New integrable hierarchy, its parametric solutions, cuspons, one-peak solitons, and M/W-shape peak solitons

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(Received 5 April 2007; accepted 22 June 2007; published online 3 August 2007)

In this paper, we propose a new completely integrable hierarchy. Particularly in the hierarchy we draw two new soliton equations: (1) $m_t = \frac{1}{2}(1/m^2)_{xxx} - \frac{1}{2}(1/m^2)_x$; (2) $m_t + m_x(u^2 - u_x^2) + 2m^2 u_x = 0$, $m = u - u_{xx}$. The first one is the second positive member in the hierarchy while the second one is the second negative member in the hierarchy. Both equations can be derived from the two-dimensional Euler equation by using the approximation procedure. All equations in the hierarchy are proven to have bi-Hamiltonian operators and Lax pairs through solving a crucial matrix equation. Moreover, we develop parametric solutions of the entire hierarchy through constructing two kinds of constraints; one is for all negative members of the hierarchy on a symplectic submanifold, and the other is for all positive members in the standard symplectic space. The most interesting things are both equations possess new type of peaked solitons—continuous and piecewise smooth "W-/M-shape peak" soliton solutions. In addition, we find new cusp solitons—cuspons for the second equation and one-single-peak solitons for the first—which are also continuous and piecewise smooth but not in the regular type $ce^{-|x-ct|}$ (c is a constant). [DOI: 10.1063/1.2759830]

I. INTRODUCTION

Solitons and integrable systems play an increasingly important role in nonlinear waves, dynamical systems, and analytical mechanics. It has been significant in soliton theory for us to find more new integrable systems. There are well-known constructions of integrable systems. The cruciality of integrable systems theory is the idea of compatibility, which is usually called Lax pair.[20] One is already at the very definition of the complete integrability of a Hamiltonian flow in the Liouville-Arnold sense, which means that the flow is able to be included into a complete family of commuting Hamiltonian flows.[3] A condition of existence of a number of commuting systems may be taken as the basis of the bi-Hamiltonian structure and Lax pair approach.[1,4,12,15,16,23] However, a key procedure is to figure out bi-Hamiltonian operators.

A general method for constructing a hereditary symmetry and bi-Hamiltonian systems was presented in Ref. 13. If one starts from an eigenvalue problem, then the computation of the gradient of the eigenvalue yields a recursion operator,[4,14,24,27] and therefore it generates a hierarchy of bi-Hamiltonian equations. However, in order to prove the integrability for all equations in the hierarchy We need to find a time part which composes of a Lax pair together with the eigenvalue problem. In the present paper, we propose a new eigenvalue problem and adopt the gradient[13] of an eigenvalue to figure out a pair of Hamiltonian operators. Then we develop a hierarchy of evolution equations through taking some kernel elements from both Hamiltonian operators. To show the complete integrability of the hierarchy, we determine the time part of the Lax pair through employing a crucial matrix equation. Furthermore, we present exact solutions of the hierarchy both parametric and peak solitary.

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The whole paper is organized as follows. In Sec. II, we first propose a new spectral problem, which stems the functional gradient of the spectral parameter. Then, we figure out a pair of operators satisfying Lenard's scheme problem. Usually, the two operators are Hamiltonian. In Sec. III, based on the two Hamiltonian operators, we develop a new hierarchy of nonlinear evolution equations associated with the new spectral problem. Particularly in the hierarchy we draw two new soliton equations:

$$m_t = \frac{1}{2}\left(\frac{1}{m^2}\right)_{xxx} - \frac{1}{2}\left(\frac{1}{m^2}\right)_x, \tag{1}$$

$$m_t + m_x(u^2 - u_x^2) + 2m^2u_x = 0, \quad m = u - u_{xx}. \tag{2}$$

The first one is the second positive member in the hierarchy while the second one is the second negative member. In particular, both equations can be reduced from the two-dimensional Euler equation by using the approximation procedure (see this in the last section). All equations both positive and negative members in the hierarchy are proven to be completely integrable in the sense of Lax pairs through solving a crucial matrix equation. Therefore, the initial value problem of the hierarchy may be solved by the inverse scattering transform (IST) method.[2,17] In Sec. IV, we restrict all negative members of the hierarchy to a symplectic submanifold, and give parametric solutions of all negative members of partial differential equations (PDEs) through constructing the constrained integrable Hamiltonian systems ordinary differential equations (ODEs). In Sec. V, we generate a new constraint between the eigenfunctions and the potential function in the standard symplectic space $\mathbb{R}^{2N}$. Under the constraint, our spectral problem is cast in another new integrable Hamiltonian system (ODE), and furthermore we provide parametric solutions of all positive members (PDEs) via a system of independently involutive functions. In Sec. VI and VII, we consider the traveling wave solutions of the two new equations (1) and (2). They are proven to have new one-single-peak solitons and cusp solitons, respectively. These solutions are continuous and piecewise smooth but not in the regular type $ce^{-|x-ct|}$ (c is a constant).[5] Their first order derivative is discontinuous at some point (see more mathematical studies about the Camassa-Holm (CH) equation in Refs. 8, 9, 21, and 25). The most interesting things are both Eqs. (1) and (2) possess new type of peaked solitons—continuous and piecewise smooth multipeak solitons—which are in the shape of "W" or "M" three-peak solitons. We will take some graphs to show how these three-peak solitons look like. The last section gives a brief derivation procedure for the two new equations (1) and (2) from the two-dimensional Euler equations. In addition, we point out the two applications of our new equations: one is closely related to Newton equation with new potentials which we solve in this paper, and the other is to use peaked solutions for explaining electrophysiological responses of visceral nociceptive neuron and sensitization of dorsal root reflex conclusions in neuroscience. We also address some remarks and open problems in the last section.

II. SPECTRAL PROBLEMS AND LENARD'S OPERATORS

Let us consider the following spectral problem:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} -1/2 & (1/2)\lambda m \\ -(1/2)\lambda m & 1/2 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \equiv U(m,\lambda)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{3}$$

where λ is a spectral parameter, m is a scalar potential function periodic or approaching the same constant at both infinities, and $\psi=(\psi_1,\psi_2)^T$ is the spectral function corresponding to the spectral parameter λ. Then, calculating the functional gradient $\delta\lambda/\delta m$ of the spectral parameter λ with respect to the potential m yields

$$\nabla\lambda := \frac{\delta\lambda}{\delta m} = \frac{\lambda}{2}(\psi_1^2 + \psi_2^2).$$

Here during our computation, we need the boundary conditions of approaching the same constant at both infinities or periodical condition for the potential function m. A general calculated method can be found in Refs. 4 and 27. Through some guesswork, we find two operators K, J satisfying

$$K\nabla\lambda = \lambda^2 J\nabla\lambda, \tag{4}$$

where $K=\partial^3-\partial$, $J=-\partial m\partial^{-1}m\partial$, $\partial=\partial/\partial x$, and ∂^{-1} is the inverse operator of ∂, namely, $\partial\partial^{-1}=\partial^{-1}\partial=1$. Such operators K, J are called a pair of Lenard'soperators, and Eq. (4) is called Lenard's eigenvalue problem. Therefore, we can figure out the inverses of K and J,

$$J^{-1} = -\partial^{-1}m^{-1}\partial m^{-1}\partial^{-1},$$

$$K^{-1} = \partial^{-1}\mathrm{e}^{-x}\partial^{-1}\mathrm{e}^{2x}\partial^{-1}\mathrm{e}^{-x}.$$

So, they lead to

$$£ = J^{-1}K = -\partial^{-1}m^{-1}\partial m^{-1}(\partial^2-1), \tag{5}$$

$$£^{-1} = K^{-1}J = -\mathrm{e}^{-x}\partial^{-1}\mathrm{e}^{2x}\partial^{-1}\mathrm{e}^{-x}m\partial^{-1}m\partial, \tag{6}$$

which are actually the two recursion operators we need in the next section.

