Born Approximation, Multiple Scattering, and Butterfly Algorithm

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ABSTRACT

Many imaging algorithms have been designed assuming the absence of multiple scattering. In the 2013 SPIE proceeding, we discussed an algorithm for removing high order scattering components from collected data. In this paper, our goal is to continue this work. First, we survey the current state of multiple scattering in SAR. Then, we revise our method and test it. Given an estimate of our target reflectivity, we compute the multi scattering effects in our target region for various frequencies. Furthermore, we propagate this energy through free space towards our antenna, and remove it from the collected data.

Keywords: Born approximation, multiple scattering, Neumann series, SAR imaging, iterative imaging.

1. INTRODUCTION

Electro-magnetism has seen applications in areas ranging from communication to astronomy. There is plenty of history behind the development of this theory. It is the model for all types of Electro-Magnetic(EM) radiation, from visible light to radio waves. RADAR (Radio Detection and Ranging) is an application of electromagnetism based on three principles: EM radiation travels at a finite constant speed while in a vacuum; radiation emitted from an antenna is contained in some localized region of space; targets scatter radiation. Initially RADAR was based on generated radio waves and was primarily concerned with detection and ranging as the name implies. In order to perform any task in RADAR we need to be able to transmit, and measure radiation. For a brief history of RADAR, and an introduction to the topic from an see.[1,2] RADAR is great for long distance high resolution ranging.

On the other hand, we know that the radiation generated propagates in some known beam pattern. The beam-width tells us how sharply focused our radiation is. If we know the beam-width of an antenna, then we have a sense of angular resolution. We know that any returns we get must be from inside the antenna beam, and so targets that are sufficiently far apart in angle can be distinguished. Sharper beams give us better angular resolution. Sadly, beam-width is tied directly to antenna size, and to cost. Moreover, if we want high angular resolution we need a very large and expensive antenna. For this reason angular resolution is limited in RADAR systems. Even if one could get the funding for a large antenna, pointing it to a given angle would be fairly difficult. These problems became evident during the early 1950s.[3] Two closely related systems were created to resolve these issues. By using multiple transmitting antennas delayed appropriately one can synthetically form and point beams. Such systems are called phased arrays.[3] By moving a single antenna while transmitting and receiving, one can create a synthetic beam along the path of the antenna by combining returns in some coherent way. These systems, called Synthetic Aperture RADAR (SAR) systems[3] are the object of our interest. A significant difference between SAR and traditional RADAR is the necessary processing time. Not much is free in life, and angular resolution does not come cheap. If we don't want to pay for better hardware, then we have to pay with computation time, and algorithmic complexity. The scattered data of a SAR system in the frequency domain takes to form of an oscillatory integral. Often, one applies the Born approximation by saying that radiation only scatters once, or more commonly replacing the total field in the Lippmann Schwinger[4] equation by the incident field. This can be justified if targets are well separated, and not terribly reflective but introduces error. In the presence of single scattering, the measured scattered field is an oscillatory integral as well.[4] In[5] the

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authors apply an algorithm developed in[6] to efficiently evaluate a generalized filtered back projection algorithm. Generally, multiple scattering effects are not negligible. If there are two objects near each other, then some of the radiation scattered from one of the objects will probably be scattered again by the other object. In[7] the authors implement a data generation software called Raider Tracer based on physical optics. From their experiments we can see that multiple scattering has noticeable effects on image reconstruction.

Our goal is to study multiple scattering in SAR image reconstruction for stationary targets. In particular, we will aim to mitigate errors introduced by the Born(single scattering) approximation. To this end, we use a rough estimate of our targets to estimate higher order scattering effects and remove them from collected data. We will use the butterfly algorithm developed in[5] for efficient evaluation of the scattering series that arises from wave equation imaging. This work has also been extended to parallel processing.[8] We provide a detailed implementation of the butterfly algorithm described in.[5] In discrete form, the integral we want to evaluate turn into a matrix, vector product. The vector represents our targets, and the matrix, the scattering operator. To compute scattered data we must evaluate this product efficiently. For every row, if we could find a low dimensional matrix, and vector whose product approximated the row of scattering data product, then we could evaluate the matrix product very efficiently. This is not trivial to do. On the other hand, if we only perform the matrix product with only one column, then depending on the properties of the matrix, we can find a low dimensional matrix, and vector to approximate the partial product for all rows. The butterfly algorithm is about propagating these partial matrix product approximation to find a full matrix product approximation. We will review the algorithm developed in[5] and discuss a serial implementation of it. This is a continuation of.[9] Other recent work in multiple scattering is done in.[10] In the following sections we present the wave model, discuss multiple scattering, discuss the butterfly algorithm, and discuss how to compensate for multiple scattering.

