Past and Current Research Summary

– Zhijun Qiao

- In recent years, my research work mainly focused on nonlinear integrable dynamical systems and hierarchies of nonlinear evolution equations, which provide a strong interdisciplinary connection between mathematics and theoretical physics. As shown by my publications list, I have already accumulated crucial experience and skills in dealing with r-matrices, Lie algebra, and explicit solution of nonlinear integrable systems.

- For finite dimensional canonical Hamiltonian systems, I proposed a new approach to show the integrability. Using the fundamental Poisson bracket, I constructed a generalized r-matrix structure and generalized Hamiltonian functions for the case where the Lax matrix is 2×2, which covered most of the constrained (c-) Hamiltonian systems such as the c-AKNS, and the c-MKdV etc, as well as generating new integrable systems. Surprisingly, I found that two distinct systems, one discrete and the other continuous, share the same constant r-matrix, Lax matrix, and involutive conserved integrals. In particular, I obtained the first constant r-matrix of the Toda symplectic map which is an important discrete system, and furthermore found several pairs of different Hamiltonian systems with common r-matrices. Moreover, I developed my Lax matrix approach to present the explicit solutions of integrable nonlinear evolution equations. This approach was proven to be valid through concrete examples. In this sense, I successfully constructed a useful procedure from finite dimensional flows to infinite dimensional systems. Please see my representative papers [1, 2, 6] and my book [4].

- Additionally, I also proposed a scheme to construct the Lie structure and r-matrix for the generalized nonlinear evolution equations. I introduced the generalized nonlinear evolution equations and presented a general framework to produce the Lax pairs. This framework embraced the isospectral and the non-isospectral hierarchy of nonlinear evolution equations as well as the positive order and negative order hierarchy. In particular, I proposed a method to generate the negative order hierarchy and to obtain its Lax pair. Therefore, I have unified the usual higher-order (i.e. positive order) and the negative order hierarchies in one generating pattern. All these hierarchies were proven to have the Lax representations. Please see my representative paper [3].

- These research achievements have already created large impacts in my research area. My articles have been quoted by experts as well as being reviewed by the worldwide review journals Mathematical Reviews (USA) and Z-Maths (Germany).

- For more details about my research work, see my Publications List.

— Research Plan

Nonlinear Integrable Complex Dynamics

– Zhijun Qiao

- ## Background

  The dynamics of interfaces separating two phases has been the subject of intense research over the years. The formation of moving interfaces is an important part of a great variety of non-equilibrium physical phenomena, such as electro-deposition, viscous fingering in a Hele-Shaw cell, diffusion-limited aggregation, as well as Kelvin-Helmholtz, Rayleigh-Taylor, and Richtmyer-Meshkov instabilities. Interfacial motion is typically much slower than the processes that take place in the bulk of both phases, such as heat transfer and diffusion. In this case, the scalar field governing the evolution of the interface satisfies the Laplace equation. For this reason we call this process Laplacian growth, and its evolution is governed by the Laplacian Growth Equation (LGE).Although the LGE has integrable structure such as infinite conservation laws, the canonical Poisson structure, and a relationship with the 2D Toda hierarchy, an open question is whether it is completely integrable? If the answer is yes, how does one find its explicit solutions? It is also important to study other interface phenomena from the same point of view. In the framework of geometrical dynamics, we propose to answer these questions by applying a new method.

- ## Proposed Work

  Our research concentrates on new types of explicit solutions of integrable dynamical equations, both real and complex, occurring in surface dynamics, fluid mechanics and other physical sciences. We will use a constraint procedure on a symplectic manifold that I have recently developed and which turned out to be very powerful and fruitful in revealing hidden mathematical structure in complex nonlinear models (technical details are in my book – Zhijun Qiao (2002): Finite-dimensional Integrable System and Nonlinear Evolution Equations (English Edition), Chinese National Higher Education Press, PR China). In short, this procedure links finite-dimensional integrable systems with infinite-dimensional integrable partial differential equations. This approach uses such powerful tools of modern mathematics such as the r-matrix method, symplectic manifolds, integrability, and algebraic geometry. Our starting point is a finite-dimensional matrix (called the Lax matrix) which leads to a representation for finite-dimensional systems in a Hamiltonian form. The Lax matrix implies the r-matrix structure, which guarantees integrability of the Hamiltonian system. We will construct reduction relations connecting finite-dimensional Hamiltonian systems to the LGE and other nonlinear partial differential equations describing complex physical phenomena. Our Lax matrix approach yields this reduction relation in an explicit form. We will generate a new class of explicit solutions for integrable nonlinear equations in terms of Riemann-theta functions, thereby generalizing the N-finger solutions for the LGE.

