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Multi-component generalization of the Camassa–Holm equation

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ABSTRACT

In this paper, we propose a multi-component system of the Camassa–Holm equation, denoted by CH(N, H), with $2N$ components and an arbitrary smooth function H. This system is shown to admit Lax pair and infinitely many conservation laws. We particularly study the case $N = 2$ and derive the bi-Hamiltonian structures and peaked soliton (peakon) solutions for some examples.

1. Introduction

In 1993, Camassa and Holm derived the well-known Camassa–Holm (CH) equation [1]

$$m_t + 2mu_x + m_x u = 0, \quad m = u - u_{xx} + k, \tag{1}$$

(with k being an arbitrary constant) with the aid of an asymptotic approximation to the Hamiltonian of the Green–Naghdi equations. Since the work of Camassa and Holm [1], more diverse studies on this equation have remarkably been developed [2–12]. The most interesting feature of the CH equation (1) is that it admits peakon solutions in the case $k = 0$. The stability and interaction of peakons were discussed in several references [13–17]. In addition to the CH equation, other similar integrable models with peakon solutions were found [18,19]. Recently, there are two integrable peakon equations found with cubic nonlinearity. They are the following cubic equation [3,20–22]

$$m_t + \frac{1}{2}\left[m(u^2 - u_x^2)\right]_x = 0, \quad m = u - u_{xx}, \tag{2}$$

and the Novikov's equation [23,24]

$$m_t = u^2 m_x + 3uu_x m, \quad m = u - u_{xx}. \tag{3}$$

There is also much attention in studying integrable multi-component peakon equations. For example, in [25–28], multi-component generalizations of the CH equation were derived from different points of view, and in [29], multi-component extensions of the cubic nonlinear equation (2) were studied.

In a previous paper [30], we proposed a two-component generalization of the CH equation (1) and the cubic nonlinear CH equation (2)

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$$\begin{cases} m_t = (mH)_x + mH - \frac{1}{2}m(u-u_x)(v+v_x), \\ n_t = (nH)_x - nH + \frac{1}{2}n(u-u_x)(v+v_x), \\ m = u - u_{xx}, \qquad n = v - v_{xx}, \end{cases} \tag{4}$$

where H is an arbitrary smooth function of u, v, and their derivatives. Such a system is interesting, since different choices of H lead to different peakon equations. We presented the Lax pair and infinitely many conservation laws of the system for the general H, and discussed the bi-Hamiltonian structures and peakon solutions of the system for the special choices of H. In [31], Li, Liu and Popowicz proposed a four-component peakon equation which also contains an arbitrary function. They derived the Lax pair and infinite conservation laws for their four-component equation, and presented a bi-Hamiltonian structure for the equation in the special case that the arbitrary function is taken to be zero.

In this paper, we propose the following multi-component system

$$\begin{cases} m_{j,t} = (m_jH)_x + m_jH + \frac{1}{(N+1)^2}\sum_{i=1}^{N}[m_i(u_j-u_{j,x})(v_i+v_{i,x}) + m_j(u_i-u_{i,x})(v_i+v_{i,x})], \\ n_{j,t} = (n_jH)_x - n_jH - \frac{1}{(N+1)^2}\sum_{i=1}^{N}[n_i(u_i-u_{i,x})(v_j+v_{j,x}) + n_j(u_i-u_{i,x})(v_i+v_{i,x})], \\ m_j = u_j - u_{j,xx}, \qquad n_j = v_j - v_{j,xx}, \quad 1 \le j \le N, \end{cases} \tag{5}$$

where H is an arbitrary smooth function of u_j, v_j, $1 \le j \le N$, and their derivatives. The system contains $2N$ components and an arbitrary function H. For $N = 1$, this system becomes the two-component system (4). Therefore, system (5) is a kind of multi-component generalization of the two-component system (4). Due to the presence of the function H, system (5) is actually a large class of multi-component equations. We show that the multi-component system (5) admits Lax representation and infinitely many conservation laws. Although having not found a unified bi-Hamiltonian structure of the system (5) for the general H yet, we demonstrate that for some special choices of H, one may find the corresponding bi-Hamiltonian structures. As examples, we derive the peakon solutions in the case $N = 2$. In particular, we obtain a new integrable model which admits stationary peakon solutions.

The whole paper is organized as follows. In Section 2, the Lax pair and conservation laws of Eq. (5) are presented. In Section 3, the Hamiltonian structures and peakon solutions of Eq. (5) in the case $N = 2$ are discussed. Some conclusions and open problems are addressed in Section 4.

2. Lax pair and conservation laws

We first introduce the N-component vector potentials $\vec{u}$, $\vec{v}$ and $\vec{m}$, $\vec{n}$ as

$$\begin{aligned} &\vec{u} = (u_1, u_2, \ldots, u_N), \qquad \vec{v} = (v_1, v_2, \ldots, v_N), \\ &\vec{m} = \vec{u} - \vec{u}_{xx}, \qquad \vec{n} = \vec{v} - \vec{v}_{xx}. \end{aligned} \tag{6}$$

Using this notation, Eq. (5) is expressed in the following vector form

$$\begin{cases} \vec{m}_t = (\vec{m}H)_x + \vec{m}H + \frac{1}{(N+1)^2}[\vec{m}(\vec{v}+\vec{v}_x)^T(\vec{u}-\vec{u}_x) + (\vec{u}-\vec{u}_x)(\vec{v}+\vec{v}_x)^T\vec{m}], \\ \vec{n}_t = (\vec{n}H)_x - \vec{n}H - \frac{1}{(N+1)^2}[\vec{n}(\vec{u}-\vec{u}_x)^T(\vec{v}+\vec{v}_x) + (\vec{v}+\vec{v}_x)(\vec{u}-\vec{u}_x)^T\vec{n}], \\ \vec{m} = \vec{u} - \vec{u}_{xx}, \qquad \vec{n} = \vec{v} - \vec{v}_{xx}, \end{cases} \tag{7}$$

where the symbol T denotes the transpose of a vector.