2. BACKGROUND AND MODEL FORMULATION

Synthetic Aperture RADAR (SAR) is an imaging scheme that involves the exploitation of scattered radiation measured along a continuous path in space. The prototype for SAR is an antenna attached to an airplane which goes along some path $\gamma(t)$. The antenna generates radiation directed towards some region, and listens for energy that is scattered back to the antenna(back scatter). After a sufficiently long time we assume that there is no more back scatter to wait for from any practical returns from the probe, so the antenna generates more radiation and listens again. This process is repeated several times during the flight path. Essentially a SAR system probes a region of interest from several different points in space, and uses collected data to form a coherent image of the region. In order to fully appreciate this scheme we need to understand the underlying forces that govern this model.

In the macroscopic regime, the appropriate model for electromagnetic radiation is Maxwell's equations. A classic treatment of the full equations is given in.[11] These equations can be difficult to work with for imaging. One can show under certain conditions(no polarization)[4,12–15] that we can work with an inhomogeneous wave equation instead. Following the literature we adopt this as our model. There are down-sides to this. Working with only one component of the EM fields ignores useful effects such as polarization but captures important aspects of electromagnetism such as finite speed, and the ability to cause interference. Simplifying the model leads to more efficient algorithms at the cost of accuracy. That said, the wave model captures several important effects regarding the scattering process, and is more manageable than the full Maxwell's equations. This is a often used model for SAR imaging, as can be seen by its proliferation in the literature.[4,5,12,16–20]

We must keep in mind, that SAR is tied very heavily to its particular application, and the application controls what approximations can be made. We have assumed that there are no moving targets. This corresponds to a short path if our antenna is moving fast enough. We also assume that our antenna is stationary during each transmission and measurement cycle. Obviously the antenna must either keep moving with constant velocity, or more undergo some acceleration is it is to cover some region of space, but this can be compensated for since we can just record (approximately) how fast the antenna moved during any transmit receive cycle. We work with one component of the EM field denoted by $u(x,t)$ which is a function of space $x \in \mathbb{R}^3$, and a time t. This is a scalar valued function. The source of the wave equation $j(x,t)$ is dependent on the change in current density as a function of time, and the spatial variation of the charge density. We could also work directly with the scalar potential that generates the em field, but it does not change much. For simplicity we suppose that we have

complete knowledge of $j(x,t)$. This is never really the case due to background radiation, but we take this view since it may lead to some insight into handling the noisy case. Typically when one processes collected data they correlate with the transmitted signal to minimize the effects of certain types of noise. The field satisfies the inhomogeneous wave equation in $3+1$ dimensions

$$\left(\nabla^2 - \frac{\partial_t^2}{c(x)^2}\right) u = -j(x,t) \tag{1}$$

with wave speed $c(x)$.

All of the information concerning a region of interest contained in measurements of $u(x,t)$ is in $c(x)$. In particular, if the field is in a vacuum, then $c(x) = c_0$. If we could measure $u(x,t)$ near a given point, then we could (mostly)reconstruct the wave speed by $c(x)^2 = \frac{\nabla^2 u(x,t)+j(x,t)}{\partial_t^2 u(x,t)}$ as long as the denominator is not zero for some time. Of course we would need local measurements on a scale comparable to what resolution we hope to attain which defeats the purpose of the system. We must consider how we treat the "functions" u, j and c. Should we treat everything as a generalized function (distribution),[21] or is a standard function powerful enough. Certainly we want our model to make sense, but distributions can be subtle. We cannot in generally multiply distributions in a meaningful way. We can multiply distributions and smooth functions though. We may also differentiate distributions, and of course tell them apart so that the equality symbol make sense. We consider u and j as distributions. Here we cannot assume $c(x)$ is continuous since many interesting objects in life have apparent discontinuities such as walls, planes, cars, and almost anything we might be interested in studying. The change from free space to target waves speed is often abrupt. We can assume that $c(x)$ is real and bounded by c_0. Sometimes $c(x)$ is considered as complex, and the imaginary part corresponds to energy lost to absorption retransmission process. As we will see soon enough, em waves do dissipate energy as they travel since they spread out in some geometric fashion. Rather than work directly with the wave speed we would like a function that accurately describes the support of our targets. That is we want a function that is zero in free space. We define the reflectivity function[4, 12, 16–20, 22] by

