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DARBOUX TRANSFORMATION AND SHOCK SOLITONS FOR COMPLEX MKDV EQUATION

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Abstract

In this paper, we use the Darboux transformation to find the solutions of the complex MKdV equation. The solutions are separated into the real and imaginary parts, and then we analyze them in different cases. We obtain smooth solution, periodic solution and shock soliton solution, and as a special case, when the imaginary part is zero, the real part satisfies the standard MKdV equation.

Mathematics Subject Classification: 37K15, 37K40

Keywords: soliton equations, Hamiltonian equation, integrable systems, Darboux transformation.

1. Introduction

Consider the MKdV equation

$$u_t + 6u^2 u_x + u_{xxx} = 0 \tag{1.1}$$

where $u = u(x,t)$. Regarding the MKdV equation, there are many works done in last decades [1, 2, 7, 8, 9, 10, 11]. Let $u = U + iV$, where

$$U = U(x,t), \quad V = V(x,t)$$

are real valued functions, called the real part and imaginary part of u, respectively. Plug $u = U + iV$ into the MKdV equation, we have the complex MKdV equation

$$U_t + iV_t + 6(U+iV)^2(U_x + iV_x) + U_{xxx} + iV_{xxx} = 0.$$

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Separating the real and imaginary part of the above complex MKdV equation, we have the so-called Coupled MKdV (CMKdV) equations:

$$U_t + 6U^2U_x - 6U_xV^2 - 12UVV_x + U_{xxx} = 0 \tag{1.2}$$
$$V_t - 6V^2V_x + 6U^2V_x + 12UU_xV + V_{xxx} = 0 \tag{1.3}$$

The CMKdV equations can be derived from two dimensional Euler equation and have applications in fluid dynamical systems [3].

The Lax pair of CMKdV equation is:

$$\phi_x = \begin{bmatrix} \lambda & u \\ -u & -\lambda \end{bmatrix} \phi \tag{1.4}$$
$$\phi_t = \begin{bmatrix} -4\lambda^3 - 2u^2\lambda & -4u\lambda^2 - 2u_x\lambda - 2u^3 - u_{xx} \\ 4u\lambda^2 - 2u_x\lambda + 2u^3 + u_{xx} & 4\lambda^3 + 2u^2\lambda \end{bmatrix} \phi \tag{1.5}$$

There are several ways to derive the Darboux transformation of the CMKdV equation [4, 5, 6]. Here we use the Darboux matrix method.

For a given solution u of the complex MKdV equation, suppose that we know a fundamental solution of its Lax pair (1.4) and (1.5):

$$\phi(x,t,\lambda) = \begin{bmatrix} \phi_{11}(x,t,\lambda) & \phi_{12}(x,t,\lambda) \\ \phi_{21}(x,t,\lambda) & \phi_{22}(x,t,\lambda) \end{bmatrix}. \tag{1.6}$$

Let λ_1, μ_1 be two arbitrary numbers and

$$\sigma = \frac{\phi_{22}(x,t,\lambda_1) + \mu_1\phi_{21}(x,t,\lambda_1)}{\phi_{12}(x,t,\lambda_1) + \mu_1\phi_{11}(x,t,\lambda_1)}.$$

Construct the matrix

$$D(x,t,\lambda) = \lambda I - \frac{\lambda_1}{1+\sigma^2} \begin{bmatrix} 1-\sigma^2 & 2\sigma \\ 2\sigma & \sigma^2 - 1 \end{bmatrix}. \tag{1.7}$$

Now, let $\phi'(x,t,\lambda) = D(x,t,\lambda)\phi(x,t,\lambda)$. Then it is easily verified that $\phi'(x,t,\lambda)$ still satisfies the Lax pair: (1.4) and (1.5), but with

$$u' = u + \frac{4\lambda_1\sigma}{1+\sigma^2}.$$

Therefore, for any solution ϕ of the Lax pair (1.4) and (1.5) with u, $D\phi$ is also a solution of the Lax pair with u'. Hence (1.4) and (1.5) are solvable with u' for any given initial data i.e. the value of ϕ' at some point (x_0, t_0). In other word, they are integrable.