III. THE HIERARCHY, BI-HAMILTONIAN STRUCTURES, AND LAX PAIRS

Now, according to Lenard's operators K and J, we construct a hierarchy of nonlinear evolution equations, and then we show the integrability of the hierarchy through solving a key matrix equation. Let $G_0 \in \ker J = \{G \in C^\infty(\mathbb{R}) \mid JG=0\}$ and $G_{-1} \in \ker K = \{G \in C^\infty(\mathbb{R}) \mid KG=0\}$. We define Lenard's sequence

$$G_j = \begin{cases} £^j G_0, & j \in \mathbb{Z} \\ £^{j+1}G_{-1}, & j \in \mathbb{Z}, \end{cases} \tag{7}$$

where £ and $£^{-1}$ are defined by Eqs. (5) and (6), respectively. Therefore, we generate a new hierarchy of nonlinear evolution equations (NLEEs):

$$m_{t_k} = JG_k, \quad \forall\, k \in \mathbb{Z}. \tag{8}$$

Apparently, this hierarchy includes the positive members ($k \geqslant 0$) and the negative members ($k<0$), and possesses the bi-Hamiltonian structure because of the Hamiltonian properties of K, J.

Let us now give special equations in the hierarchy (8).

- Choosing $G_0 = 1/2m^2 \in \ker J$ leads to the second positive member of the hierarchy:

$$m_t = \frac{1}{2}\left(\frac{1}{m^2}\right)_{xxx} - \frac{1}{2}\left(\frac{1}{m^2}\right)_x. \tag{9}$$

This is a new integrable equation. Later in this section, we give its Lax pair. We will also study the soliton solution for this new equation. Equation (9) has the following Hamiltonian structure:

$$m_t = \frac{1}{2}\left(\frac{1}{m^2}\right)_{xxx} - \frac{1}{2}\left(\frac{1}{m^2}\right)_x = J\frac{\delta H_1^+}{\delta m} = K\frac{\delta H_0^+}{\delta m}, \tag{10}$$

where

$$H_0^+ = -\frac{1}{2}\int_\Omega \frac{1}{m}\mathrm{d}x,$$

$$H_1^+ = -\frac{1}{2}\int_\Omega \left(\frac{1}{4m^3} + \left(\frac{4}{5m^5} + \frac{4}{7m^7}\right)m_x^2\right)\mathrm{d}x.$$

- Choosing $G_{-1}=1 \in \ker K$ yields $G_{-2}=2u$, and therefore, we obtain the second negative member of the hierarchy:

$$m_t = JG_{-2} = -(m(u^2-u_x^2))_x, \quad m = u - u_{xx}. \tag{11}$$

This equation is another new integrable equation with a new type of soliton solution—W-shape soliton solution—which is proposed by Qiao in 2005, see Ref. 22 for more details. Equation (11) can be cast in the following Hamiltonian structure:

$$m_t = -(m(u^2-u_x^2))_x = J\frac{\delta H_0^-}{\delta m} = K\frac{\delta H_1^-}{\delta m}, \tag{12}$$

where

$$J = -\partial m \partial^{-1} m \partial, \tag{13}$$

$$K = \partial^3 - \partial,$$

$$H_0^- = 2\int_\Omega mu\mathrm{d}x,$$

$$H_1^- = \frac{1}{4}\int_\Omega (u^4 + 2u^2u_x^2)\mathrm{d}x, \tag{14}$$

$\Omega=(x_0,x_0+T)$ or $\Omega=(-\infty,+\infty)$ is the domain of u which needs to be periodic with T or to approach a constant. Apparently, both operator J and operator K are Hamiltonian. So, our Eqs. (11) and (9) are bi-Hamiltonian.

Of course, we may generate further nonlinear equations by selecting different members in the hierarchy. In the following, we will see that all equations in the hierarchy (8) are integrable. Particularly, the above two equations (9) and (11) are integrable.

Let us return to the spectral problem (3) and continue using the notations in Sec. II. Apparently, the Gateaux derivative matrix $U_*(\xi)$ of the spectral matrix U in the direction $\xi \in C^\infty(\mathbb{R})$ at point m is

$$U_*(\xi) \triangleq \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} U(u+\epsilon\xi) = \begin{pmatrix} 0 & (1/2)\lambda\xi \\ -(1/2)\lambda\xi & 0 \end{pmatrix}, \tag{15}$$

which is obviously an injective homomorphism, i.e., $U^*(\xi)=0 \Leftrightarrow \xi=0$.

For any given C^∞-function G, we construct the following 2×2 matrix equation with respect to $V=V(G)$:

$$V_x - [U,V] = U_*(KG - \lambda^2 JG). \tag{16}$$

Theorem 1: For the spectral problem (3) and an arbitrary C^∞-function G, the matrix equation (16) has the following solution:

$$V = -\frac{1}{2}\lambda\begin{pmatrix} \lambda\partial^{-1}mG_x & G_x - G_{xx} - \lambda^2 m\partial^{-1}mG_x \\ G_x + G_{xx} + \lambda^2 m\partial^{-1}mG_x & -\lambda\partial^{-1}mG_x \end{pmatrix}, \tag{17}$$

where $\partial = \partial_x = \partial/\partial x, \partial^{-1}\partial = \partial\partial^{-1} = 1$, and subscripts stand for the partial derivatives in x.

Proof: Let us set

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & -V_{11} \end{pmatrix}$$

and substitute it into Eq. (16), which is an overdetermined equation. Using calculation techniques in Ref. 23, we obtain the following results:

$$V_{11} = -\frac{1}{2}\lambda^2\partial^{-1}mG_x,$$

$$V_{12} = -\frac{1}{2}\lambda(G_x - G_{xx}) + \frac{1}{2}\lambda^3 m\partial^{-1}mG_x,$$

$$V_{21} = -\frac{1}{2}\lambda(G_x + G_{xx}) - \frac{1}{2}\lambda^3 m\partial^{-1}mG_x,$$

which complete the proof.

Theorem 2: Let $G_0 \in \ker J$, $G_{-1} \in \ker K$, and let each G_j be given through Eq. (7). Then the following are obtained.

1. *Each new vector field $X_k = JG_k$, $k \in \mathbb{Z}$ satisfies the following commutator representation:*

$$V_{k,x} - [U, V_k] = U_*(X_k), \quad \forall\, k \in \mathbb{Z}. \tag{18}$$

2. *The new hierarchy (8), i.e.,*

$$m_{t_k} = X_k = JG_k, \quad \forall\, k \in \mathbb{Z}, \tag{19}$$

possesses the zero curvature representation

$$U_{t_k} - V_{k,x} + [U, V_k] = 0, \quad \forall\, k \in \mathbb{Z}, \tag{20}$$

where

$$V_k = \sum V(G_j)\lambda^{2(k-j-1)}, \quad \sum = \begin{cases} \sum_{j=0}^{k-1}, & k > 0 \\ 0, & k = 0 \\ -\sum_{j=k}^{-1}, & k < 0, \end{cases} \tag{21}$$

and $V(G_j)$ is given by Eq. (17) with $G = G_j$.

