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A finite-dimensional integrable system and the involutive solutions of the higher-order Heisenberg spin chain equations

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Abstract

By the use of the spectral problem nonlinearization method, a finite-dimensional integrable system and the involutive solutions of the higher-order Heisenberg spin chain equations are presented. In particular, the involutive solution of the well-known Heisenberg spin chain equation $u_t=\frac{1}{2}\mathrm{i}(u_{xx}w-w_{xx}u)$, $v_t=\frac{1}{2}\mathrm{i}(w_{xx}v-v_{xx}w)$ $(w^2+uv=1)$ is obtained.

In the middle of the 1970s, the continuous Heisenberg spin chain aroused considerable interest [1–4]. Tjon and Wright obtained the explicit formula for the single-soliton solution in the isotropic case [3]. Takhtajan studied the integration of the continuous Heisenberg spin chain equation through the inverse scattering transform method and obtained its Lax representation [4]. Afterwards, Chen and Li gave the higher-order Heisenberg spin chain equations [5]. All of these studies about the Heisenberg spin chain are admirable. However, within the author's knowledge, there have not been any reports on the solution representations of the higher-order Heisenberg spin chain equations.

In this Letter, using the spectral problem and Lax pair nonlinearization method [6,7], which was first suggested by Cao [8] in 1988 and was successfully applied to produce completely integrable finite-dimensional Hamiltonian systems in the Liouville sense, we first give a finite-dimensional integrable Hamiltonian system associated with the Heisenberg spin chain, and then through this completely integrable system in the Liouville sense and its involutive system, we present the solution representations of the higher-order Heisenberg spin chain equations. In particular, the solution representation of the well-known Heisenberg spin chain equation $u_t=\frac{1}{2}\mathrm{i}(u_{xx}w-w_{xx}u)$, $v_t=\frac{1}{2}\mathrm{i}(w_{xx}v-v_{xx}w)$ $(w^2+uv=1)$ is obtained.

Consider the Heisenberg spin chain spectral problem [4]

$$y_x=-\mathrm{i}\lambda Sy,\qquad S=\begin{pmatrix} w & u \\ v & -w\end{pmatrix},\qquad w^2+uv=1\,, \tag{1}$$

in which $y=(y_1,y_2)^{\mathrm{T}}$, u and v are two potentials, λ is a spectral parameter. Let λ_j $(1\leqslant j\leqslant N)$ be N different spectral parameters, and $y=(q_j,p_j)^{\mathrm{T}}$ be the associated spectral functions. Define $A_j\triangleq(\lambda_j p_j^2,\ -\lambda_j q_j^2)^{\mathrm{T}}$. Then A_j satisfies

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$$KA_j = \lambda_j \cdot JA_j\,, \tag{2}$$

where K and J are two operators ($\partial = \partial/\partial X$, $\partial\partial^{-1} = \partial^{-1}\partial = 1$),

$$K = \mathrm{i}\begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \quad J = \begin{pmatrix} u\partial^{-1}(u/w)\partial & u\partial^{-1}(v/w)\partial + 2w \\ -v\partial^{-1}(u/w)\partial - 2w & -v\partial^{-1}(v/w)\partial \end{pmatrix}, \tag{3}$$

which are called the pair of Lenard operators of the spectral problem (1).

Now, we recursively define the Lenard gradient sequence $\{G_j\}$ of (1) as follows,

$$G_0 = \alpha(v, u)^{\mathrm{T}} \in \operatorname{Ker} J, \qquad \alpha = \text{const}\,, \qquad KG_j = JG_{j+1}\,, \qquad j = 0, 1, 2, \ldots\,. \tag{4}$$

It is easy to see the recursion operator $\mathscr{L} = J^{-1}K$ is

$$\mathscr{L} = \frac{1}{2\mathrm{i}}\begin{pmatrix} (1/w)\partial - \frac{1}{2}v\partial^{-1}u\partial(1/w)\partial & \frac{1}{2}v\partial^{-1}v\partial(1/w)\partial \\ -\frac{1}{2}u\partial^{-1}u\partial(1/w)\partial & -(1/w)\partial + \frac{1}{2}u\partial^{-1}v\partial(1/w)\partial \end{pmatrix}. \tag{5}$$