The integrability condition of (1.4) and (1.5) with u' implies that u' is also a solution of the complex MKdV equation. Using this method, we can obtain a new solution of the CMKdV equations together with the corresponding fundamental solution of its Lax pair from a known one.

2. The Darboux Transformation

Starting from the trivial solution $u = 0$ of the complex MKdV equation, one can use the Darboux transformation to obtain a new non-trivial solutions. For $u = 0$, the fundamental solution of the Lax pair (1.4) and (1.5) are:

$$\Phi(x,t,\lambda) = \begin{bmatrix} e^{\lambda x - 4\lambda^3 t + \delta} & 0 \\ 0 & e^{-\lambda x + 4\lambda^3 t + \delta} \end{bmatrix} \tag{2.1}$$

where $\delta = \delta_1 + i\delta_2$ is an arbitrary complex number, and δ_1, δ_2 are two real numbers.

Let $\lambda_1 \neq 0$ be any complex number and μ_1 be defined as

$$\lambda_1 = \alpha_1 + i\beta_1,$$

$$\mu_1 = \delta = e^{2(\alpha_2 - i\beta_2)}.$$

Then

$$\sigma = \frac{\Phi_{22}(x,t,\lambda_1) + \mu_1 \Phi_{21}(x,t,\lambda_1)}{\Phi_{12}(x,t,\lambda_1) + \mu_1 \Phi_{11}(x,t,\lambda_1)}$$

or

$$\sigma = e^{8t\alpha_1^3 - 8it\beta_1^3 - 24t\alpha_1\beta_1^2 - 2ix\beta_1 - 2x\alpha_1 + 24it\alpha_1^2\beta_1 + 2i\beta_2 - 2\alpha_2}.$$

Thus, by

$$u' = u + \frac{4\lambda_1\sigma}{1+\sigma^2}, \tag{2.2}$$

a new solution of the complex MKdV equation may be given by

$$u' = \frac{4(\alpha_1 + i\beta_1)e^{8t\alpha_1^3 - 8it\beta_1^3 - 24t\alpha_1\beta_1^2 - 2ix\beta_1 - 2x\alpha_1 + 24it\alpha_1^2\beta_1 + 2i\beta_2 - 2\alpha_2}}{1 + (e^{8t\alpha_1^3 - 8it\beta_1^3 - 24t\alpha_1\beta_1^2 - 2ix\beta_1 - 2x\alpha_1 + 24it\alpha_1^2\beta_1 + 2i\beta_2 - 2\alpha_2})^2}. \tag{2.3}$$

This solution may have singularity points at

$$x = \frac{-4\alpha_1^3\beta_2 - 12\alpha_1^2\alpha_2\beta_1 + \pi\alpha_1^3 + 12\alpha_1\beta_1^2\beta_2 + 4\alpha_2\beta_1^3 - 3\pi\alpha_1\beta_1^3}{8\alpha_1\beta_1(\alpha_1^2 + \beta_2^2)} \tag{2.4}$$

and

$$t = \frac{-4\alpha_1^3\beta_2 - 4\alpha_2\beta_1 + \pi\alpha_1}{32\alpha_1\beta_1(\alpha_1^2 + \beta_2^2)} \tag{2.5}$$

Letting $u' = U + iV$ and separating the real and imaginary part of u', we have

$$U = \frac{(4\alpha_1 e^A \cos B - 4\beta_1 e^A \sin B)(1 + e^{2A}\cos 2B) + (4\alpha_1 e^A \sin B + 4\beta_1 e^A \cos B)e^{2A}\sin 2B}{(1 + e^{2A}\cos 2B)^2 + e^{4A}\sin 4B^2} \tag{2.6}$$

$$V = \frac{(4\alpha_1 e^A \sin B + 4\beta_1 e^A \cos B)(1 + e^{2A}\cos 2B) - (4\alpha_1 e^A \cos B - 4\beta_1 e^A \sin B)e^{2A}\sin 2B}{(1 + e^{2A}\cos 2B)^2 + e^{4A}\sin 4B^2} \tag{2.7}$$

where

$$A = 8t\alpha_1^3 - 24t\alpha_1\beta_1^2 - 2x\alpha_1 - 2\alpha_2, \quad B = 2\beta_2 - 8t\beta_1^3 - 2x\beta_1 + 24t\alpha_1^2\beta_1.$$