- ## Impact of Work

  Successful accomplishment of the proposed research will reveal crucial hidden relations between complex nonlinear systems, which are very important for unstable interface fluid dynamics programs dealing with nonlinear partial differential equations and complex integrable systems. This research will also provide a constructive method for obtaining explicit solutions of integrable complex dynamical systems.

Statement of Research Interests

– Zhijun Qiao

As a mathematician, my research experience is broad. Since I began my Masters graduate study, I have been working on two main directions in nonlinear mathematical physics: finite-dimensional integrable systems and nonlinear integrable equations. They have strong interdisciplinary connections between mathematics and theoretical or experimental physics. My past and current research work is composed of two main parts **A: Finite dimensional case** and **B: Infinite dimensional case**. More details are in my book [4].

A. Finite dimensional case

1. Construction of new integrable dynamical systems

An important and very active topic in the theory of integrability is to search for as many new integrable dynamical systems as possible and discuss their algebraic and geometric structures. It has been shown that the nonlinearization technique for Lax pair is an valid approach to generate finite dimensional integrable systems. This method is used for each concrete Lax pair. Each integrable system produced by the nonlinearization method has its own feature. Applying the nonlinearization method to some spectral problems, we found many new integrable Hamiltonian systems, involutive systems and involutive solutions of integrable *nonlinear evolution equations (NLEEs)*. All these involutive systems were presented in the explicit forms. Also we developed some new spectral problems and Lax pairs. Please see my papers in references [1, 2, 5, 22, 24, 30, 33, 34, 35, 36, 38, 39, 40, 41, 47, 48, 50].

2. C. Neumann type dynamical flows on symplectic submanifolds

An earlier C. Neumann system was used to describe a Hamiltonian mechanical system with a harmonic N-oscillator constraint on sphere S^N. The C. Neumann type flows we considered are restricted on some symplectic submanifold via a constraint between potentials and eigenfunctions. Of course this is a natural generalization of the earlier C. Neumann system. The integrability problem on some symplectic submanifold of R^{2N} is usually more difficult than on R^{2N} because it needs some guesses and technical calculations. Here we proved the integrability of some concrete C. Neumann type flows mainly by the use of Moser constraint method (see papers [13, 21, 31, 44, 46, 51]) and by introducing the dynamical r-matrix structure and Dirac-Poisson bracket (see papers [1, 5, 9]). For Dirac-Poisson bracket our starting point is Lax matrix other than usual Lax pair. This is a different idea. We shall apply this method to some numerical lattice systems. This is our newest task and will involve computer simulation.

3. r-matrix structures for continuous and discrete dynamical systems

The r-matrix method is quite an effective approach to the Lie-Poisson structure in infinite dimensional system as well as proving fundamental commutator relation in the quantum inverse scattering. Here, we applied the r-matrix method to finite dimensional constrained flows. It is worthwhile to stress that our starting point is Lax matrix instead of usual Lax pair (see papers [1, 6, 7, 10]). In the papers [6, 7, 10], we proposed a unisonant r-matrix structure of integrable systems. We found some new r-matrix structures being non-dynamical which reduces the calculation of integrability. Those examples include the constrained MKdV, the constrained AKNS, the constrained CKdV, the constrained Toda etc (see papers [6, 9, 12, 14, 16, 17, 18]). We first time obtained the nondynamical r-matrix of Toda symplectic map which is a basic integrability condition of discrete lattice system. Furthermore, an amazing fact was revealed that a pair of very different (one is discrete, the other continuous) systems shares the same nondynamical r-matrix, Lax matrix, and even involutive system (see papers [6, 17]). Besides, we developed three other pairs with such property (see papers [6, 14]). Recently, we still found another interesting fact: a pair of Hamiltonian systems produced by two gauge equivalent spectral problems has two different r-matrices (see papers [6, 11]). This will be helpful for us to classify integrable systems, which will involve further research in the future.