Let us introduce a pair of $(N+1) \times (N+1)$ matrix spectral problems

$$\phi_x = U\phi, \qquad \phi_t = V\phi, \tag{8}$$

with

$$\begin{aligned} &\phi = (\phi_1, \phi_{21}, \ldots, \phi_{2N})^T, \\ &U = \frac{1}{N+1}\begin{pmatrix} -N & \lambda\vec{m} \\ \lambda\vec{n}^T & I_N \end{pmatrix}, \\ &V = \frac{1}{N+1}\begin{pmatrix} -N\lambda^{-2} + \frac{1}{N+1}(\vec{u}-\vec{u}_x)(\vec{v}+\vec{v}_x)^T & \lambda^{-1}(\vec{u}-\vec{u}_x) + \lambda\vec{m}H \\ \lambda^{-1}(\vec{v}+\vec{v}_x)^T + \lambda\vec{n}^T H & \lambda^{-2}I_N - \frac{1}{N+1}(\vec{v}+\vec{v}_x)^T(\vec{u}-\vec{u}_x) \end{pmatrix}, \end{aligned} \tag{9}$$

where λ is a spectral parameter, I_N is the $N \times N$ identity matrix, $\vec{u}$, $\vec{v}$, $\vec{m}$ and $\vec{n}$ are the vector potentials shown in (6).

Proposition 1. *(8) provides the Lax pair for the multi-component system (5).*

Proof. It is easy to check that the compatibility condition of (8) generates

$$U_t - V_x + [U, V] = 0. \tag{10}$$

From (9), we have

$$\begin{aligned} U_t &= \frac{1}{N+1}\begin{pmatrix} 0 & \lambda \vec{m}_t \\ \lambda \vec{n}_t^T & 0 \end{pmatrix}, \\ V_x &= \frac{1}{N+1}\begin{pmatrix} \frac{1}{N+1}[\vec{m}(\vec{v}+\vec{v}_x)^T - (\vec{u}-\vec{u}_x)\vec{n}^T] & \lambda^{-1}(\vec{u}_x - \vec{u}_{xx}) + \lambda(\vec{m}H)_x \\ \lambda^{-1}(\vec{v}_x+\vec{v}_{xx})^T + \lambda(\vec{n}^T H)_x & \frac{1}{N+1}[\vec{n}^T(\vec{u}-\vec{u}_x) - (\vec{v}+\vec{v}_x)^T\vec{m}] \end{pmatrix}, \end{aligned} \tag{11}$$

and

$$[U, V] = UV - VU = \frac{1}{(N+1)^2}\begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned} \Gamma_{11} &= \vec{m}(\vec{v}+\vec{v}_x)^T - (\vec{u}-\vec{u}_x)\vec{n}^T, \\ \Gamma_{12} &= (N+1)[\lambda^{-1}(\vec{u}_x - \vec{u}_{xx}) - \lambda \vec{m}H] - \frac{\lambda}{N+1}[\vec{m}(\vec{v}+\vec{v}_x)^T(\vec{u}-\vec{u}_x) + (\vec{u}-\vec{u}_x)(\vec{v}+\vec{v}_x)^T\vec{m}], \\ \Gamma_{21} &= (N+1)[\lambda^{-1}(\vec{v}_x+\vec{v}_{xx})^T + \lambda \vec{n}^T H] + \frac{\lambda}{N+1}[\vec{n}^T(\vec{u}-\vec{u}_x)(\vec{v}+\vec{v}_x)^T + (\vec{v}+\vec{v}_x)^T(\vec{u}-\vec{u}_x)\vec{n}^T], \\ \Gamma_{22} &= \vec{n}^T(\vec{u}-\vec{u}_x) - (\vec{v}+\vec{v}_x)^T\vec{m}. \end{aligned}$$

We remark that (12) is written in the form of block matrix. As shown above, the element Γ_{11} is a scalar function, the element Γ_{12} is a N-component row vector function, the element Γ_{21} is a N-component column vector function, and the element Γ_{22} is a $N \times N$ matrix function.

Substituting the expressions (11) and (12) into (10), we find that (10) gives rise to

$$\begin{cases} \vec{m}_t = (\vec{m}H)_x + \vec{m}H + \frac{1}{(N+1)^2}[\vec{m}(\vec{v}+\vec{v}_x)^T(\vec{u}-\vec{u}_x) + (\vec{u}-\vec{u}_x)(\vec{v}+\vec{v}_x)^T\vec{m}], \\ \vec{n}_t^T = (\vec{n}^T H)_x - \vec{n}^T H - \frac{1}{(N+1)^2}[\vec{n}^T(\vec{u}-\vec{u}_x)(\vec{v}+\vec{v}_x)^T + (\vec{v}+\vec{v}_x)^T(\vec{u}-\vec{u}_x)\vec{n}^T], \\ \vec{m} = \vec{u} - \vec{u}_{xx}, \qquad \vec{n}^T = \vec{v}^T - \vec{v}_{xx}^T, \end{cases} \tag{13}$$

which is nothing but the vector equation (7). Hence, (8) exactly gives the Lax pair of multi-component equation (5).