3. Analysis of the Solutions

There are four arbitrary constants $\alpha_1, \beta_1, \alpha_2$ and β_2 in solution (2.6) and (2.7). In this section, we give several special cases by taking different values of $\alpha_1, \beta_1, \alpha_2$ and β_2, and analyze the properties of different solutions.

3.1. Case 1: $\alpha_1 = 1, \beta_1 = 1, \alpha_2 = 1, \beta_2 = 1$

We have the smooth solution $U1$ and $V1$ of equations (1.2) and (1.3)

$$U1 = \frac{(4e^{-16t-2x-2}(\cos(-16t-2x-2) - \sin(-16t-2x-2))(1+e^{-32t-4x-4}\cos(32t-4x+4)}{(1+e^{-32t-4x-4}\cos(-32t-4x-4))^2 + e^{-64t-8x-8}\sin(32t-4x+4)}$$

$$+\frac{(4e^{-48t-6x--6}(\sin(-16t-2x-2) + \cos(-16t-2x-2))\sin(32t-4x+4)}{(1+e^{-32t-4x-4}\cos(-32t-4x-4))^2 + e^{-64t-8x-8}\sin(32t-4x+4)^2}$$

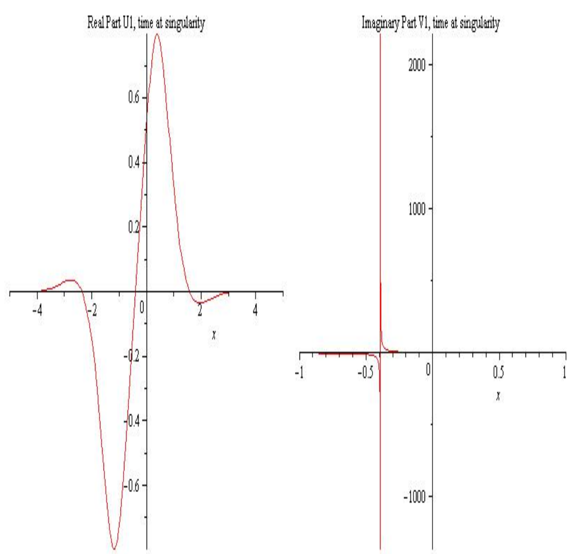
$$V1 = \frac{(4e^{-16t-2x-2}(\sin(-16t-2x-2) + \cos(-16t-2x-2))(1+e^{-32t-4x-4}\cos(32t-4x+4)}{(1+e^{-32t-4x-4}\cos(-32t-4x-4))^2 + e^{-64t-8x-8}\sin(32t-4x+4)}$$

$$-\frac{(4e^{-48t-6x--6}(\cos(-16t-2x-2) - \sin(-16t-2x-2))\sin(32t-4x+4)}{(1+e^{-32t-4x-4}\cos(-32t-4x-4))^2 + e^{-64t-8x-8}\sin(32t-4x+4)^2}$$

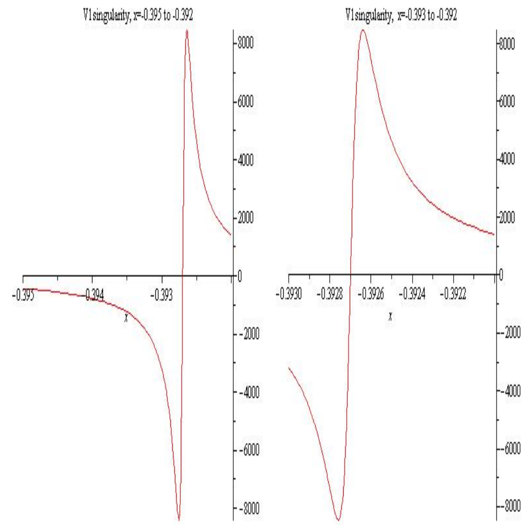
Apparently, both $U1$ and $V1$ decay at infinities. But they have the following singularity point:

$$x = -\frac{\pi}{8}, \quad t = -\frac{1}{8} + \frac{\pi}{64}.$$

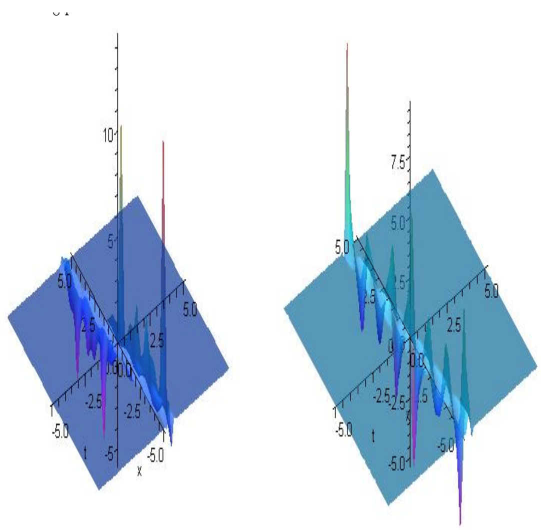
The graphs of the real and imaginary part of the solution look like:



The real part is smooth, but the imaginary part looks like discontinuous. But actually, both of them are smooth. The following picture gives us a close look at it.



The following pictures are three-dimensional for $U1$ and $V1$:



The real part U1 in Case 1   The Imainary Part V1 in Case 1

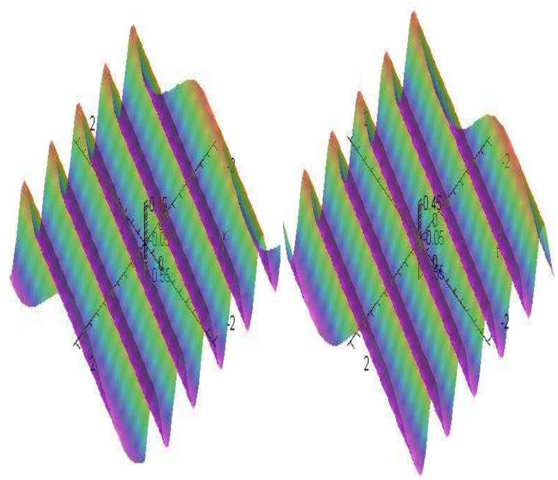
3.2. Case 2: $\alpha_1 = 1, \beta_1 = 1, \alpha_2 = 1, \beta_2 = 1$

In this case, we obtain the periodic solution for both the real and imaginary parts:

$$U2 = \frac{-4e^{-2}\sin(-8t-2x+2)(1+e^{-4}\cos(-16t-4x+4))}{(1+e^{-4}\cos(-16t-4x+4))^2 + e^{-8}\sin(-16t-4x+4)}$$
$$+\frac{-4e^{-6}\cos(-8t-2x+2)\sin(-16t-4x+4)}{(1+e^{-4}\cos(-16t-4x+4))^2 + e^{-8}\sin(-16t-4x+4)}$$
$$V2 = \frac{-4e^{-2}\sin(-8t-2x+2)(1+e^{-4}\cos(-16t-4x+4))}{(1+e^{-4}\cos(-16t-4x+4))^2 + e^{-8}\sin(-16t-4x+4)}$$

$$+\frac{-4e^{-6}\sin(-8t-2x+2)\sin(-16t-4x+4)}{(1+e^{-4}\cos(-16t-4x+4))^2+e^{-8}\sin(-16t-4x+4)}$$

with the following 3D pictures



The Real Part U2 in Case 2 The Imaginary Part V2 in Case 2

3.3. Case 3: $\alpha_1 = 1, \beta_1 = 1, \alpha_2 = 0, \beta_2 = 1$

In this case, we obtain a smooth solution $U3$ for the real part, but imagery part $V3$ has singularity at $x = 0.1073009182, t = -0.01341261478$.