4. Explicit solutions for integrable systems both discrete and continuous

The ideal aim for integrable NLEEs is to obtain their explicit solutions. Usually this is difficult. Here we successfully connected finite dimensional integrable flows to infinite dimensional integrable systems (i.e. NLEEs) through some constraint relations. Using separation of variables and algebro-geometric tools on Riemann surface, we solved integrable NLEEs with an explicit form of algebro-geometric solutions (see papers [1, 6, 7, 12]). A couple of successful examples include the discrete Toda lattice, the well-known Ablowitz-Kaup-Newell-Segur (AKNS) equations, and the shallow water Camassa-Holm (CH) equation. In paper [1], we constrained the CH equation to a symplectic submanifold, and it was proven to have a parametric solution on the submanifold. Solving the parametric representation of the solution, we successfully obtained a new solution of the CH equation. The new solution has more simpler form than the results in other literaturs related to the CH equation. This is our important discovery, see paper [1] for more details. In addition, very recently we developed the CH equation to a more generalized integrable hierarchy [2] which has parametric solution, and traveling wave solutions. Our current research involves continuous equations available for numerical analysis by computer software.

The above Items 1 – 4 have already formed a systematical work lifting from finite dimensional integrable systems up to infinite dimensional systems (i.e. NLEEs). See my book [4] for more details.

B. Infinite dimensional case

1. Construction of new spectral problems and their hierarchies

The major features of evolution equations integrable by the inverse scattering transformation (IST) are determined by the corresponding spectral problem. However, the main difficulty is how to connect the given NLEEs to a suitable spectral problem. Thus it is important and meaningful to find some new spectral problems and the corresponding hierarchies of NLEEs. In papers [3, 19, 30, 33], we proposed several new spectral problems and corresponding hierarchies of NLEEs.

2. Commutator representation structure

Lax representation is a very important tool to study the integrable NLEEs. We presented Lax representation through considering the commutator representation structure of evolution equations. That procedure needs to solve some key operator or matrix equations. In that procedure, we proposed a general structure equation of Lax operator, which is able to be fit for any hierarchy of NLEEs (see papers [25, 27]). All Lax pairs are given in the form of operator. We gave many examples of the hierarchy of NLEEs, both the well-known and the new, to show the procedure (see papers [22, 26, 28, 29, 37, 43, 45, 49, 52, 53, 54]). In particular, we supplied an approach how to obtain a new Lax pair from a given Lax pair (see paper [19]), discussed the equivalence of NLEEs and Lax representations (see paper [23]), and by using the gradient method found that a spectral problem can generate two different hierarchies (one is positive, the other negative) of NLEEs with the standard Lax representations (see papers [27, 32, 42]).

3. Generalized Lax representation and Category of nonlinear evolution equations

For a given spectral operator, we introduced category of nonlinear evolution equations[3], which contains usual positive and negative order hierarchies of nonlinear evolution equations (NLEEs) and has the generalized Lax representation. This is a new generalization and a new development in the theory of nonlinear evolution equations. A given spectral problem can produce all possible hierarchies (called universal hierarchies) of NLEEs associated with it, and universal hierarchies have the generalized Lax representations. Here the method we used is the spectral gradient and the generalized generator satisfying a operator equation other than the expansion of spectral parameter. Please see papers [3, 8, 15, 20, 25, 27] for more details.

4. Algebraic structures related to nonlinear evolution equations

By using our generalized Lax representation and category of nonlinear evolution equations (NLEEs), we found two kinds of algebraic structures, one Lie algebra and the other non-Lie algebra. Both of them are associated with the nonlinear stationary systems (see papers [8, 25]). A further aim is to construct the algebraic structure related to the category of NLEEs and to connect it to the r-matrix in infinite dimensional case, which has been dealt with in paper [3]. But, I hope to continue pursuing this research at your institute.

In summary, I expect to tap into my last few years working experiences as a postdoc, Humboldt research fellow, and university researcher in establishing a comprehensive research effort geared primarily towards exploring the exact solutions of some nonlinear wave equations and in studying the mathematical context of nonlinear dynamical system. Both research directions can begin with a comprehensive effort comprising mathematical techniques, theory and methods. The rough time-scale to get command of the theory and methods in this research is about one year (depending on the commitment of students). Within 2 years I anticipate that I will be carrying out solutions analysis and comparing my theory and methods with the results in the literature. On the basis of my research experience and academic achievements, I am very confident that I can attract external funding. I already successfully applied for and received five grants as the principal investigator. When appropriate, I will seek to collaborate with researchers at Los Alamos National Laboratory, at University of Kassel, and at other organizations so as to carry out a broad research effort.

References

[1] Qiao Zhijun (2003): The Camassa-Holm hierarchy, N-dimensional integrable systems, and algebro-geometric solution on a symplectic submanifold, **Communications in Mathematical Physics**, 239, 309-341.

[2] Qiao Zhijun (2003): A new integrable hierarchy, parametric solution, and traveling wave solution, to appear in **Mathematical Physics, Analysis and Geometry**.