Proof:

1. For $k=0$, it is obvious. For $k<0$, we have

$$
\begin{aligned}
V_{k,x}-[U,V_k] &= -\sum_{j=k}^{-1}(V_x(G_j)-[U,V(G_j)])\lambda^{2(k-j-1)} = -\sum_{j=k}^{-1}U_*(KG_j-\lambda^2 KG_{j-1})\lambda^{2(k-j-1)} \\
&= U_*\left(\sum_{j=k}^{-1}KG_{j-1}\lambda^{2(k-j)}-KG_j\lambda^{2(k-j-1)}\right) = U_*(KG_{k-1}-KG_{-1}\lambda^{2k}) = U_*(KG_{k-1}) \\
&= U_*(X_k).
\end{aligned}
$$

For the case of $k>0$, it is similar to prove.

2. Noticing $U_{t_k}=U_*(m_{t_k})$, we obtain

$$U_{t_k}-V_{k,x}+[U,V_k]=U_*(m_{t_k}-X_k).$$

The injectiveness of U_* implies that the second result holds.

So, the hierarchy (8) has Lax pair and all equations in the hierarchy are therefore integrable. In particular, our new equations (9) and (11) have the following Lax pairs:

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x = U(m,\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_t = V_1(m,\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

and

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x = U(m,\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_t = V_{-2}(m,u,\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \quad m=u-u_{xx},$$

respectively, where $U(m,\lambda)$ is defined by Eq. (3), and

$$V_1(m,\lambda)=\frac{\lambda}{2}\begin{pmatrix} -\lambda/m & \lambda^2+[m(m_x-m_{xx})+3m_x^2/m^4] \\ -\lambda^2+[m(m_x+m_{xx})-3m_x^2/m^4] & \dfrac{\lambda}{m}\end{pmatrix},$$

$$V_{-2}(m,u,\lambda)=\begin{pmatrix} \lambda^{-2}+(1/2)(u^2-u_x^2) & -\lambda^{-1}(u-u_x)-(1/2)\lambda m(u^2-u_x^2) \\ \lambda^{-1}(u+u_x)+(1/2)\lambda m(u^2-u_x^2) & -\lambda^{-2}-(1/2)(u^2-u_x^2)\end{pmatrix}.$$

Thus, both of them are integrable.

IV. PARAMETRIC SOLUTION OF THE NEGATIVE MEMBERS IN THE HIERARCHY

To get the parametric solution of the hierarchy, we use the constrained method[4,23] which connects finite dimensional integrable systems to the PDEs.

A. Hamiltonian systems on a symplectic submanifold

Let λ_j $(j=1,\ldots,N)$ be N distinct spectral values of spectral problem (3), and q_j,p_j be the spectral functions corresponding to λ_j, respectively. Then we have

$$q_x = -\frac{1}{2}q + \frac{1}{2}m\Lambda p,$$

$$p_x = \frac{1}{2}p - \frac{1}{2}m\Lambda q, \tag{22}$$

where $\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_N)$, $q=(q_1,q_2,\ldots,q_N)^T$, $p=(p_1,p_2,\ldots,p_N)^T$.

Now, we restrict the ODE system (22) to the following $(2N-2)$-dimensional symplectic submanifold M of $\mathbb{R}^{2N}$:

$$M = \{(p,q)^T \in \mathbb{R}^{2N} | F=0, G=0\}, \tag{23}$$

where $F=\frac{1}{2}(\langle\Lambda q,q\rangle-1)$, $G=\frac{1}{2}(\langle\Lambda p,p\rangle-1)$, and $\langle\cdot,\cdot\rangle$ stands for the standard inner product in $\mathbb{R}^N$. Thus, on the submanifold M, we obtain a constraint of m related to the spectral functions p, q:

$$m = \frac{1}{\langle\Lambda^2 p,q\rangle}. \tag{24}$$

Remark 1: $\langle\Lambda^2 p,q\rangle \neq 0$ is necessary because it guarantees that M is a symplectic submanifold of $\mathbb{R}^{2N}$.

Under the constraint (24), on the symplectic submanifold M the finite dimensional system (22) is changed to

$$q_x = -\frac{1}{2}q + \frac{1}{2}\frac{1}{\langle\Lambda^2 p,q\rangle}\Lambda p,$$

$$p_x = \frac{1}{2}p - \frac{1}{2}\frac{1}{\langle\Lambda^2 p,q\rangle}\Lambda q, \tag{25}$$

which form a $(2N-2)$-dimensional nonlinear system on M with respect to p, q. Is it integrable? To see this, in $\mathbb{R}^{2N}$ the standard Poisson bracket[3] of two functions F_1, F_2 is defined as follows:

$$\{F_1,F_2\} = \left\langle \frac{\partial F_1}{\partial q},\frac{\partial F_2}{\partial p}\right\rangle - \left\langle \frac{\partial F_1}{\partial p},\frac{\partial F_2}{\partial q}\right\rangle, \tag{26}$$

which is antisymmetric, bilinear, and satisfies the Jacobi identity.

Obviously,

$$\{F,G\} = \langle\Lambda^2 p,q\rangle \neq 0. \tag{27}$$

Because the system (25) is imposed on the submanifold M, we need to introduce the so-called Dirac-Poisson bracket of two functions f, g on M:

$$\{f,g\}_D = \{f,g\} + \frac{1}{\{F,G\}}(\{f,F\}\{G,g\} - \{f,G\}\{F,g\}), \tag{28}$$

which is still a Poisson bracket, namely, antisymmetric, bilinear, and satisfies, the Jacobi identity.

Let us now choose a very simple Hamiltonian

$$H^- = -\frac{1}{2}\langle p,q\rangle, \tag{29}$$

then, the system (25) is able to be cast in a canonical Hamiltonian form in the support of Dirac-Poisson bracket on M:

$$q_x = \{q,H^-\}_D,$$

$$p_x=\{p,H^-\}_D. \tag{30}$$

It is easy to check that $\mathrm{d}H^-/\mathrm{d}x=0$, i.e., H^- is invariant along the flow (30).

To show the integrability of canonical system (30), we need to figure out a system of independent functions in involution.

B. Integrability on the symplectic submanifold

For the Hamiltonian canonical system (30), we set up the following independent functions:

$$F_k^-=\frac{1}{4}(\langle\Lambda^{2k+1}p,p\rangle-\langle\Lambda^{2k+1}q,q\rangle)+\frac{1}{4}\sum_{j=k}^{-1}(\langle\Lambda^{2j+2}p,q\rangle\langle\Lambda^{2(k-j)}p,q\rangle-\langle\Lambda^{2j+3}q,q\rangle\langle\Lambda^{2(k-j)-1}p,p\rangle)$$

$$k=-1,-2,-3,\ldots,$$

where Λ^{-1} is the inverse of Λ. According to the Dirac-Poisson bracket (28), each F_k^- produces a canonical Hamiltonian system on M:

$$q_{t_k}=\{q,F_k^-\}_D,$$

$$p_{t_k}=\{p,F_k^-\}_D. \tag{31}$$

Therefore, on submanifold M we have the following theorem.