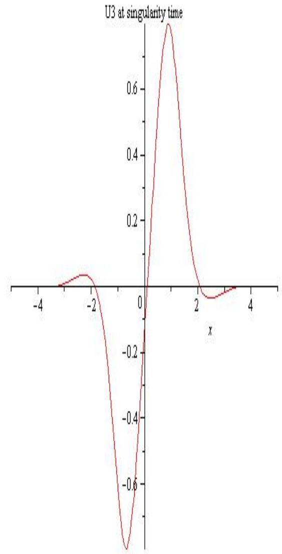
$$U3 = \frac{4e^{-16t-2x}(\cos(16t-2x+2)-\sin(16t-2x+2))(1+e^{-32t-4x}\cos(32t-4x+4))}{(1+e^{-32t-4x}\cos(32t-4x+4))^2+e^{-64t-8x}\sin^2(32t-4x+4)}$$

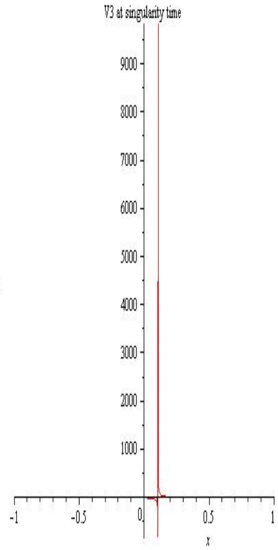
$$+\frac{4e^{-48t-6x}(\sin(16t-2x+2)+\cos(16t-2x+2))\sin(32t-4x+4)}{(1+e^{-32t-4x}\cos(32t-4x+4))^2+e^{-64t-8x}\sin^2(32t-4x+4)}$$

$$V3 = \frac{-4e^{-16t-2x}(\sin(16t-2x+2)+\cos(16t-2x+2))(1+e^{-32t-4x}\cos(32t-4x+4))}{(1+e^{-32t-4x}\cos(32t-4x+4))^2+e^{-64t-8x}\sin^2(32t-4x+4)}$$

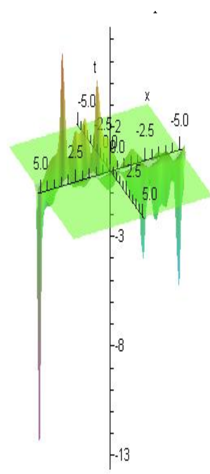
$$-\frac{4e^{-48t-6x}(\cos(16t-2x+2)-\sin(16t-2x+2))\sin(32t-4x+4)}{(1+e^{-32t-4x}\cos(32t-4x+4))^2+e^{-64t-8x}\sin^2(32t-4x+4)}$$

The left limit of $V3$ at the singularity point is $-\infty$, and the right limit at the singularity point is ∞. So, $V3$ is kind of shock soliton. The following pictures show the graphs of $U3$ and $V3$.

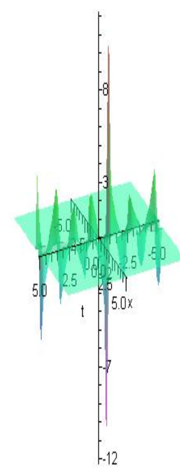




These are three-dimensional pictures:



The Real Part U3 in Case 3



The Imaginary Part V3 in Case 3

3.4. Case 4: $\alpha_1 = 2, \beta_1 = 1, \alpha_2 = 1, \beta_2 = 1$

In this case, we again obtain a smooth solution $U4$ for the real part U, but the imagery part $V4$ has singularity at $x = -0.5714601836, t = -0.01786504591$.

$$U4 = \frac{e^{16t-4x-2}(8\cos(88t-2x+2) - 4\sin(88t-2x+2))(1+e^{32t-8x-4}\cos(176t-4x+4))}{(1+e^{32t-8x-4}\cos(176t-4x+4))^2 + e^{64t-16x-8}\sin^2(176t-4x+4)}$$