[3] Qiao Zhijun, Cao C., Strampp W. (2003): Category of nonlinear evolution equations, algebraic structure, and r-matrix, **Journal of Mathematical Physics** 44, 701-722.

[4] Qiao Zhijun (2002): **Finite-dimensional Integrable System and Nonlinear Evolution Equations**, Chinese National Higher Education Press, PR China.

[5] Qiao Zhijun and Strampp W. (2002): Negative order MKdV hierarchy and a new integrable Neumann-like system, *Physica A* 313, 365-380.

[6] Qiao Zhijun (2001): Generalized r-matrix structure and algebro-geometric solution for integrable system, **Reviews in Mathematical Physics** 13, 545-586.

[7] Qiao Zhijun (2001): r-matrix and algebro-geometric solution for the AKNS system, *Theoretical and Mathematical Physics* 127, 827-834.

[8] Qiao Zhijun and Strampp W. (2001): $L - A - B$ representation of nonlinear evolution equations and its applications, *Physica A* 293, 143-156.

[9] Qiao Zhijun and Zhou R. (2000): On restricted c-KdV and Toda flows of Neumann type, *Communication in Theoretical Physics* 34, 229-234.

[10] Qiao Zhijun and Strampp W. (2000): A unisonant r-matrix structure of integrable systems and its reductions, *Chinese Physics Letters* 17, 235-237.

[11] Qiao Zhijun and Strampp W. (1999): A gauge equivalent pair with two different r-matrices, *Physics Letters A* 263, 365-372.

[12] Qiao Zhijun (1999): r-matrix and algebraic-geometric solution for the integrable symplectic map, *Chinese Science Bulletin* 44, 114-119.

[13] Qiao Zhijun (1999): C. Neumann system associated with the Levi hierarchy, *Northeastern Mathematics Journal* 15, 269-273.

[14] Qiao Zhijun and Strampp W.(1998): On different integrable systems sharing the same non-dynamical r-matrix, *Journal of Mathematical Physics* 39, 3271-3280.

[15] Qiao Zhijun (1998): New hierarchies of isospetral and non-isospectral integrable NLEEs derived from the Harry-Dym spectral problem, *Physica A* 252, 377-387.

[16] Qiao Zhijun (1998): Non-dynamical r-matrix and algebraic-geometric solution for the integrable symplectic map, preprint, 1996, *Chinese Science Bulletin* 43, 1149-1153.

[17] Qiao Zhijun and Zhou R. (1997): Discrete and continuous integrable systems possessing the same non-dynamical r-matrix. *Physics Letters A* 235, 35-40.

[18] Qiao Zhijun (1997): Modified r-matrix and separation of variables for the MKdV hierarchy. *Physica A* 243, 129-140.

[19] Qiao Zhijun (1997): Two new hierarchies containing the sine-Gordon and sinh-Gordon equations, and their Lax representations. *Physica A* 243, 141-151.

[20] Qiao Zhijun (1997): Generalized Structure of Lax Representations for Nonlinear Evolution Equation. *Applied Mathematics and Mechanics* 18, 671-677.

[21] Qiao Zhijun (1997): Completely integrable generalized C. Neumann systems on several symplectic submanifolds. *Journal of Nonlinear Mathematical Physics* 4, 221-230.

[22] Qiao Zhijun (1997): Involutive solutions and commutator representations of the Dirac hierarchy. *Chinese Quarterly Journal of Mathematics* 12, 1: 1-4.

[23] Qiao Zhijun and Wu X. (1996): Equivalence of spectral problems, evolution equations, and Lax representations. *Journal of Exploration and Research Mathematics* 16, 256-258.

[24] Qiao Zhijun (1995): A completely integrable system and the parametric representations of solutions of the WKI hierarchy. *Journal of Mathematical Physics* 36, 3535-3560.

[25] Qiao Zhijun (1995): Algebraic structure of operator related to the stationary systems. *Physics Letters A* 206, 347-358.

[26] Qiao Zhijun (1995): Commutator representations of nonlinear evolution equations: Harry-Dym and Kaup-Newell cases. *Journal of Nonlinear Mathematical Physics* 2, 151-157.

[27] Qiao Zhijun (1995): Generation of the hierarchies of solitons and generalized structure of the commutator representation. *Acta Applied Mathematics Sinica* 18, 287-301.

[28] Qiao Zhijun (1995): Commutator representations of the Tu hierarchy of evolution equations. *Mathematica Applicata* 8, 177-182.