Theorem 3: All canonical Hamiltonian flows (30) and (31) mutually commute on M.

Proof: Through a lengthy calculation, we have

$$\{H,F_k^-\}_D=0,$$

$$\{F_k^-,F_l^-\}_D=0,$$

which complete the proof.

So, all Hamiltonian flows (30) and (31) are integrable on M.

C. Parametric solutions of all negative members

Since Hamiltonian flows (H^-) and (F_k^-) are completely integrable on M and their Dirac-Poisson bracket $\{H^-,F_k^-\}=0$ $(k=-1,-2,\ldots)$., their phase flows $g_{H^-}^x$, $g_{F_k^-}^{t_k}$ are commutable.[3] Thus, we can define their compatible solution as follows:

$$\begin{pmatrix}q(x,t_k)\\ p(x,t_k)\end{pmatrix}=g_{H^-}^x g_{F_k^-}^{t_k}\begin{pmatrix}q(x^0,t_k^0)\\ p(x^0,t_k^0)\end{pmatrix},\quad k=-1,-2,\ldots, \tag{32}$$

where x^0, t_k^0 are the initial values of phase flows $g_{H^-}^x$, $g_{F_k^-}^{t_k}$.

Theorem 4: Let $p(x,t_k),q(x,t_k)$ be the compatible solution of the two integrable flows (30) and (31), then

$$m=\frac{1}{\langle\Lambda^2p(x,t_k),q(x,t_k)\rangle} \tag{33}$$

satisfies the kth negative member $m_{t_k}=JG_k$ $(k<0,k\in\mathbb{Z})$ in the hierarchy (19).

Proof: Noticing the following formulas

$$G_k=\frac{1}{2}(\langle\Lambda^{2k+3}q,q\rangle+\langle\Lambda^{2k+3}p,p\rangle),\quad k=-1,-2,-3,\ldots,$$

$$G_{k,x}=\frac{1}{2}(\langle\Lambda^{2k+3}p,p\rangle-\langle\Lambda^{2k+3}q,q\rangle),$$

$$\partial^{-1}mG_{k,x}=\langle\Lambda^{2k+2}p,q\rangle,$$

and Eq. (31), we directly compute and find that Eq. (33) satisfies $m_{t_k}=JG_k=-\partial m\partial^{-1}mG_{k,x}$ ($k<0,k\in\mathbb{Z}$) which completes the proof.

In particular, we obtain the following corollary.

Corollary 1: Let $p(x,t),q(x,t)$ be the compatible solution of the two integrable flows (30) and (31) with $k=-2$ on the symplectic submanifold M, then

$$m=\frac{1}{\langle\Lambda^2p(x,t),q(x,t)\rangle}, \tag{34}$$

$$u=-\frac{\langle\Lambda^{-1}q(x,t),q(x,t)\rangle+\langle\Lambda^{-1}p(x,t),p(x,t)\rangle}{2\langle p(x,t),q(x,t)\rangle}, \tag{35}$$

satisfy our new equation

$$m_t+m_x(u^2-u_x^2)+2m^2u_x=0,\quad m=u-u_{xx}. \tag{36}$$

Proof: A direct substitution leads to the result on M.

Similarly, we can discuss the parametric solution of the positive members $m_{t_k}=JG_k=-\partial m\partial^{-1}mG_{k,x}$ ($k>0,k\in\mathbb{Z}$) in the hierarchy (8). That needs us to consider a new kind of constraint and related integrable system, which we deal with in the next section.

V. PARAMETRIC SOLUTION OF THE POSITIVE MEMBERS IN THE HIERARCHY

Let us consider the following constraint:

$$G_0=\sum_{j=1}^{N}\nabla\lambda_j, \tag{37}$$

where $\nabla\lambda_j=\frac{1}{2}\lambda_j(q_j^2+p_j^2)$ is the functional gradient of λ_j for the spectral problem (3), and q_j, p_j are the eigenfunctions corresponding to λ_j. By $G_0=1/2m^2$, Eq. (37) is saying that

$$m=\frac{1}{\sqrt{\langle\Lambda q,q\rangle+\langle\Lambda p,p\rangle}}, \tag{38}$$

which forms a new constraint between the potential function m and the eigenfunctions q_j, p_j ($j=1,\ldots,N$) in the whole space $\mathbb{R}^{2N}$. Under this constraint, the spectral problem (3) is cast in another canonical Hamiltonian system in $\mathbb{R}^{2N}$:

$$(H^+):\quad \begin{aligned} q_x&=-\frac{1}{2}q+\frac{1}{2}\frac{1}{\sqrt{\langle\Lambda q,q\rangle+\langle\Lambda p,p\rangle}}=\{q,H^+\},\\ p_x&=-\frac{1}{2}\frac{1}{\sqrt{\langle\Lambda q,q\rangle+\langle\Lambda p,p\rangle}}+\frac{1}{2}q=\{p,H^+\}, \end{aligned} \tag{39}$$

with the Hamiltonian

$$H^+=\frac{1}{2}\sqrt{\langle\Lambda q,q\rangle+\langle\Lambda p,p\rangle}-\frac{1}{2}\langle p,q\rangle. \tag{40}$$

To see the integrability of the system (39), like the negative case, we take into account of the following independent functions:

$$F_k^+ = \frac{1}{4}\frac{\langle p,q\rangle}{\sqrt{\langle\Lambda p,p\rangle + \langle\Lambda q,q\rangle}}(\langle\Lambda^{2k+1}p,p\rangle + \langle\Lambda^{2k+1}q,q\rangle) + \frac{1}{4}(\langle p,q\rangle - 2\sqrt{\langle\Lambda p,p\rangle + \langle\Lambda q,q\rangle})\langle\Lambda^{2k}p,q\rangle$$
$$+ \frac{1}{4}\sum_{j=0}^{k-1}(\langle\Lambda^{2j+1}q,q\rangle\langle\Lambda^{2(k-j)-1}p,p\rangle - \langle\Lambda^{2j}p,q\rangle\langle\Lambda^{2(k-j)}p,q\rangle),$$

$$k = 1,2,3\ldots, \tag{41}$$

which generates the canonical Hamiltonian system in $\mathbb{R}^{2N}$ for each k:

$$(F_k^+): \begin{matrix} q_{t_k} = \{q,F_k^+\}, \\ p_{t_k} = \{p,F_k^+\}, \end{matrix} \quad k = 1,2,3,\ldots. \tag{42}$$

Through a lengthy computation, we obtain

$$\{H^+,F_k^+\} = 0, \quad \{F_l^+,F_k^+\} = 0, \quad k,l = 1,2,\ldots, \tag{43}$$

which show that each Hamiltonian t_k-flow (F_k^+) commutes with both each other and Hamiltonian x-flow (H^+). Thus, all Hamiltonian canonical systems (F_k^+) are integrable in $\mathbb{R}^{2N}$. Particularly, the constrained system (39) is integrable.

Using a derivation procedure similar to the last section, we obtain the following theorem.

Theorem 5: Let $p(x,t_k),q(x,t_k)$ $(k=1,2,3,\ldots)$ be the compatible solution of the two integrable flows (39) and (42) in $\mathbb{R}^{2N}$, then for each k,

$$m = \frac{1}{\sqrt{\langle\Lambda q(x,t_k),q(x,t_k)\rangle + \langle\Lambda p(x,t_k),p(x,t_k)\rangle}}, \quad k = 1,2,3,\ldots \tag{44}$$

satisfies the kth positive member $m_{t_k}=JG_k$, $(k>0, k\in\mathbb{Z})$ in the hierarchy (19).