Now let us construct the conservation laws of Eq. (5). We write the spacial part of the spectral problems (8) as

$$\begin{cases} \phi_{1,x} = \frac{1}{N+1}\left(-N\phi_1 + \lambda \sum_{i=1}^N m_i \phi_{2i}\right), \\ \phi_{2j,x} = \frac{1}{N+1}\left(\lambda n_j \phi_1 + \phi_{2j}\right), \end{cases} \qquad 1 \le j \le N. \tag{14}$$

Let $\Omega_j = \frac{\phi_{2j}}{\phi_1}$, $1 \le j \le N$, we obtain the following system of Riccati equations

$$\Omega_{j,x} = \frac{1}{N+1}\left[\lambda n_j + (N+1)\Omega_j - \lambda \Omega_j \sum_{i=1}^N m_i \Omega_i\right], \quad 1 \le j \le N. \tag{15}$$

Making use of the relation $(\ln \phi_1)_{xt} = (\ln \phi_1)_{tx}$ and (8), we arrive at the conservation law

$$\left(\sum_{i=1}^N m_i \Omega_i\right)_t = \left(\lambda^{-2}\sum_{i=1}^N (u_i - u_{i,x})\Omega_i + \frac{1}{N+1}\lambda^{-1}\sum_{i=1}^N (u_i - u_{i,x})(v_i + v_{i,x}) + H\sum_{i=1}^N m_i \Omega_i\right)_x. \tag{16}$$

Eq. (16) means that $\sum_{i=1}^N m_i \Omega_i$ is a generating function of the conserved densities. To derive the explicit forms of conserved densities, we expand Ω_j into the negative power series of λ as

$$\Omega_j = \sum_{k=0}^{\infty} \omega_{jk}\lambda^{-k}, \quad 1 \le j \le N. \tag{17}$$

Substituting (17) into the Riccati system (15) and equating the coefficients of powers of λ, we obtain

$$\begin{aligned} \omega_{j0} &= n_j\left(\sum_{i=1}^{N} m_i n_i\right)^{-\frac{1}{2}}, \\ \omega_{j1} &= (N+1)\left[\omega_{j0} - \omega_{j0,x} - \frac{1}{2}n_j\left(\sum_{i=1}^{N} m_i(\omega_{i0}-\omega_{i0,x})\right)\left(\sum_{i=1}^{N} m_i n_i\right)^{-1}\right]\left(\sum_{i=1}^{N} m_i n_i\right)^{-\frac{1}{2}}, \end{aligned} \tag{18}$$

and the recursion relations for $\omega_{j(k+1)}$, $k \geq 1$,

$$\omega_{j(k+1)} = (N+1)\left[\omega_{jk} - \omega_{jk,x} - \frac{1}{2}n_j\left(\sum_{i=1}^{N} m_i(\omega_{ik}-\omega_{ik,x})\right)\left(\sum_{i=1}^{N} m_i n_i\right)^{-1}\right]\left(\sum_{i=1}^{N} m_i n_i\right)^{-\frac{1}{2}}. \tag{19}$$

Inserting (17)–(19) into (16), we finally obtain the following infinitely many conserved densities ρ_j and the associated fluxes F_j:

$$\begin{aligned} &\rho_0 = \sum_{i=1}^{N} m_i\omega_{i0} = \left(\sum_{i=1}^{N} m_i n_i\right)^{\frac{1}{2}}, \qquad F_0 = H\sum_{i=1}^{N} m_i\omega_{i0} = H\left(\sum_{i=1}^{N} m_i n_i\right)^{\frac{1}{2}}, \\ &\rho_1 = \sum_{i=1}^{N} m_i\omega_{i1}, \qquad F_1 = \frac{1}{N+1}\sum_{i=1}^{N}(u_i-u_{i,x})(v_i+v_{i,x}) + H\sum_{i=1}^{N} m_i\omega_{i1}, \\ &\rho_2 = \sum_{i=1}^{N} m_i\omega_{i2}, \qquad F_2 = \sum_{i=1}^{N}(u_i-u_{i,x})\omega_{i0} + H\sum_{i=1}^{N} m_i\omega_{i2}, \\ &\rho_j = \sum_{i=1}^{N} m_i\omega_{ij}, \qquad F_j = \sum_{i=1}^{N}(u_i-u_{i,x})\omega_{i(j-2)} + H\sum_{i=1}^{N} m_i\omega_{ij}, \quad j \geq 3, \end{aligned} \tag{20}$$

where ω_{ij}, $1 \leq i \leq N, j \geq 0$ is given by (18) and (19).

Remark 1. The $2N$-component system (5) with an arbitrary function H does possess Lax representation and infinitely many conservation laws. Such a system is interesting since different choices of H lead to different peakon equations. Let us look back why an arbitrary smooth function may be involved in system (5). System (5) is produced by the compatibility condition (10) of the spectral problems (8) where such an arbitrary function is included in V part (see the formula (9)). The appearance of this arbitrary function can be explained as that the Lax equation is an over determined system by choosing the appropriate V to match U.