$$\nu(x) = \frac{1}{c_0^2} - \frac{1}{c(x)^2}. \tag{2}$$

The reflectivity function gives us an idea of what our targets looks like. It is zero in the absence of targets, and non zero when targets are present. The goal of RADAR imaging in this wave model is to solve for (2). The reflectivity function is very closely tied to the field $u(x,t)$, and as we will see, a given reflectivity function determines u under a weak scattering assumption. The reflectivity is also related to the index of refraction of a material by

$$\nu(x) = 1 - n(x)^2. \tag{3}$$

It is very difficult to get light to completely stop moving, so unless we are imaging a very expensive science experiment, we can assume that $c(x) \neq 0$. We can also assume that are targets are not infinitely large so that $c(x) = c_0$ outside of some compact set $K \subset \mathbb{R}^3$. This means that ν is compactly supported. For this reason ν must have finite energy. That is ν is square integrable: $\nu(x) \in L^2(\mathbb{R}^3)$. We may also assume that $\nu \in L^p$ for any p since it bounded and compactly supported, but L^2 is the most useful space. Similarly we assume that any generated wave must contain a finite amount of energy, and so can also be found in $L^2\mathbb{R}^4$. There are some subtleties involved when working with these spaces. We must keep in mind that the L^p space are not sets of functions strictest sense, since two functions are equivalent if they are equal almost everywhere. The L^p spaces are sets of classes of functions that all look the same with respect to integration. Continuous functions are point wise objects so we can see what they look like at a given point in space and time. Distributions are not point wise at all, they are the linear continuous functions from a test space to the real numbers(functionals). Distributions act on other functions, so it does not make too much sense to represent evaluate a distribution at a single point. The test space we take generally affects the properties of the distributions. Most useful functions(those with finite energy) define unique distributions, so distributions are basically extended classes of functions. If the test space is smooth, and consist of fast decaying functions we can define differentiation(weak) on our distributions. For simplicity we may regard the source term j as a distribution. Distributions gives us the ability to understand point like objects in a useful way, since integrating over a single point gives the value 0.

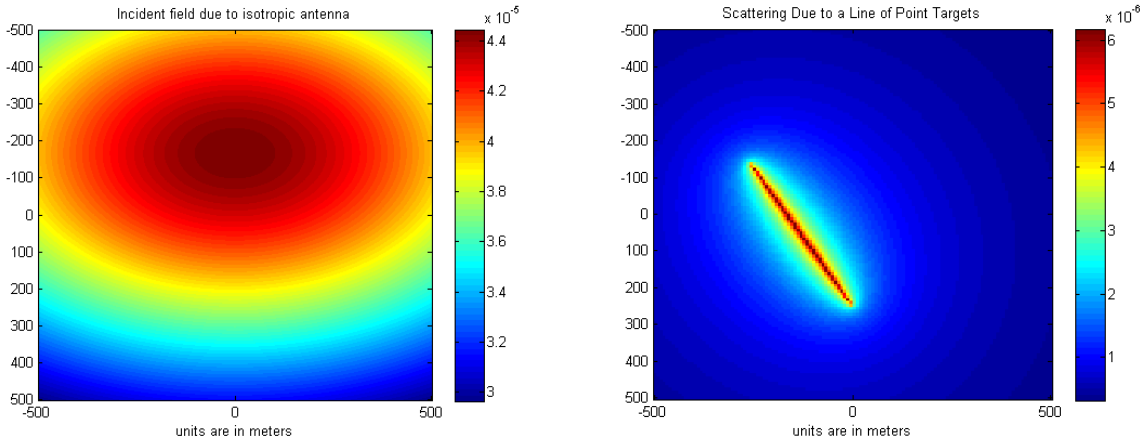

Figure 1. Incident and Born approximated scattered field on left and right, respectively.

