frac{8\alpha_1\beta_1 \exp 2\eta_1 (\exp 4\eta_1 - 1)\sin 2\xi_1}{\beta_1^2 + \beta_1^2 \exp 8\eta_1 + 2(\alpha_1^2-\beta_1^2)\exp 4\eta_1 - 2\alpha_1^2 \exp 4\eta_1 \cos 4\xi_1}. \tag{4.26}$$

The graph of the complexiton solution (4.26) for the integrable equation (1.10) is shown in Fig. 6.

In this way, with the help of the DT (2.20), we are able to generate soliton and other solutions of the integrable equations (1.9) and (1.10) from a trivial solution.

5. Conclusion

In this paper, the soliton and complexiton solutions to the integrable equations (1.9) and (1.10) have been presented by using the N-fold Darboux transformation. The formula (2.41) is a unified and explicit formulation of multiple soliton solution. The expression (4.23) is a formulation of all k-complexiton solution ($1 \leqslant k \leqslant N$), from which we can easily get complexiton and other solutions of the integrable equations (1.9) and (1.10). It is important to point out that the soliton and complexiton solutions for one of potentials u and v have singularities. The profiles of soliton and complexiton solutions are graphically figured out. Within our knowledge, these solutions may be of significance for the explanation of some physical phenomenon. Also, we may try to find cuspons and peakons for the integrable equations (1.9) and (1.10), using the procedure we studied in [25,26].

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References

[1] M.J. Ablowitz, P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, 1991.
[2] R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge, 2004.
[3] C.H. Gu, H.S. Hu, Z.X. Zhou, Darboux Transformations in Integrable Systems. Theory and Their Applications to Geometry, Springer, Dortrecht, 2005.
[4] V.B. Matveev, M.A. Salle, Darboux Transformations and Solitons, Springer, Berlin, 1991.
[5] C. Rogers, W.K. Schief, Bäcklund and Darboux Transformations Geometry and Modern Applications in Soliton Theory, Cambridge University Press, Cambridge, 2002.
[6] Y.S. Li, J.E. Zhang, Chaos Solitons Fractals 16 (2003) 271–277.
[7] H.C. Hu, X.Y. Tang, S.Y. Lou, Q.P. Liu, Chaos Solitons Fractals 22 (2004) 327–334.
[8] Z.J. Qiao, Chinese Sci. Bull. 43 (1998) 1149–1153.
[9] Z.J. Qiao, Rev. Math. Phys. 13 (2001) 545–586.
[10] J.S. Zhang, Y.T. Wu, X.M. Li, Phys. A 319 (2003) 213–232.
[11] G. Neugebauer, R. Meinel, Phys. Lett. A 100 (1984) 467–470.
[12] D. Levi, G. Neugebauer, R. Meinel, Phys. Lett. A 102 (1984) 1–6.
[13] X.G. Geng, H.W. Tam, J. Phys. Soc. Japan 68 (1999) 1508–1512.
[14] E.G. Fan, Commun. Theor. Phys. 36 (2001) 401–404.
[15] X.M. Li, A.H. Chen, Phys. Lett. A 342 (2005) 413–420.
[16] D.J. Huang, D.S. Li, H.Q. Zhang, Chaos Solitons Fractals 33 (2007) 1677–1685.
[17] Zhaqilao, Z.B. Li, Modern Phys. Lett. B 22 (2008) 2945–2966.
[18] Zhaqilao, Sirendaoreji, J. Math. Phys. 51 (2010) 073501.
[19] H.Z. Li, B. Tian, L.L. Li, H.Q. Zhang, T. Xu, Phys. Scr. 78 (2008) 065001.
[20] Z.J. Qiao, Phys. Lett. A 192 (1994) 316–322.
[21] Z.J. Qiao, Finite-Dimensional Integrable System and Nonlinear Evolution Equations, Higher Education Press, PR China, 2002.
[22] P.A. Clarkson, C.M. Cosgrove, J. Phys. A 20 (1987) 2003–2024.
[23] V.S. Gerdjikov, M.I. Ivanov, Bulg. J. Phys. 10 (1983) 130.
[24] V.S. Gerdjikov, M.I. Ivanov, JINR Preprint E2-82-595, Dubna, USSR, 1982.
[25] Z.J. Qiao, Comm. Math. Phys. 239 (2003) 309–341.
[26] Z.J. Qiao, G.P. Zhang, Europhys. Lett. 73 (2006) 657–663.