In particular, we have the following corollary.

Corollary 2: Let $p(x,t),q(x,t)$ be the compatible solution of the two integrable flows (H^+) and (F_1^+) given by Eqs. (39) and (41) with $k=1$, respectively. Then

$$m = \frac{1}{\sqrt{\langle\Lambda q(x,t),q(x,t)\rangle + \langle\Lambda p(x,t),p(x,t)\rangle}} \tag{45}$$

is a parametric solution of our new equation

$$m_t = \frac{1}{2}(m^{-2})_{xxx} - \frac{1}{2}(m^{-2})_x. \tag{46}$$

Proof: A direct verification completes the proof.

VI. W-/M-SHAPE PEAK SOLITONS

Let us consider the traveling wave solutions of two new equations (9) and (11) through a generic setting $m(x,t)$ or $u(x,t)=U(x-ct)$, where c is the wave speed.

A. W-/M shape peak solitons of Eq. (9)

Let us first solve Eq. (9). Set $m=1/\sqrt{v(x,t)}$, then Eq. (9) becomes

$$-\frac{(\partial/\partial t)v(x,t)}{v(x,t)^{(3/2)}} = \frac{\partial^3}{\partial x^3}v(x,t) - \frac{\partial}{\partial x}v(x,t). \tag{47}$$

Denote $\xi=x-ct$ and let $v(x,t)=U(\xi)$. Substituting it into Eq. (47) yields the following ODE:

$$U_{\xi\xi\xi} - U_{\xi} = cU^{-3/2}U_{\xi}. \tag{48}$$

Generally, we have the following trivial facts.

1. Any constant function is a solution of Eq. (47) and the ODE (48).
2. Any translation $U(\xi-\xi_0)$ of a solution $U(\xi)$ of ODE (48) is still a solution.
3. If $v(x,t)$ is a solution of Eq. (47), then any translation $v(x-x_0,t-t_0)$ in space x and time t is a solution, too.

Because of the translation invariance of the differential equation (48), without any loss of generality, we choose ξ_0 as vanishing, namely, $\xi_0=0$. Apparently U=const is a solution, which is not interesting for us. Let us find nontrivial solutions. Taking indefinite integral twice on both sides of the ODE (48), we obtain

$$\frac{2c}{\sqrt{U}} + U_{\xi\xi} - U + C_1 = 0, \tag{49}$$

$$4c\sqrt{U} + C_1 U - \frac{U^2}{2} + \frac{1}{2}U_{\xi}^2 + C_2 = 0, \tag{50}$$

where C_1 and C_2 are two constants to be determined.

To have solitary traveling wave solutions, we set $U=V^2$ and impose the boundary condition

$$\lim_{\xi\to\pm\infty} V = A, \quad A > 0, \tag{51}$$

which implies $m \to 1/A$ as x approaches $\pm\infty$ (see Ref. 26 for more details). We can figure out the two constants C_1, C_2 through substituting the boundary condition (51) into the ODEs (49) and (50), which generate the following two constants:

$$C_1 = A^2 - \frac{2c}{A}, \tag{52}$$

$$C_2 = -\frac{1}{2}A^4 - 2cA. \tag{53}$$

So the ODE (50) becomes

$$\frac{dV}{d\xi} = -\operatorname{sign}(\xi)\frac{(V-A)\sqrt{AV^2 + 2A^2V + A^3 + 4c}}{2\sqrt{A}V}.$$

Taking integral on both sides of the above equation, we arrive at

$$2\ln\left(A + V + \sqrt{(A+V)^2 + \frac{4c}{A}}\right) - \frac{\sqrt{A^3}}{\sqrt{A^3+c}}\left(2\ln 2 + \ln\frac{A^3 + 2c + A^2V + \sqrt{(A^3+c)(AV^2 + 2A^2V + A^3 + 4c)}}{A(V-A)}\right) = -|\xi|.$$

In general, we cannot get an explicit form of V. But, if $\sqrt{A^3}/\sqrt{A^3+c}=2$, namely, $c=-3/4A^3$, then we have

$$2\ln(A + V + \sqrt{V^2 + 2AV - 2A^2}) - 2\ln 2 - 2\ln\frac{A(-A + 2V + \sqrt{V^2 + 2AV - 2A^2})}{V-A} = -|\xi|,$$

which implies

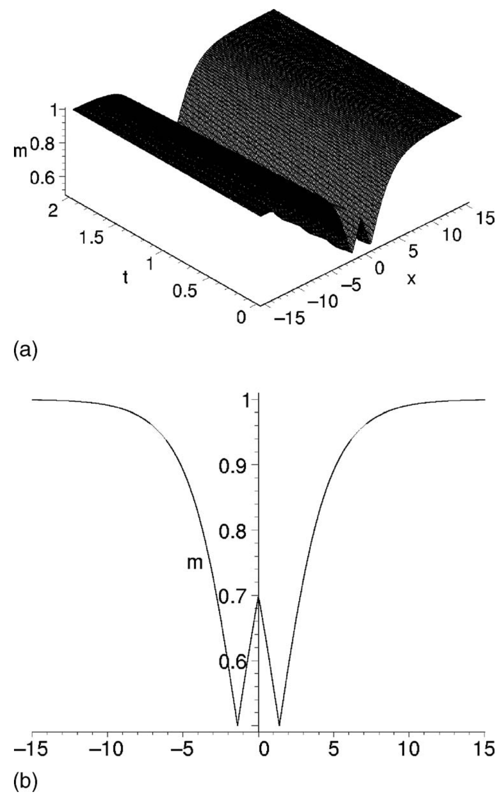


FIG. 1. (a) 3D graph of the explicit solution $m(x,t)$ defined by Eq. (55) when $B=1$, wave speed $c=-3/4$, and intervals of x, t, m: $-15\leqslant x\leqslant 15$, $0\leqslant t\leqslant 2$, $0\leqslant m\leqslant 1$. (b) 2D graph of the explicit solution $m(x,t)$ defined by Eq. (55) at $t=0$. This is a W-shape peak soliton solution.

$$V = A\frac{3+2X+3X^2-\sqrt{3(3+2X+3X^2)(X-1)^2}}{4X},$$

$$X = \mathrm{e}^{-1/2|\xi|+\ln 2},$$

$$\xi = x + \frac{3}{4}A^3 t.$$

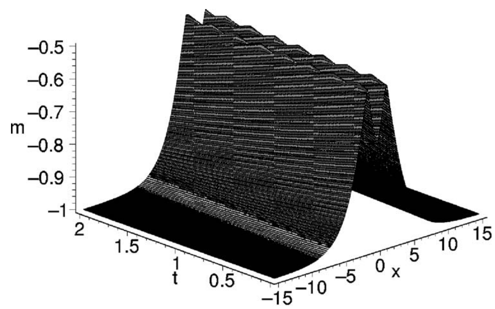
Since $m=1/V$, we denote $B=1/A\neq 0$, then $m\to B$ as $\xi\to\pm\infty$, therefore we obtain the following explicit solution of Eq. (9):

$$m(x,t) = B\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\sqrt{\frac{(X-1)^2}{3X^2+2X+3}}\right),$$