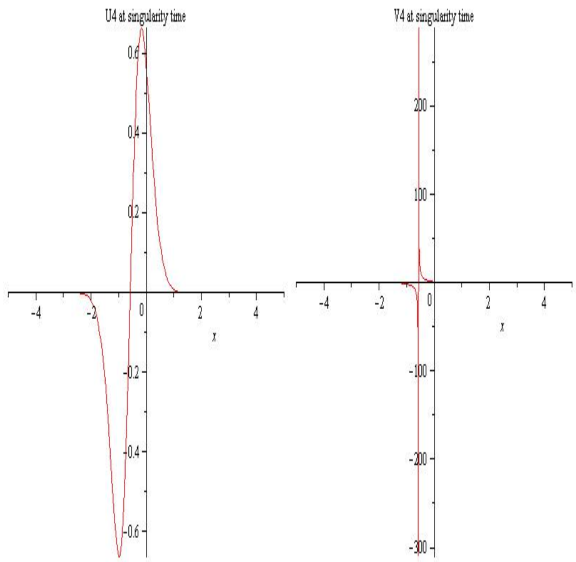
$$+\frac{e^{48t-12x-6}(8\sin(88t-2x+2)+4\cos(88t-2x+2))\sin(176t-4x+4)}{(1+e^{32t-8x-4}\cos(176t-4x+4))^2 + e^{64t-16x-8}\sin^2(176t-4x+4)}$$

$$V4 = \frac{e^{16t-4x-2}(8\sin(88t-2x+2)+4\cos(88t-2x+2))(1+e^{32t-8x-4}\cos(176t-4x+4))}{(1+e^{32t-8x-4}\cos(176t-4x+4))^2+e^{64t-16x-8}\sin^2(176t-4x+4)}$$

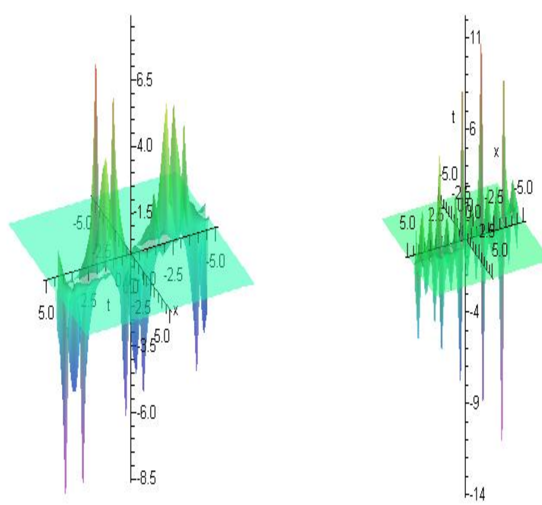
$$-\frac{e^{48t-12x-6}(8\cos(88t-2x+2)-4\sin(88t-2x+2))\sin(176t-4x+4)}{(1+e^{32t-8x-4}\cos(176t-4x+4))^2+e^{64t-16x-8}\sin^2(176t-4x+4)}$$

The left limit of $V4$ at the singularity point is $-\infty$, and the right limit at the singularity point is ∞. So, $V4$ is also kind of shock soliton.

The following pictures show the graphs of $U4$ and $V4$.



These are three-dimensional pictures:



The Real Part U4 in Case 4 The Imaginary Part V4 in Case 4

3.5. Case 5: $\alpha_1 = -1, \beta_1 = 1, \alpha_2 = 1, \beta_2 = 1$

In this case, we again obtain a smooth solution $U5$ for the real part U, but the imagery part $V5$ has singularity at $x = 0.6073009182, t = 0.04908738522$.

$$U5 = \frac{-4e^{16t+2x-2}(\cos(16t-2x-2)+\sin(16t-2x-2))(1+e^{32t+4x-4}\cos(32t-4x+4))}{(1+e^{32t+4x-4}\cos(32t-4x+4))^2+e^{64t+8x-8}\sin^2(32t-4x+4)}$$