3. Examples for $N = 2$

In the case $N = 2$, Eq. (5) is cast into the following four-component model

$$\begin{cases} m_{1,t} = (m_1H)_x + m_1H \\ \quad + \frac{1}{9}\{m_1[2(u_1-u_{1,x})(v_1+v_{1,x}) + (u_2-u_{2,x})(v_2+v_{2,x})] + m_2(u_1-u_{1,x})(v_2+v_{2,x})\}, \\ m_{2,t} = (m_2H)_x + m_2H \\ \quad + \frac{1}{9}\{m_1(u_2-u_{2,x})(v_1+v_{1,x}) + m_2[(u_1-u_{1,x})(v_1+v_{1,x}) + 2(u_2-u_{2,x})(v_2+v_{2,x})]\}, \\ n_{1,t} = (n_1H)_x - n_1H \\ \quad - \frac{1}{9}\{n_1[2(u_1-u_{1,x})(v_1+v_{1,x}) + (u_2-u_{2,x})(v_2+v_{2,x})] + n_2(u_2-u_{2,x})(v_1+v_{1,x})\}, \\ n_{2,t} = (n_2H)_x - n_2H \\ \quad - \frac{1}{9}\{n_1(u_1-u_{1,x})(v_2+v_{2,x}) + n_2[(u_1-u_{1,x})(v_1+v_{1,x}) + 2(u_2-u_{2,x})(v_2+v_{2,x})]\}, \\ m_1 = u_1 - u_{1,xx}, \qquad m_2 = u_2 - u_{2,xx}, \qquad n_1 = v_1 - v_{1,xx}, \qquad n_2 = v_2 - v_{2,xx}, \end{cases} \tag{21}$$

where H is an arbitrary smooth function of u_1, u_2, v_1, v_2, and their derivatives. This system admits the following 3×3 Lax pair

$$U = \frac{1}{3}\begin{pmatrix} -2 & \lambda m_1 & \lambda m_2 \\ \lambda n_1 & 1 & 0 \\ \lambda n_2 & 0 & 1 \end{pmatrix}, \qquad V = \frac{1}{3}\begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}, \tag{22}$$

where

$$
\begin{aligned}
&V_{11} = -2\lambda^{-2} + \frac{1}{3}[(u_1 - u_{1,x})(v_1 + v_{1,x}) + (u_2 - u_{2,x})(v_2 + v_{2,x})],\\
&V_{12} = \lambda^{-1}(u_1 - u_{1,x}) + \lambda m_1 H, \qquad V_{13} = \lambda^{-1}(u_2 - u_{2,x}) + \lambda m_2 H,\\
&V_{21} = \lambda^{-1}(v_1 + v_{1,x}) + \lambda n_1 H, \qquad V_{22} = \lambda^{-2} - \frac{1}{3}(u_1 - u_{1,x})(v_1 + v_{1,x}),\\
&V_{23} = -\frac{1}{3}(u_2 - u_{2,x})(v_1 + v_{1,x}), \qquad V_{31} = \lambda^{-1}(v_2 + v_{2,x}) + \lambda n_2 H,\\
&V_{32} = -\frac{1}{3}(u_1 - u_{1,x})(v_2 + v_{2,x}), \qquad V_{33} = \lambda^{-2} - \frac{1}{3}(u_2 - u_{2,x})(v_2 + v_{2,x}).
\end{aligned}
\tag{23}
$$

Due to the appearance of arbitrary function H, we do not know yet whether (21) is bi-Hamiltonian in general. But we find that it is possible to figure out the bi-Hamiltonian structures for some special cases, which we will show in the following examples.

Example 1. A new integrable model with stationary peakon solutions

Taking $H = 0$, Eq. (21) becomes the following system

$$
\begin{cases}
m_{1,t} = \frac{1}{9}\{m_1[2(u_1 - u_{1,x})(v_1 + v_{1,x}) + (u_2 - u_{2,x})(v_2 + v_{2,x})] + m_2(u_1 - u_{1,x})(v_2 + v_{2,x})\},\\
m_{2,t} = \frac{1}{9}\{m_1(u_2 - u_{2,x})(v_1 + v_{1,x}) + m_2[(u_1 - u_{1,x})(v_1 + v_{1,x}) + 2(u_2 - u_{2,x})(v_2 + v_{2,x})]\},\\
n_{1,t} = -\frac{1}{9}\{n_1[2(u_1 - u_{1,x})(v_1 + v_{1,x}) + (u_2 - u_{2,x})(v_2 + v_{2,x})] + n_2(u_2 - u_{2,x})(v_1 + v_{1,x})\},\\
n_{2,t} = -\frac{1}{9}\{n_1(u_1 - u_{1,x})(v_2 + v_{2,x}) + n_2[(u_1 - u_{1,x})(v_1 + v_{1,x}) + 2(u_2 - u_{2,x})(v_2 + v_{2,x})]\},\\
m_1 = u_1 - u_{1,xx}, \qquad m_2 = u_2 - u_{2,xx}, \qquad n_1 = v_1 - v_{1,xx}, \qquad n_2 = v_2 - v_{2,xx}.
\end{cases}
\tag{24}
$$