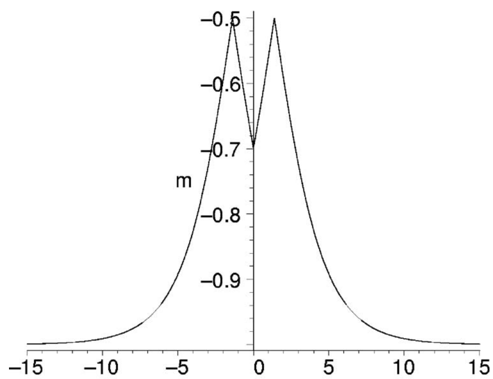
$$X = \mathrm{e}^{-1/2|x+3/4B^3t|+\ln 2}, \tag{54}$$

which is able to be converted to

$$m(x,t) = \frac{B}{2}\left(1+\sqrt{6}\frac{\sinh|s/2|}{\sqrt{3\cosh s+1}}\right),$$



(a)



(b)

FIG. 2. (a) 3D graph of solution (57) when wave speed $c=3/4$ and intervals of x, t, m: $-15\leqslant x\leqslant 15$, $0\leqslant t\leqslant 2$, $0\leqslant m\leqslant 1$. (b) 2D graph of solution (57) at $t=0$. This is an M-shape peak soliton solution.

$$s=\frac{1}{2}\left|x+\frac{3}{4B^3}t\right|-\ln 2. \tag{55}$$

We take $B=1$, then

$$m(x,t)=\frac{1}{2}\left(1+\sqrt{6}\frac{\sinh|s/2|}{\sqrt{3\cosh s+1}}\right),$$

$$s=\frac{1}{2}\left|x+\frac{3}{4}t\right|-\ln 2, \tag{56}$$

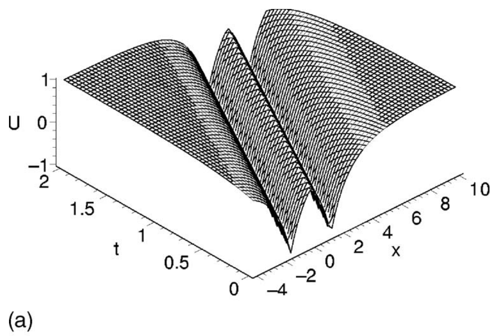
whose three-dimensional (3D) and two-dimensional (2D) graphs are plotted in Fig. 1. This solution is of W-shape peak soliton.[22]

We can also set $m=-1/V$ and take negative $B\neq 0$ as its infinity, limit. For instance, $B=-1$, then $m\rightarrow -1$ as $\xi\rightarrow\pm\infty$ and we have

$$m(x,t)=-\frac{1}{2}\left(1+\sqrt{6}\frac{\sinh|s/2|}{\sqrt{3\cosh s+1}}\right),$$

$$s=\frac{1}{2}\left|x-\frac{3}{4}t\right|-\ln 2, \tag{57}$$

This is an M-shape peak soliton solution of Eq. (9). See Fig. 2 for the details.



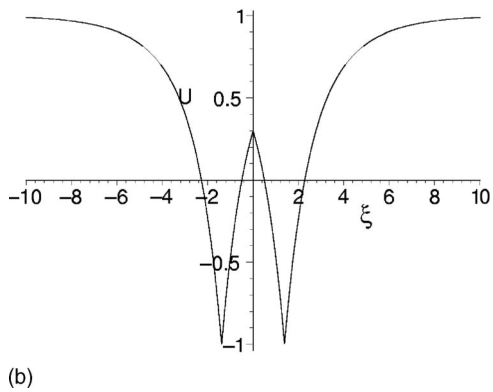


FIG. 3. 3D graph of the explicit solution $u(x,t)$ for Eq. (58), defined by Eq. (59) when $A=1$, $s=8/3$, wave speed $c=11/3$, and the range of x, tu: $-4\leqslant x\leqslant 10$, $0\leqslant x\leqslant 2$, $-1\leqslant u\leqslant 1$. (b) 2D graph of the explicit solution $u(x,t)$ defined by Eq. (59) when $A=1$, $s=8/3$, the wave speed $c=11/3$, and the range of ξ: $-10\leqslant\xi\leqslant 10$. This is a W-shape peak soliton solution.

B. W/M shape peak solitons of Eq. (11)

Similarly, we apply the above procedure to our new equation (11), namely,

$$m_t + m_x(u^2 - u_x^2) + 2m^2u_x = 0, \quad m = u - u_{xx}. \tag{58}$$

We arrive at the following explicit solutions:

$$u(x,t) = A\left(\frac{5}{3} - (3z+2)\left(z - \sqrt{z^2 - \frac{4}{9}}\right)\right),$$

$$z = \cosh\left(\frac{|x-(11/3)A^2t|}{2} - \ln 2\right) - \frac{1}{3}, \tag{59}$$

Since $A\neq 0$, there is no peaked soliton for homogeneous boundary conditions. Let us select a special $A=1$, then the solution reads

$$u(x,t) = 2 - 3\cosh^2 X + \left(\cosh X + \frac{1}{3}\right)\sqrt{3(3\cosh X+1)(\cosh X-1)},$$

$$X = \frac{|x-(11/3)t|}{2} - \ln 2.$$

This solution has three peaks and its profile looks like a W-type wave. So, we called it W-shape peak soliton. Three peaks occur at $x=\frac{11}{3}t_0$, $x=\frac{11}{3}t_0-2\ln 2$, $x=\frac{11}{3}t_0+2\ln 2$, for each time t_0. See Fig. 3 for more details.

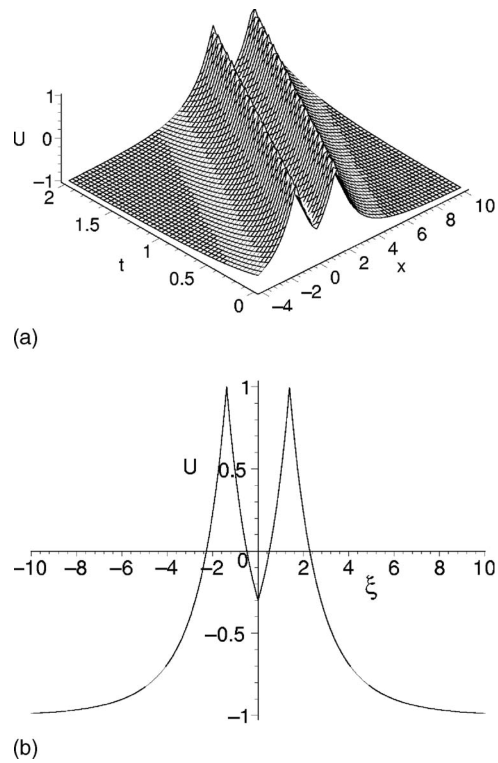


FIG. 4. (a) 3D graph of the M-shape peaks soliton $u(x,t)$ defined by Eq. (59) with $A=-1$ and $c=11/3$. (b) 2D graph of the M-shape peaks soliton $u(x,t)$ defined by Eq. (59) with $A=-1$ and $c=11/3$.

If we select the boundary constant $A=-1$, we are able to get the anti-W-shape peak soliton, called M-shape peak soliton. See a 3D and a 2D graph in Fig. 4 for more details.