The Lenard recursive sequence $\{G_j\}$ can be calculated through (4) and (5). $X_m = KG_m$ ($m = 0, 1, 2, \ldots$) are called the Heisenberg vector fields of (1), which yield the hierarchy of nonlinear evolution equations associated with (1),

$$(u, v)^{\mathrm{T}}_{t_m} = X_m(u, v) = KG_m = K\mathscr{L}^m G_0\,, \qquad m = 0, 1, 2, \ldots\,. \tag{6}$$

The first few terms in the hierarchy (6) are

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_0} = X_0(u, v) = \mathrm{i}\alpha\begin{pmatrix} u_x \\ v_x \end{pmatrix} \tag{7}$$

(this is trivial),

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_1} = X_1(u, v) = \tfrac{1}{2}\alpha\begin{pmatrix} w_{xx}u - u_{xx}w \\ v_{xx}w - w_{xx}v \end{pmatrix} \tag{8}$$

and

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = X_2(u, v) = -\tfrac{1}{4}\mathrm{i}\alpha\begin{pmatrix} u_{xxx} + \frac{3}{2}[u(u_x v_x + w_x^2)]_x \\ v_{xxx} + \frac{3}{2}[v(u_x v_x + w_x^2)]_x \end{pmatrix}. \tag{9}$$

Here, Eq. (8) is exactly the famous Heisenberg spin chain equation [4] when $\alpha = -\mathrm{i}$. Thus, (6) stands for the hierarchy of Heisenberg spin chain equations. It is not difficult to see that the Heisenberg hierarchy (6) possesses the Lax representations

$$Ly = \lambda y, \qquad L = \mathrm{i}\begin{pmatrix} w & u \\ v & -w \end{pmatrix}\partial, \qquad w^2 + uv = 1\,, \tag{10}$$

$$y_{t_m} = W_m y = -\mathrm{i}\sum_{j=0}^{m}\begin{pmatrix} -\frac{1}{2}\partial^{-1}[(u/w)G^{(1)}_{jx} + (v/w)G^{(2)}_{jx}] & G_j^{(2)} \\ G_j^{(1)} & \frac{1}{2}\partial^{-1}[(u/w)G^{(1)}_{jx} + (v/w)G^{(2)}_{jx}] \end{pmatrix}\lambda^{m+1-j}y\,, \tag{11}$$

where $G_j = (G_j^{(1)}, G_j^{(2)})^{\mathrm{T}}$ is determined by (4).

Now, we introduce a constraint relation [7] between the eigenfunctions and the potentials of (1),

$$G_0|_{\alpha=1} = \sum_{j=1}^{N} A_j\,, \tag{12}$$

which is equivalent to

$$u=-\langle\Lambda q,q\rangle,\qquad v=\langle\Lambda p,p\rangle\,. \tag{13}$$

Hence, $w=\sqrt{1-uv}=\sqrt{1+\langle\Lambda q,q\rangle\langle\Lambda p,p\rangle}$. Here, $p=(p_1,...,p_N)^T$, $q=(q_1,...,q_N)^T$, $\Lambda=\mathrm{diag}(\lambda_1,...,\lambda_N)$, $\langle\ \rangle$ stands for the standard inner-product in $\mathbb{R}^N$.

Under the constraint (13), (1) is nonlinearized as

$$q_x=-\mathrm{i}\sqrt{1+\langle\Lambda q,q\rangle\langle\Lambda p,p\rangle}\Lambda q+\mathrm{i}\langle\Lambda q,q\rangle\Lambda p,\qquad p_x=\mathrm{i}\langle\Lambda p,p\rangle\Lambda q+\mathrm{i}\sqrt{1+\langle\Lambda q,q\rangle\langle\Lambda p,p\rangle}\Lambda p\,. \tag{14}$$

Proposition 1. Suppose $(p,q)^T$ satisfies Eq. (14), then

$$(\sqrt{1+\langle\Lambda q,q\rangle\langle\Lambda p,p\rangle}-\langle\Lambda q,p\rangle)_x=0\,. \tag{15}$$

Proof.