Let us introduce a Hamiltonian pair

$$
J = \begin{pmatrix} 0 & 0 & \partial + 1 & 0 \\ 0 & 0 & 0 & \partial + 1 \\ \partial - 1 & 0 & 0 & 0 \\ 0 & \partial - 1 & 0 & 0 \end{pmatrix}, \qquad K = \frac{1}{9}\begin{pmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{pmatrix},
\tag{25}
$$

where

$$
\begin{aligned}
&K_{11} = -2m_1\partial^{-1}m_1, \qquad K_{12} = -m_2\partial^{-1}m_1 - m_1\partial^{-1}m_2,\\
&K_{13} = 2m_1\partial^{-1}n_1 + m_2\partial^{-1}n_2, \qquad K_{14} = m_1\partial^{-1}n_2,\\
&K_{21} = -K_{12}^* = -m_1\partial^{-1}m_2 - m_2\partial^{-1}m_1, \qquad K_{22} = -2m_2\partial^{-1}m_2,\\
&K_{23} = m_2\partial^{-1}n_1, \qquad K_{24} = m_1\partial^{-1}n_1 + 2m_2\partial^{-1}n_2,\\
&K_{31} = -K_{13}^* = 2n_1\partial^{-1}m_1 + n_2\partial^{-1}m_2, \qquad K_{32} = -K_{23}^* = n_1\partial^{-1}m_2,\\
&K_{33} = -2n_1\partial^{-1}n_1, \qquad K_{34} = -n_1\partial^{-1}n_2 - n_2\partial^{-1}n_1,\\
&K_{41} = -K_{14}^* = n_2\partial^{-1}m_1, \qquad K_{42} = -K_{24}^* = n_1\partial^{-1}m_1 + 2n_2\partial^{-1}m_2,\\
&K_{43} = -K_{34}^* = -n_2\partial^{-1}n_1 - n_1\partial^{-1}n_2, \qquad K_{44} = -2n_2\partial^{-1}n_2.
\end{aligned}
\tag{26}
$$

By direct but tedious calculations, we arrive at

Proposition 2. *Eq. (24) can be rewritten in the following bi-Hamiltonian form*

$$
\left(m_{1,t},\ m_{2,t},\ n_{1,t},\ n_{2,t}\right)^T = J\left(\frac{\delta H_2}{\delta m_1}, \frac{\delta H_2}{\delta m_2}, \frac{\delta H_2}{\delta n_1}, \frac{\delta H_2}{\delta n_2}\right)^T = K\left(\frac{\delta H_1}{\delta m_1}, \frac{\delta H_1}{\delta m_2}, \frac{\delta H_1}{\delta n_1}, \frac{\delta H_1}{\delta n_2}\right)^T,
\tag{27}
$$

where J and K are given by (25), and

$$
\begin{aligned}
H_1 &= \int_{-\infty}^{+\infty} [(u_{1,x} - u_1)n_1 + (u_{2,x} - u_2)n_2]dx,\\
H_2 &= \frac{1}{9}\int_{-\infty}^{+\infty} [(u_1 - u_{1,x})^2(v_1 + v_{1,x})n_1 + (u_1 - u_{1,x})(u_2 - u_{2,x})(v_2 + v_{2,x})n_1\\
&\quad + (u_1 - u_{1,x})(u_2 - u_{2,x})(v_1 + v_{1,x})n_2 + (u_2 - u_{2,x})^2(v_2 + v_{2,x})n_2]dx.
\end{aligned}
\tag{28}
$$

Suppose an N-peakon solution of (24) is in the form

$$\begin{aligned} u_1 &= \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|}, \qquad u_2 = \sum_{j=1}^{N} r_j(t) e^{-|x-q_j(t)|}, \\ v_1 &= \sum_{j=1}^{N} s_j(t) e^{-|x-q_j(t)|}, \qquad v_2 = \sum_{j=1}^{N} w_j(t) e^{-|x-q_j(t)|}. \end{aligned} \tag{29}$$

Then, in the distribution sense, one can get

$$\begin{aligned} u_{1,x} &= -\sum_{j=1}^{N} p_j sgn(x-q_j) e^{-|x-q_j|}, \qquad m_1 = 2\sum_{j=1}^{N} p_j \delta(x-q_j), \\ u_{2,x} &= -\sum_{j=1}^{N} r_j sgn(x-q_j) e^{-|x-q_j|}, \qquad m_2 = 2\sum_{j=1}^{N} r_j \delta(x-q_j), \\ v_{1,x} &= -\sum_{j=1}^{N} s_j sgn(x-q_j) e^{-|x-q_j|}, \qquad n_1 = 2\sum_{j=1}^{N} s_j \delta(x-q_j), \\ v_{2,x} &= -\sum_{j=1}^{N} w_j sgn(x-q_j) e^{-|x-q_j|}, \qquad n_2 = 2\sum_{j=1}^{N} w_j \delta(x-q_j). \end{aligned} \tag{30}$$

Substituting (29) and (30) into (24) and integrating against test functions with compact support, we arrive at the N-peakon dynamical system as follows:

$$\begin{aligned} q_{j,t} &= 0, \\ p_{j,t} &= -\frac{1}{9}\Bigg\{ \frac{2}{3} p_j (p_j s_j + r_j w_j) - \sum_{i,k=1}^{N} \big(p_j(2p_i s_k + r_i w_k) + r_j p_i w_k\big)\big(sgn(q_j-q_i) - sgn(q_j-q_k)\big) e^{-|q_j-q_i|-|q_j-q_k|} \\ &\quad + \sum_{i,k=1}^{N} \big(p_j(2p_i s_k + r_i w_k) + r_j p_i w_k\big)\big(sgn(q_j-q_i) sgn(q_j-q_k) - 1\big) e^{-|q_j-q_i|-|q_j-q_k|} \Bigg\}, \\ r_{j,t} &= -\frac{1}{9}\Bigg\{ \frac{2}{3} r_j (r_j w_j + p_j s_j) - \sum_{i,k=1}^{N} \big(r_j(2r_i w_k + p_i s_k) + p_j r_i s_k\big)\big(sgn(q_j-q_i) - sgn(q_j-q_k)\big) e^{-|q_j-q_i|-|q_j-q_k|} \\ &\quad + \sum_{i,k=1}^{N} \big(r_j(2r_i w_k + p_i s_k) + p_j r_i s_k\big)\big(sgn(q_j-q_i) sgn(q_j-q_k) - 1\big) e^{-|q_j-q_i|-|q_j-q_k|} \Bigg\}, \\ s_{j,t} &= \frac{1}{9}\Bigg\{ \frac{2}{3} s_j (p_j s_j + r_j w_j) - \sum_{i,k=1}^{N} \big(s_j(2p_i s_k + r_i w_k) + w_j r_i s_k\big)\big(sgn(q_j-q_i) - sgn(q_j-q_k)\big) e^{-|q_j-q_i|-|q_j-q_k|} \\ &\quad + \sum_{i,k=1}^{N} \big(s_j(2p_i s_k + r_i w_k) + w_j r_i s_k\big)\big(sgn(q_j-q_i) sgn(q_j-q_k) - 1\big) e^{-|q_j-q_i|-|q_j-q_k|} \Bigg\}, \\ w_{j,t} &= \frac{1}{9}\Bigg\{ \frac{2}{3} w_j (p_j s_j + r_j w_j) - \sum_{i,k=1}^{N} \big(w_j(2r_i w_k + p_i s_k) + s_j p_i w_k\big)\big(sgn(q_j-q_i) - sgn(q_j-q_k)\big) e^{-|q_j-q_i|-|q_j-q_k|} \\ &\quad + \sum_{i,k=1}^{N} \big(w_j(2r_i w_k + p_i s_k) + s_j p_i w_k\big)\big(sgn(q_j-q_i) sgn(q_j-q_k) - 1\big) e^{-|q_j-q_i|-|q_j-q_k|} \Bigg\}. \end{aligned} \tag{31}$$

The formula $q_{j,t} = 0$ in (31) implies that the peakon position is stationary and the solution is in the form of separation of variables. Especially, for $N = 1$, (31) becomes

$$\begin{cases} q_{1,t} = 0, \\ p_{1,t} = \frac{4}{27} p_1(p_1 s_1 + r_1 w_1), \\ r_{1,t} = \frac{4}{27} r_1(p_1 s_1 + r_1 w_1), \\ s_{1,t} = -\frac{4}{27} s_1(p_1 s_1 + r_1 w_1), \\ w_{1,t} = -\frac{4}{27} w_1(p_1 s_1 + r_1 w_1), \end{cases} \tag{32}$$

which has the solution

$$q_1 = C_1, \qquad p_1 = A_4 e^{\frac{4}{27}(A_2+A_3)t}, \qquad r_1 = \frac{1}{A_1} p_1, \qquad s_1 = \frac{A_2}{A_4} e^{-\frac{4}{27}(A_2+A_3)t}, \qquad w_1 = \frac{A_3}{r_1}, \tag{33}$$

where C_1 and $A_1, \cdots, A_4$ are integration constants. Thus, the stationary single-peakon solution becomes

$$\begin{aligned} u_1 &= A_4 e^{\frac{4}{27}(A_2+A_3)t} e^{-|x-C_1|}, \qquad u_2 = \frac{u_1}{A_1}, \\ v_1 &= \frac{A_2}{A_4} e^{-\frac{4}{27}(A_2+A_3)t} e^{-|x-C_1|}, \qquad v_2 = \frac{A_1 A_3}{A_2} v_1. \end{aligned} \tag{34}$$

See Fig. 1 for the stationary single-peakon of the potentials $u_1(x, t)$ and $v_1(x, t)$ with $C_1 = 0, A_2 = A_4 = 1$ and $A_3 = 2$.

Example 2. A new integrable four-component system with peakon solutions

Let us choose

$$H = -\frac{1}{9}[(u_1 - u_{1,x})(v_1 + v_{1,x}) + (u_2 - u_{2,x})(v_2 + v_{2,x})],$$

then Eq. (21) is cast into

$$\begin{cases} m_{1,t} = (m_1 H)_x + \frac{1}{9} m_1 (u_1 - u_{1,x})(v_1 + v_{1,x}) + \frac{1}{9} m_2 (u_1 - u_{1,x})(v_2 + v_{2,x}), \\ m_{2,t} = (m_2 H)_x + \frac{1}{9} m_1 (u_2 - u_{2,x})(v_1 + v_{1,x}) + \frac{1}{9} m_2 (u_2 - u_{2,x})(v_2 + v_{2,x}), \\ n_{1,t} = (n_1 H)_x - \frac{1}{9} n_1 (u_1 - u_{1,x})(v_1 + v_{1,x}) - \frac{1}{9} n_2 (u_2 - u_{2,x})(v_1 + v_{1,x}), \\ n_{2,t} = (n_2 H)_x - \frac{1}{9} n_1 (u_1 - u_{1,x})(v_2 + v_{2,x}) - \frac{1}{9} n_2 (u_2 - u_{2,x})(v_2 + v_{2,x}), \\ m_1 = u_1 - u_{1,xx}, \qquad m_2 = u_2 - u_{2,xx}, \qquad n_1 = v_1 - v_{1,xx}, \qquad n_2 = v_2 - v_{2,xx}. \end{cases} \tag{35}$$