VII. ONE-SINGLE-PEAK SOLITON

A. One-single-peak soliton for Eq. (9)

We just know that Eq. (9) has a three-peak (either W-shape peak or M-shape peaks) soliton solution. For Eq. (9), namely, $m_t=\frac{1}{2}(1/m^2)_{xxx}-\frac{1}{2}(1/m^2)_x$, let us consider the solution $m(x,t)$, defined by Eq. (54), without the absolute value of $x+(3/4B^3)t$. So, we create

$$M(x,t)=B\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\sqrt{\frac{X^2-2X+1}{3X^2+2X+3}}\right),$$

$$X=e^{-1/2(x+3/4B^3t)}, \quad B>0. \tag{60}$$

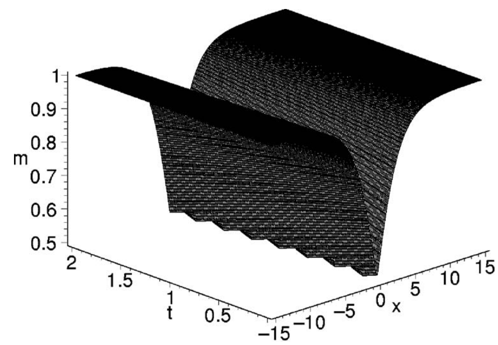
Note that there is no absolute value in X's expression. A direct verification reveals that $M(x,t)$ still satisfies Eq. (9).

We view solution (60) as a function of $\xi=x+3/4B^3t$. Then apparently, $M(\xi)$ has the following properties:

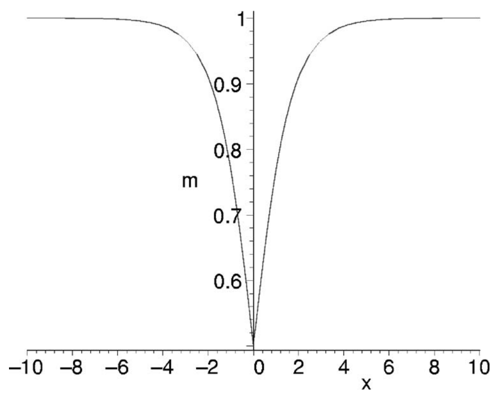
$$M(0)=\frac{1}{2}B, \quad M'(0+)=\frac{\sqrt{6}}{8}B, \quad M'(0-)=-\frac{\sqrt{6}}{8}B.$$

So, we found a continuous and piecewise-smooth (but not smooth) soliton solution for our new equation (9). See the graphs of $M(x,t)$ in Fig. 5.

Regarding negative $B<0$, we have similar consequence. For instance, $B=-1$, we have



(a)



(b)

FIG. 5. (a) 3D graph of the explicit solution $M(x,t)$ defined by Eq. (60) when $B=1$, wave speed $c=-3/4$, and intervals of x, t, M: $-15\leqslant x\leqslant 15$, $0\leqslant t\leqslant 2$, $0\leqslant M\leqslant 1$. (b) 2D graph of the explicit solution $M(x,t)$ defined by Eq. (60) at $t=0$. This is a one-single-peak soliton solution.

$$M(\xi)=-\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\sqrt{\frac{X^2-2X+1}{3X^2+2X+3}}\right),$$

$$X=\mathrm{e}^{-1/2\xi},\quad \xi=x-\frac{3}{4}t.$$

In this case, $M(0)=-\frac{1}{2}$, $M'(0+)=-\sqrt{6}/8$, $M'(0-)=\sqrt{6}/8$ imply that $M(\xi)$ is another continuous and piecewise-smooth soliton solution. See Fig. 6 for more details.

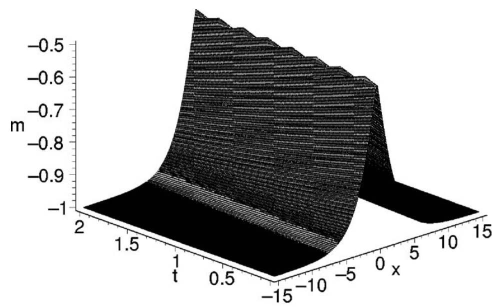
B. Cusp solitons for Eq. (11)

The above procedure is also available for Eq. (11), namely, $m_t+m_x(u^2-u_x^2)+2m^2u_x=0$, $m=u-u_{xx}$. This equation has the following solution:

$$U(\xi)=A\left(\frac{5}{3}-(3z+2)\left(z-\sqrt{z^2-\frac{4}{9}}\right)\right),$$

$$z=\cosh\left(\frac{x}{2}-\frac{11}{6}A^2t\right)-\frac{1}{3},\quad A\neq 0. \tag{61}$$

Let us take $A=\pm 1$. Then the corresponding solutions read



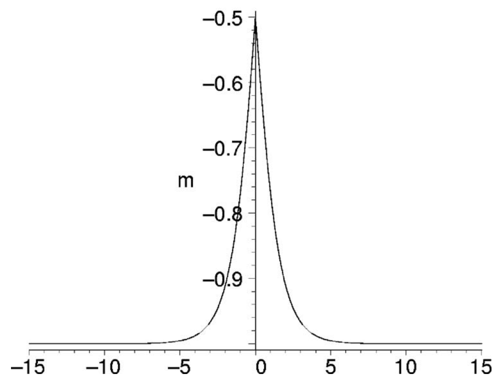


FIG. 6. 3D and 2D graphs of a continuous and piecewise-smooth soliton solution for Eq. (9) with negative amplitude. This is a one-single-peak soliton solution.

$$U(X) = \pm\left(2 - 3\cosh^2 X + \left(\cosh X + \frac{1}{3}\right)\sqrt{3(3\cosh X + 1)(\cosh X - 1)}\right),$$

$$X = \frac{x}{2} - \frac{11}{6}t,$$

which have the following characteristic features:

$$U(0) = \mp 1, \quad U'(0+) = \pm\infty, \quad U'(0-) = \mp\infty.$$

Apparently, they differ from the regular peakons.[5] So, the solution (61) is actually a cusp soliton of our new equation (11). See Figs. 7 and 8 for more details.

VIII. CONCLUSIONS AND OPEN PROBLEMS

In the paper, we present two new integrable systems (11) and (9), which are two special members in the whole hierarchy (8). Through solving a crucial matrix equation, we give the Lax pairs of the hierarchy, which guarantee the integrability of the whole hierarchy. Through placing the hierarchy in a symplectic manifold or submanifold, we investigate the constraint between spectral functions and potential functions, and obtain the parametric solutions of all equations in the hierarchy. In particular, we obtain the parametric solutions of Eqs. (11) and (9).