$$\langle\Lambda p,q\rangle_x=\mathrm{i}(\langle\Lambda q,q\rangle\langle\Lambda^2p,p\rangle-\langle\Lambda p,p\rangle\langle\Lambda^2q,q\rangle)\,,$$

$$\langle\Lambda q,q\rangle_x=2\mathrm{i}(-\sqrt{1+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle}\langle\Lambda^2q,q\rangle+\langle\Lambda p,p\rangle\langle\Lambda^2p,q\rangle)\,,$$

$$\langle\Lambda p,p\rangle_x=2\mathrm{i}(-\langle\Lambda p,p\rangle\langle\Lambda^2p,q\rangle+\sqrt{1+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle}\langle\Lambda^2p,p\rangle)\,,$$

$$(\sqrt{1+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle})_x=(1+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle)^{-1/2}(\langle\Lambda q,q\rangle\langle\Lambda p,p_x\rangle+\langle\Lambda p,p\rangle\langle\Lambda q,q_x\rangle)$$
$$=\langle\Lambda p,q\rangle_x\,.$$

(15) implies $\sqrt{1+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle}=\langle\Lambda p,q\rangle+\beta$, $\beta=$const. Let $\beta=0$, then (14) can be rewritten as

$$q_x=-\mathrm{i}\langle\Lambda p,q\rangle\Lambda q+\mathrm{i}\langle\Lambda q,q\rangle\Lambda p,\qquad p_x=-\mathrm{i}\langle\Lambda p,p\rangle\Lambda q+\mathrm{i}\langle\Lambda p,q\rangle\Lambda p\,. \tag{16}$$

Proposition 2. (16) can be expressed in the Hamiltonian form

$$(H):\qquad q_x=\partial H/\partial p,\qquad p_x=-\partial H/\partial q\,, \tag{17}$$

with the Hamiltonian function

$$H=\tfrac{1}{2}\mathrm{i}\langle\Lambda q,q\rangle\langle\Lambda p,p\rangle-\tfrac{1}{2}\mathrm{i}\langle\Lambda p,q\rangle^2\,. \tag{18}$$

Proof. This is obvious.

In order to show the integrability of the Hamiltonian system (17), we introduce a set of functions F_m as follows,

$$F_m=\tfrac{1}{2}\mathrm{i}\sum_{j=0}^{m}(\langle\Lambda^{j+1}q,q\rangle\langle\Lambda^{m+1-j}p,p\rangle-\langle\Lambda^{j+1}p,q\rangle\langle\Lambda^{m+1-j}p,q\rangle)\,, \tag{19}$$

Note $H=F_0$.

The Poisson bracket of two Hamiltonian functions F, C in the symplectic space $(\mathbb{R}^{2N},\mathrm{d}p\wedge\mathrm{d}q)$ is defined by [9]

$$(F,G)=\sum_{j=1}^{N}\left(\frac{\partial F}{\partial q_j}\frac{\partial F}{\partial p_j}-\frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j}\right)=\langle F_q,G_p\rangle-\langle F_p,G_q\rangle\,. \tag{20}$$

F, G are called involutive if $(F,G)=0$.

According to formula (20), through some calculations, we can easily get

Proposition 3.

$$(F_m, F_n)=0,\ \forall m, n\in\mathbb{Z}^+ . \tag{21}$$

Proposition 4. $(H, F_m)=0$, $\forall m\in\mathbb{Z}^+$, thus the Hamiltonian system (17) is completely integrable in the Liouville sense and its involutive system is $\{F_m\}$.

Proposition 5. F_m defined by (19) is actually generated through nonlinearization of the time part (11) of the Lax pair for the Heisenberg hierarchy (6) under the constraint (13).

Proof. Acting with the recursive operator $\mathscr{L}$ upon (12), and noticing (2) and (4), we have

$$G_j=\mathscr{L}^j G_0\,|_{\alpha=1}=\sum_{k=1}^{N}\lambda_k^j A_k=\begin{pmatrix}\langle \Lambda^{j+1}p, p\rangle\\ -\langle \Lambda^{j+1}q, q\rangle\end{pmatrix}. \tag{22}$$