Let us set

$$K = \frac{1}{9} \begin{pmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{pmatrix}, \tag{36}$$

where

$$\begin{aligned} & K_{11} = \partial m_1 \partial^{-1} m_1 \partial - m_1 \partial^{-1} m_1, \qquad K_{12} = \partial m_1 \partial^{-1} m_2 \partial - m_2 \partial^{-1} m_1, \\ & K_{13} = \partial m_1 \partial^{-1} n_1 \partial + m_1 \partial^{-1} n_1 + m_2 \partial^{-1} n_2, \qquad K_{14} = \partial m_1 \partial^{-1} n_2 \partial, \\ & K_{21} = -K_{12}^* = \partial m_2 \partial^{-1} m_1 \partial - m_1 \partial^{-1} m_2, \qquad K_{22} = \partial m_2 \partial^{-1} m_2 \partial - m_2 \partial^{-1} m_2, \\ & K_{23} = \partial m_2 \partial^{-1} n_1 \partial, \qquad K_{24} = \partial m_2 \partial^{-1} n_2 \partial + m_1 \partial^{-1} n_1 + m_2 \partial^{-1} n_2, \\ & K_{31} = -K_{13}^* = \partial n_1 \partial^{-1} m_1 \partial + n_1 \partial^{-1} m_1 + n_2 \partial^{-1} m_2, \qquad K_{32} = -K_{23}^* = \partial n_1 \partial^{-1} m_2 \partial, \\ & K_{33} = \partial n_1 \partial^{-1} n_1 \partial - n_1 \partial^{-1} n_1, \qquad K_{34} = \partial n_1 \partial^{-1} n_2 \partial - n_2 \partial^{-1} n_1, \\ & K_{41} = -K_{14}^* = \partial n_2 \partial^{-1} m_1 \partial, \qquad K_{42} = -K_{24}^* = \partial n_2 \partial^{-1} m_2 \partial + n_1 \partial^{-1} m_1 + n_2 \partial^{-1} m_2, \\ & K_{43} = -K_{34}^* = \partial n_2 \partial^{-1} n_1 \partial - n_1 \partial^{-1} n_2, \qquad K_{44} = \partial n_2 \partial^{-1} n_2 \partial - n_2 \partial^{-1} n_2. \end{aligned} \tag{37}$$

By direct calculations, we arrive at

Proposition 3. *Eq. (35) can be rewritten in the following Hamiltonian form*

$$\left(m_{1,t},\ m_{2,t},\ n_{1,t},\ n_{2,t}\right)^T = K\left(\frac{\delta H_1}{\delta m_1},\ \frac{\delta H_1}{\delta m_2},\ \frac{\delta H_1}{\delta n_1},\ \frac{\delta H_1}{\delta n_2}\right)^T, \tag{38}$$

where K are given by (36), and

$$H_1 = \int_{-\infty}^{+\infty} [(u_{1,x} - u_1)n_1 + (u_{2,x} - u_2)n_2]dx. \tag{39}$$

We believe that Eq. (35) could be cast into a bi-Hamiltonian system. But we did not find another Hamiltonian operator yet that is compatible with the Hamiltonian operator (36).

Suppose N-peakon solution of (35) is expressed also in the form of (29). Then, we obtain the N-peakon dynamical system of (35):

$$\begin{aligned}
q_{j,t} &= \frac{1}{9}\Bigg\{-\frac{1}{3}(p_js_j + r_jw_j) + \sum_{i,k=1}^{N}(p_is_k + r_iw_k)\left(sgn(q_j - q_i) - sgn(q_j - q_k)\right)e^{-|q_j-q_i|-|q_j-q_k|} \\
&\quad + \sum_{i,k=1}^{N}(p_is_k + r_iw_k)\left(1 - sgn(q_j - q_i)sgn(q_j - q_k)\right)e^{-|q_j-q_i|-|q_j-q_k|}\Bigg\}, \\
p_{j,t} &= \frac{1}{9}\Bigg\{-\frac{1}{3}p_j(p_js_j + r_jw_j) + \sum_{i,k=1}^{N}\left(p_jp_is_k + r_jp_iw_k\right)\left(sgn(q_j - q_i) - sgn(q_j - q_k)\right)e^{-|q_j-q_i|-|q_j-q_k|} \\
&\quad + \sum_{i,k=1}^{N}\left(p_jp_is_k + r_jp_iw_k\right)\left(1 - sgn(q_j - q_i)sgn(q_j - q_k)\right)e^{-|q_j-q_i|-|q_j-q_k|}\Bigg\}, \\
r_{j,t} &= \frac{1}{9}\Bigg\{-\frac{1}{3}r_j(p_js_j + r_jw_j) + \sum_{i,k=1}^{N}\left(r_jr_iw_k + p_jr_is_k\right)\left(sgn(q_j - q_i) - sgn(q_j - q_k)\right)e^{-|q_j-q_i|-|q_j-q_k|} \\
&\quad + \sum_{i,k=1}^{N}\left(r_jr_iw_k + p_jr_is_k\right)\left(1 - sgn(q_j - q_i)sgn(q_j - q_k)\right)e^{-|q_j-q_i|-|q_j-q_k|}\Bigg\}, \\
s_{j,t} &= \frac{1}{9}\Bigg\{\frac{1}{3}s_j(p_js_j + r_jw_j) - \sum_{i,k=1}^{N}\left(w_jr_is_k + s_jp_is_k\right)\left(sgn(q_j - q_i) - sgn(q_j - q_k)\right)e^{-|q_j-q_i|-|q_j-q_k|} \\
&\quad + \sum_{i,k=1}^{N}\left(s_jp_is_k + w_jr_is_k\right)\left(sgn(q_j - q_i)sgn(q_j - q_k) - 1\right)e^{-|q_j-q_i|-|q_j-q_k|}\Bigg\}, \\
w_{j,t} &= \frac{1}{9}\Bigg\{\frac{1}{3}w_j(p_js_j + r_jw_j) - \sum_{i,k=1}^{N}\left(s_jp_iw_k + w_jr_iw_k\right)\left(sgn(q_j - q_i) - sgn(q_j - q_k)\right)e^{-|q_j-q_i|-|q_j-q_k|} \\
&\quad + \sum_{i,k=1}^{N}\left(w_jr_iw_k + s_jp_iw_k\right)\left(sgn(q_j - q_i)sgn(q_j - q_k) - 1\right)e^{-|q_j-q_i|-|q_j-q_k|}\Bigg\}.
\end{aligned} \tag{40}$$