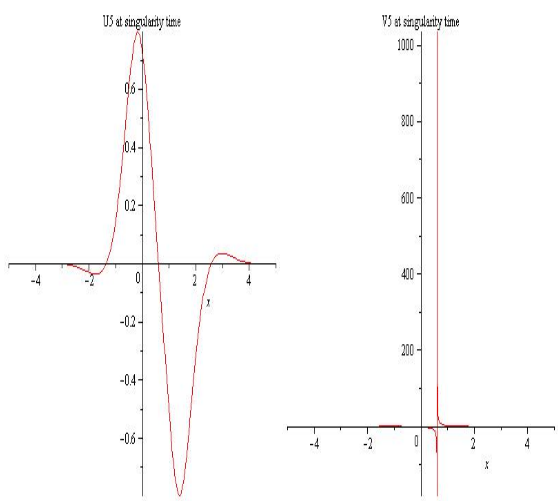
$$-\frac{4e^{48t+6x-6}(\sin(16t-2x+2)-\cos(16t-2x+2))\sin(32t-4x+4)}{(1+e^{32t+4x-4}\cos(32t-4x+4))^2+e^{64t+8x-8}\sin^2(32t-4x+4)}$$

$$V5 = \frac{-4e^{16t+2x-2}(\sin(16t-2x-2)-\cos(16t-2x-2))(1+e^{32t+4x-4}\cos(32t-4x+4))}{(1+e^{32t+4x-4}\cos(32t-4x+4))^2+e^{64t+8x-8}\sin^2(32t-4x+4)}$$

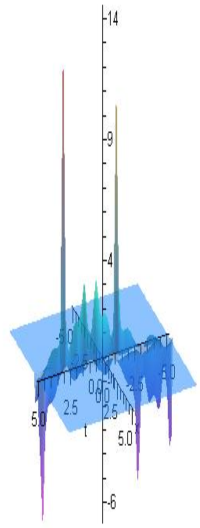
$$+\frac{4e^{48t+6x-6}(\cos(16t-2x+2)+\sin(16t-2x+2))\sin(32t-4x+4)}{(1+e^{32t+4x-4}\cos(32t-4x+4))^2+e^{64t+8x-8}\sin^2(32t-4x+4)}$$

The left limit of $V5$ at the singularity point is $-\infty$, and the right limit at the singularity point is ∞. So, $V5$ is also kind of shock soliton.

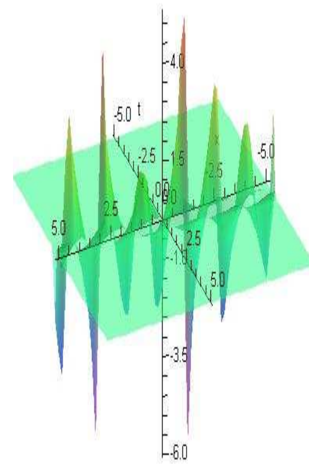
The following pictures show the graphs of $U5$ and $V5$.



These are three-dimensional pictures:



The Real Part U5 in Case 5



The Imaginary Part V5 in Case 5

3.6. Case 6: $\alpha_1 = 1, \beta_1 = 1, \alpha_2 = 0, \beta_2 = 0$

In this case, the imaginary part $V = 0$, and the real part

$$U = \frac{4e^{8t-2x-2}}{1 + e^{16t-4x-4}}$$

exactly solves the standard MKdV equation: $U_t + 6U^2 U_x + U_{xxx} = 0$. Using the Darboux transformation, one can easily restore multi-soliton solutions of the MKdV equation.

4. Conclusion

In the paper, we deal with the complex MKdV systems by using Darboux matrix approach and obtain some explicit solutions both smooth and shock solitons. In case 3.1, both U and V are smooth solitons vanishing at infinities; in case 3.2 they are periodic smooth solutions; in case 3.6 $V = 0$ and U just gives the classical solitons and multi-solitons of the standard MKdV equation. But in cases 3.3 - 3.5, we find new kind of solitons - shock solitons for the imagery part V of the complex MKdV systems. In the future work, we will figure out multi-shock-soliton's interactions as well as the real part solution's interaction.

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