In virtue of (13), (14), (22) and $w=\sqrt{1+\langle \Lambda p, p\rangle\langle \Lambda q, q\rangle}=\langle \Lambda p, q\rangle$, we can deduce

$$-\tfrac{1}{2}\partial^{-1}\left(\frac{u}{w}G_{jx}^{(1)}+\frac{v}{w}G_{jx}^{(2)}\right)=\langle \Lambda^{j+1}p, q\rangle . \tag{23}$$

So, under the constraint (13), the time part (11) is nonlinearized as

$$q_{t_m}=-\mathrm{i}\sum_{j=0}^{m}\left(\langle \Lambda^{j+1}p, q\rangle \Lambda^{m+1-j}q-\langle \Lambda^{j+1}q, q\rangle \Lambda^{m+1-j}p\right),$$

$$p_{t_m}=-\mathrm{i}\sum_{j=0}^{m}\left(\langle \Lambda^{j+1}p, p\rangle \Lambda^{m+1-j}q-\langle \Lambda^{j+1}p, q\rangle \Lambda^{m+1-j}p\right). \tag{24}$$

After expressing (24) in Hamiltonian form, we immediately know its Hamiltonian function is none other than F_m. The proof is complete.

Proposition 6. Let $(q, p)^{\mathrm{T}}$ be a solution of the Hamiltonian system (17). Then $u=-\langle \Lambda q, q\rangle$, $v=\langle \Lambda p, p\rangle$ and $w=\sqrt{1+\langle \Lambda p, p\rangle\langle \Lambda q, q\rangle}=\langle \Lambda p, q\rangle$ satisfy a stationary Heisenberg evolution equation

$$K\mathscr{L}^N G_0\,|_{\alpha=1}+\sum_{j=0}^{N-1}\alpha_{N-j}K\mathscr{L}^j G_0\,|_{\alpha=1}=0 , \tag{25}$$

where the α_j are determined by $\lambda_1, ..., \lambda_N$.

Proof. Consider the polynomial ($\alpha_0=1$)

$$p(\lambda)=\prod_{l=1}^{N}(\lambda-\lambda_l)=\alpha_0\lambda^N+\alpha_1\lambda^{N-1}+...+\alpha_N . \tag{26}$$

Acting with the operator $K\sum_{j=0}^{N}\alpha_{N-j}$ upon (22) and using (26), we get (25). The Poisson bracket $(H, F_m)=0$ implies the Hamiltonian systems (H) and (F_m) are consistent, and their solution operators g_0^x, $g_m^{t_m}$ of the corresponding initial-value problems commute (see Ref. [9]). Denote the involutive solution [10] of the compatible equations (H) and (F_m) $q_{t_m}=\partial F_m/\partial p$, $p_{t_m}=-\partial F_m/\partial q$ by

$$\begin{pmatrix} q(x,t_m) \\ p(x,t_m) \end{pmatrix} = g_0^x g_m^{t_m} \begin{pmatrix} q(0,0) \\ p(0,0) \end{pmatrix}, \tag{27}$$

which are smooth functions of (x, t_m).

Proposition 7. Let $(q(x,t_m), p(x,t_m))^{\mathrm{T}}$ be an involutive solution of the compatible systems (H) and (F_m). Then

$$u(x,t_m) = -\langle \Lambda q, q\rangle, \qquad v(x,t_m) = \langle \Lambda p, p\rangle, \qquad w(x,t_m) = \sqrt{1 + \langle \Lambda p, p\rangle \langle \Lambda q, q\rangle} = \langle \Lambda p, q\rangle \tag{28}$$

satisfy the higher-order Heisenberg evolution equation

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_m} = -\mathrm{i}K\mathscr{L}^m G_0|_{\alpha=1}, \qquad G_0|_{\alpha=1} = (v, u)^{\mathrm{T}}, \qquad m = 0, 1, 2, \dots . \tag{29}$$

Proof. On the one hand, substituting (24) into the following two equalities, we have

$$\frac{\partial u}{\partial t_m} = -2\langle \Lambda q, q_{t_m}\rangle = 2\mathrm{i}(\langle \Lambda p, q\rangle \langle \Lambda^{m+2} q, q\rangle - \langle \Lambda^{m+2} p, q\rangle \langle \Lambda q, q\rangle),$$

$$\frac{\partial V}{\partial t_m} = 2\langle \Lambda p, p_{t_m}\rangle = 2\mathrm{i}(\langle \Lambda p, q\rangle \langle \Lambda^{m+2} p, p\rangle - \langle \Lambda^{m+2} p, q\rangle \langle \Lambda p, p\rangle).$$