For $N = 1$, (40) becomes

$$\begin{cases}
q_{1,t} = \frac{2}{27}(p_1s_1 + r_1w_1), \\
p_{1,t} = \frac{2}{27}p_1(p_1s_1 + r_1w_1), \\
r_{1,t} = \frac{2}{27}r_1(p_1s_1 + r_1w_1), \\
s_{1,t} = -\frac{2}{27}s_1(p_1s_1 + r_1w_1), \\
w_{1,t} = -\frac{2}{27}w_1(p_1s_1 + r_1w_1).
\end{cases} \tag{41}$$

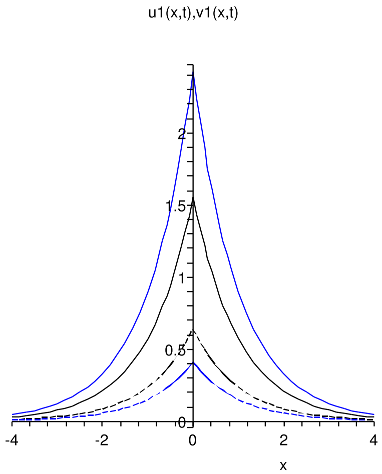

Fig. 1. The stationary single-peakon solution of the potentials $u_1(x,t)$ and $v_1(x,t)$ given by (34) with $C_1=0, A_2=A_4=1$ and $A_3=2$. Solid line: $u_1(x,t)$; Dashed line: $v_1(x,t)$; Black: $t=1$; Blue: $t=2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

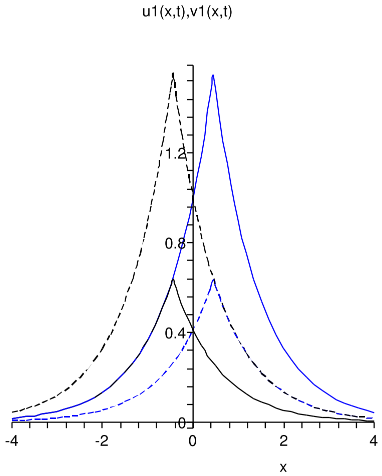

Fig. 2. The single-peakon solution of the potentials $u_1(x,t)$ and $v_1(x,t)$ given by (42) with $A_4=0, A_2=A_5=1$ and $A_3=2$. Solid line: $u_1(x,t)$; Dashed line: $v_1(x,t)$; Black: $t=-2$; Blue: $t=2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

We may solve this equation as

$$q_1=\frac{2}{27}(A_2+A_3)t+A_4,\qquad p_1=A_5e^{\frac{2}{27}(A_2+A_3)t},\qquad r_1=\frac{1}{A_1}p_1,\qquad s_1=\frac{A_2}{A_5}e^{-\frac{2}{27}(A_2+A_3)t},\qquad w_1=\frac{A_3}{r_1},\tag{42}$$

where $A_1,\cdots,A_5$ are integration constants. See Fig. 2 for the single-peakon of the potentials $u_1(x,t)$ and $v_1(x,t)$ with $A_4=0$, $A_2=A_5=1$ and $A_3=2$.

4. Conclusions and discussions

In our paper, we propose a multi-component generalization of the Camassa–Holm equation, and provide its Lax representation and infinitely many conservation laws. This system contains an arbitrary smooth function H, thus it is actually a large class of multi-component peakon equations. We show it is possible to find the bi-Hamiltonian structures for the special choices of H. In particular, we study the peakon solutions of this system in the case $N=2$, and obtain a new integrable system which admits stationary peakon solutions.

In contrast with the usual soliton equations, the peakon equations with arbitrary functions seem to be unusual. We believe that there are much investigations deserved to do for both our generalized peakon system and Li–Liu–Popowicz's system [31]. The following topics seem to be interesting:

- Is there a gauge transformation that can be applied to the Lax pair to remove the arbitrary function H?
- Does there exist a unified (bi-)Hamiltonian structure for the system (5) for the general H?
- Can the inverse scattering transforms be applied to solve the systems in general?

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