Actually, both Eqs. (11) and (9) can be reduced from the two-dimensional Euler equation by using the approximation procedure. For example, in two dimensional Euler equations[18] $\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla(p+gy)$, div $\mathbf{v}=0$, where p is a pressure and g is the gravitational acceleration constant, we take the velocity $\mathbf{v}=(-\psi_y,\psi_x)^T$, where ψ is a stream function. Then, the following equation

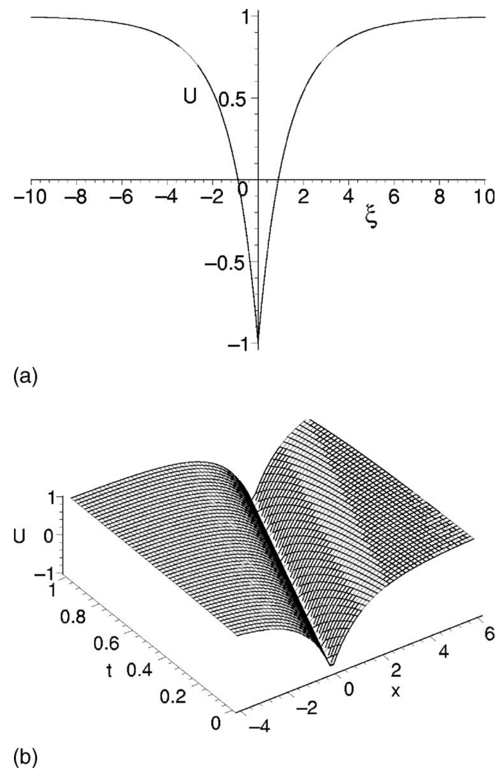


FIG. 7. (a) 2D graph of new cusp soliton solution $U(\xi)$ defined by Eq. (61) with amplitude $A=1$ and wave speed $c=11/3$. (b) 3D graph of new cusp soliton solution $U(\xi)$ defined by Eq. (61) with $A=1$ and $c=11/3$.

$$r_t+\psi_x r_y-\psi_y r_x=0, \quad r:=\psi_{xx}+\psi_{yy}$$

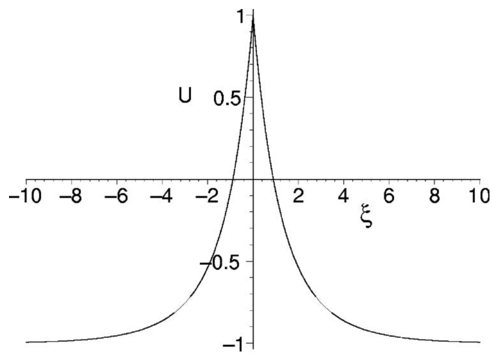
is generated from the Euler equations, where r stands for the vorticity. Employing $\psi=\phi(\xi,y,\tau)$, $\xi=\epsilon(x-c_0t)$, $\tau=\epsilon^3 t$, imposing $\phi(\xi,y,\tau)=\epsilon\phi_1(\xi,y,\tau)+\epsilon^2\phi_2(\xi,y,\tau)+\epsilon^3\phi_3(\xi,y,\tau)$ and $\phi_1(\xi,y,\tau)=B_1(y)F(\xi,\tau)$, $\phi_2(\xi,y,\tau)=B_2(y)F(\xi,\tau)+B_3(y)F(\xi,\tau)^2$, and picking up the coefficient of ϵ^4 term in the approximation expansion of the equation, we will eventually arrive at

$$F_\tau-a_1F_{\xi\xi\tau}+(3a_2F^2-a_3F_\xi^2)F_\xi-(2a_4F-2a_5F_{\xi\xi})F_\xi F_{\xi\xi}-((a_4-a_3)F^2-a_5F_\xi^2)F_{\xi\xi\xi}=0, \tag{62}$$

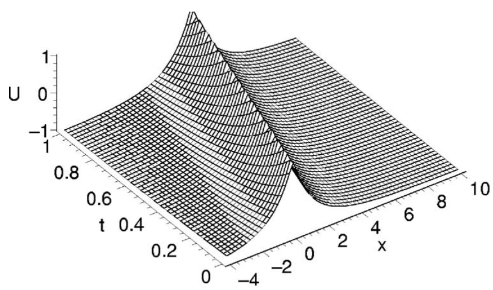
where $a_1,\ldots,a_5$ are constants. If we take $a_1=a_2=a_3=a_5=1$ and $a_4=2$, Eq. (62) exactly gives the new equation (11). In a similar way, Eq. (9) is also able to be generated from the two-dimensional Euler equation.

So, Eqs. (11) and (9) are the two new integrable equations derived from the 2D Euler equations. They very likely pertain also to the free surface problem in a water flow with vorticity, as it is the case for the CH equation.[10,19] This property may be the intrinsic difference between the two new equations and classical solitary equations. Another notable feature is that two new integrable equations (11) and (9) have no classical smooth solitons, because the first order derivatives of their traveling wave solitary solutions do not exist (see those derivative expressions on pages 16 and 18).

Moreover, through studying the two equations (11) and (9), we develop a new type of soliton solutions—W-shape peaks/M-shape peaks (three peaks, continuous, and piecewise smooth, but not smooth) soltions (see Figs. 1–4). Not only this but our new equations have new one-peak-soliton solutions (see Figs. 5 and 6) and new cusp solitons as well (see Figs. 7 and 8), which are apparently different from regular peakons. No smooth solitons are found for our equations, but our



(a)



(b)

FIG. 8. 2D graph of new cusp soliton solution $U(\xi)$ defined by Eq. (61) when $A=-1$ and $c=11/3$. (b) 3D graph of new cusp soliton solution $U(\xi)$ defined by Eq. (61) when $A=-1$ and $c=11/3$.

equations are completely integrable. The parametric solutions of our new equations cannot include their one-peak and W-/M-shape peak solitons since their parametric solutions are smooth and peaked solitons are not.

In the study of the CH traveling wave solutions, the CH peakons are orbitally stable,[11] namely, peakons are stable under small perturbations. But for our two equations (11) and (9), we do not know whether their cuspons, one-peak solitons, and M-/W-shape peak solitons are stable or not. Later, we will deal with this problem and construct the interaction of the two cuspons, two one-peak solitons, two W-shape peak solitons, two M-shape peak solitons, or one W-shape and the other M-shape solitons.

Furthermore, we suggest a more general partial differential equation: $m_t+m_x(u^2-u_x^2)+km^2u_x=0$, $m=u-u_{xx}$ with any constant $k\in\mathbb{R}$. When $k=2$, the equation is integrable, which is already discussed in this paper. Any other integrable cases? We do not know yet. Also for the water equation (62), what conditions can be driven so that it is integrable in addition to $a_1=a_2=a_3=a_5=1$ and $a_4=2$?

The ODE (50) has a physical meaning and can be cast into the Newton equation $U'^2=S(U)-S(A^2)$ of a particle with a new potential $S(U)=U^2+[2(2c-A^3)/A]U-8c\sqrt{U}$, or can be converted to $V'^2=T(V)-T(A)$ with $U=V^2$, $T(V)=(V^2/4)-(2c/V)+[A(A^3+4c)/4V^2]$. Likewise, our new equation (11) is able to be transformed to the following Newton equation $U'^2=P(U)-P(A^2)$ of a particle with a new potential $P(U)=U^2+\text{sign}(s)\sqrt{s(s+4A^2-4AU)}$, $s=c-A^2$, or to be changed to $V'^2=Q(V)-Q(A)$ with $U=A+(1/4sA)(s^2-V^2)$, $Q(V)=(V^2/4)+(4s|s|A^2/V)+[s^3(s-8A^2)/4V^2]$, $s=c-A^2$.

In the paper, we successfully solve those two new Newton systems with new one-single-peak solitons, cusp solitons, and M-/W-shape peak solitons. Those peaked and cusped solutions might be applied to neuroscience for providing a mathematical model and explaining electrophysiological responses of visceral nociceptive neurons and sensitization of dorsal root reflexes.[6,7]

ACKNOWLEDGMENT

The author's work was partially supported by UTPA-FRC and UTPA-URI grants.

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