On the other hand, from (3), (22) and (16), we get

$$-\mathrm{i}K\mathscr{L}^m G_0|_{\alpha=1} = 2\begin{pmatrix} -\langle \Lambda^{m+1} q, q_x\rangle \\ \langle \Lambda^{m+1} p, p_x\rangle \end{pmatrix} = 2\begin{pmatrix} -\langle \Lambda^{m+1} q, -\mathrm{i}\langle \Lambda p, q\rangle \Lambda q + \mathrm{i}\langle \Lambda q, q\rangle \Lambda p\rangle \\ \langle \Lambda^{m+1} p, -\mathrm{i}\langle \Lambda p, p\rangle \Lambda q + \mathrm{i}\langle \Lambda p, q\rangle \Lambda p\rangle \end{pmatrix}.$$

Thus, Proposition 7 holds.

As applications of Proposition 7, we give two examples below.

Example 1. When $m=1$, (29) exactly becomes the well known Heisenberg spin chain equation (HSCE) (i.e. Eq. (8) as $\alpha = -\mathrm{i}$)

$$u_{t_1} = \tfrac{1}{2}\mathrm{i}(u_{xx}w - w_{xx}u), \qquad v_{t_1} = \tfrac{1}{2}\mathrm{i}(w_{xx}v - v_{xx}w). \tag{30}$$

So, according to Proposition 7 and (27), we can know that the HSCE (30) possesses the solution representation

$$u(x,t_1) = -\langle \Lambda g_0^x g_1^{t_1} q(0,0), g_0^x g_1^{t_1} q(0,0)\rangle, \qquad v(x,t_1) = \langle \Lambda g_0^x g_0^{t_1} p(0,0), g_0^x g_1^{t_1} p(0,0)\rangle,$$

$$\omega(x,t_1) = [1 + \langle \Lambda g_0^x g_1^{t_1} p(0,0), g_0^x g_1^{t_1} p(0,0)\rangle \langle \Lambda g_0^x g_1^{t_1} q(0,0), g_0^x g_1^{t_1} q(0,0)\rangle]^{1/2}$$

$$= \langle \Lambda g_0^x g_1^{t_1} p(0,0), g_0^x g_1^{t_1} q(0,0)\rangle, \tag{31}$$

where g_0^x and $g_1^{t_1}$ stand for the solution operators of the initial-value problems of Hamiltonian systems (H) and (F_1), respectively.

Example 2. In (29), setting $m=2$, we may easily obtain

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = -\mathrm{i}K\mathscr{L}^2 G_0|_{\alpha=1} = -\tfrac{1}{4}\begin{pmatrix} u_{xxx} + \tfrac{3}{2}[u(u_x v_x + w_x^2)]_x \\ v_{xxx} + \tfrac{3}{2}[v(u_x v_x + w_x^2)]_x \end{pmatrix}, \tag{32}$$

which is none other than Eq. (9) when $\alpha = -\mathrm{i}$.

Thus, Eq. (32) (i.e. the third-order Heisenberg spin chain equation) has the solution representation

$$u(x,t_2) = -\langle \Lambda g_0^x g_2^{t_2} q(0,0), g_0^x g_2^{t_2} q(0,0)\rangle, \qquad v(x,t_2) = \langle \Lambda g_0^x g_2^{t_2} p(0,0), g_0^x g_2^{t_2} p(0,0)\rangle,$$

$$w(x,t_2) = [1 + \langle \Lambda g_0^x g_2^{t_2} p(0,0), g_0^x g_2^{t_2} p(0,0)\rangle \langle \Lambda g_0^x g_2^{t_2} q(0,0), g_0^x g_2^{t_2} q(0,0)\rangle]^{1/2}$$
$$= \langle \Lambda g_0^x g_2^{t_2} p(0,0), g_0^x g_2^{t_2} q(0,0)\rangle, \tag{33}$$

where g_0^x and $g_2^{t_2}$ are the solution operators of the initial-value problems of the systems (H) and (F_2), respectively.

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