[29] Qiao Zhijun (1995): On commutator representation of soliton equation. *Proceedings of the 2nd Youth Academic Annual Conference at Dalian*, P. R. China, October 1995, 18-21.

[30] Qiao Zhijun (1994): A hierarchy of nonlinear evolution equations and finite-dimensional involutive systems. *Journal of Mathematical Physics* 35, 2971-2977.

[31] Qiao Zhijun (1994): Involutive system and integrable C. Neumann system associated with the MKdV hierarchy. *Journal of Mathematical Physics* 35, 2978-2983.

[32] Qiao Zhijun (1994): A general approach for getting the commutator representations of the hierarchies of nonlinear evolution equations. *Physics Letters A* 195, 319-329.

[33] Qiao Zhijun (1994): Completely integrable system related to a new hierarchy of isospectral evolution equations. *Physics Letters A* 192, 316-322.

[34] Qiao Zhijun (1994): A finite-dimensional integrable system and the involutive solutions of the higher-order Heisenberg spin chain equations. *Physics Letters A* 186, 97-103.

[35] Qiao Zhijun (1994): A completely integrable system associated with the Harry-Dym hierarchy. *Journal of Nonlinear Mathematical Physics* 1, 65-75.

[36] Qiao Zhijun (1994): On the involutive solutions of soliton hierarchy. *Chinese Quarterly Journal of Mathematics* 9, 2: 44-53.

[37] Qiao Zhijun (1994): Commutator representations of the L-C-Z hierarchy of evolution equations. *Mathematica Applicata* 7, 320-325.

[38] Qiao Zhijun (1993): New completely integrable Liouville's system produced by the Kaup-Newell eigenvalue problem. *Journal of Mathematical Physics* 34, 3110-3120.

[39] Qiao Zhijun (1993): Bargmann system and the involutive solutions of the Levi hierarchy. *Journal of Physics A* 26, 4407-4417.

[40] Qiao Zhijun (1993): Two new integrable systems in Liouville's sense. *Physics Letters A* 172, 224-228.

[41] Qiao Zhijun (1993): Four new completely integrable finite-dimensional Hamiltonian systems. *Proceedings of the 2nd International Conference on Nonlinear Mechanics at Beijing*, P. R. China, August 1993, 729-733.

[42] Qiao Zhijun (1993): Commutator representations of three isospectral equation hierarchies. *Chinese Journal of Contemporary Mathematics* 14, 41-51.

[43] Qiao Zhijun (1993): Commutator Representations for Two Hierarchies of Nonlinear Isospectral Evolution Equations. *Northeastern Mathematical Journal* 9, 215-225.

[44] Qiao Zhijun (1993): C. Neumann Constraint and Involutive Solutions of the Tu Hierarchy. *Northeastern Mathematical Journal* 9, 325-330.

[45] Qiao Zhijun (1993): Commutator representations for the hierarchies of isospectral evolution equations. *Chinese Annual Mathematics A* 14, 31-38.

[46] Qiao Zhijun (1993): A kind of Hamiltonian systems with the C. Neumann constraint and WKI hierarchy. *Journal of Exploration and Research Mathematics* 13, 377-343.

[47] Qiao Zhijun (1993): Nonlinearization of Lax pair for Levi hierarchy. *Mathematica Applicata* 6, 472-476.

[48] Qiao Zhijun (1993): Two kinds of new completely integrable Hamiltonian systems in Liouville's sense. *Modern Mathematics and Mechanics (MMM–V)*, Ed. Chen Zhida (Xuzhou: Publishing Company of China Mining University Press), 663-689.

[49] Qiao Zhijun (1992): Commutator representation of WKI hierarchy. *Chinese Science Bulletin* 37, 763-764.

[50] Mou Weihua and Qiao Zhijun (1992): A new polynomial involutive system and a classical completely integrable system. *Acta Applied Mathematics Sinica* 8, 348-356.

[51] Qiao Zhijun (1992): A new kind of C. Neumann constraint and C. Neumann integrable system produced by the Tu spectral problem. *Proceedings of the 1st Youth Academic Annual Conference at Shenyang*, P. R. China, December 1992, 11-18.

[52] Qiao Zhijun (1991): Representation of the Lax system for the Levi hierarchy. *Chinese Science Bulletin* 36, 1756-1757.

[53] Qiao Zhijun (1991): Commutator representations for the D-AKNS hierarchy of evolution equations. *Mathematica Applicata* 4, 64-70.

[54] Qiao Zhijun (1990): Lax representations of Levi hierarchy. *Chinese Science Bulletin* 35, 1353-1354.