We must consider how to represent recovered images for the user, since graphing is a point wise operation. To do this we can simply look at the action on a smooth function whose support is smaller than a pixel. In a sense this means we want an average value of our targets, and corresponds to expanding the target function into a localized basis and finding the corresponding coefficients. For now we consider c as square integrable. These are nice spaces to work with since it gives us an inner product, and a norm. Moreover, we have a well defined Fourier transform for ν, and the field u which will be useful for image formation latter. This is good news, since any kind of approximation to this needs to be made in some metric space, and vector spaces allow one to impose a finite structure on the problem in a straightforward manner. A normalised vector space is almost essential to any approximation problem, and we have that structure for reflectivity and wave functions. The solution space for the field u is not so clear. We can establish at least one solution based on a perturbation argument. We can rewrite the total field equation as

$$\left(\nabla^2 - \frac{\partial_t^2}{c_0^2}\right) u = -(j(x,t) + \nu(x)\partial_t^2 u). \tag{4}$$

The left hand side is the free space wave equation and can be solved by convolving with the Green's "function"

$$g(x,t) = \frac{\delta(t - |x|/c_0)}{4\pi|x|}. \tag{5}$$

Thus we have

$$u(x,t) = g(x,t) * (j(x,t) + \nu(x)\partial_t^2 u). \tag{6}$$

where $*$ denotes convolution over space and time. We call $g * j$ the incident field, and denote it by u^{in}. Note that if we are in free space, $\nu = 0$, and $u = u^{in}$. That is, we think of the incident field as the portion of u that satisfies the wave equation in free space. We can solve this for u by first defining the scattering operator L which takes a reflectivity function ν and an EM field u and produces a new EM field $L\{\nu, u\}$. It is evident that L that is bilinear. A bilinear operator may also be viewed as a family of linear operators indexed by one of its arguments. That is, for every fixed reflectivity our scattering operator is simply a linear operator. We define the scattering operator as

$$L\{\nu, u\} = g(x,t) * \nu(x)\partial_t^2 u. \tag{7}$$

Under this notation we have

$$u(x,t) - L\{\nu, u\} = u^{in}. \tag{8}$$

Adopting the convention that

$$L^n\{\nu, u\} = L\{\nu, L^{n-1}\{\nu, u\}\} \tag{9}$$

where $L^0 = I$ the identity map, we can solve for u as by computing the inverse of $I - L\{v, \cdot\}$ and applying to u^{in} to get

$$u(x,t) = \sum_{n=0}^{\infty} L^n\{v, u^{in}\}. \tag{10}$$

Such a series does not always converge. The series converges for any incident field if the norm of the scattering operator is less than 1, analogous to the geometric series. On the other hand, this is not necessary in order for the series to converge for a given incident field. It can be shown[23] that it is enough for $L^n\{v, u^{in}\} \to 0$. If the reflectivity scatters weakly enough, and the incident field does not vary too quickly, the terms will decay as needed. To solve for v is a very nonlinear problem. The purpose of the Born approximation is to remove this non-linearity by truncating the series to the first two terms. Physically, the first term in the series corresponds to radiation scattered away from the transmitting antenna. The second term corresponds to this first scattered field again interacting with targets and getting scattered again. In general the n-th term is the $n-1$-th term being scattered. Adding all of these together we get the scattered field. The scattered field is then the sum of these re-scattered fields. In SAR, we measure the field u at several points along a known path, and use the collected data to estimate v. If we measure far from our targets, the targets are far from each other, and our signal is not too high frequency, then the multiple scattering effects will be weak. This is not the only solution to the wave equation. There are different Green's functions for the wave equation. We have chosen one that propagates energy away from a source in a spherical manner. This is the unique solution if we assume the Sommerfeld radiation condition which is that the solution we consider decays inversely proportional to the distance from the source, and the change of energy radially at a given wave number is equal to the wave number times u times i plus a term inversely proportional to the distance from the source.[22] We will use the series to compute the multiple scattering effects a coarse estimate of the target reflectivity creates, and then subtract this from the data. Basically the model corresponds to a sphere emanating from each point on the antenna and propagating outward at the speed of light, which eventually will contact a target. Every point on the sphere that is contacting a target will start a new sphere emanating from that point with energy proportional to the second derivative of the incoming wave. We must truncate the series after some finite number of scattering events. We want to be able get a large number of terms in the series for as few applications of the scattering operator as we can. For this reason, we look at factored forms of the series. Consider the geometric series to k terms

$$s_k = 1 + x + x^2 + x^3 + \cdots + x^k. \tag{11}$$

How fast can we compute this series? For a general k degree polynomial, Horner's method may be used to evaluate it at a single point in k multiplications and additions and multiplications by writing the sum as

$$s_k = 1 + x(1 + x(1 + \cdots (1 + x) \cdots))) \tag{12}$$

and evaluating it from the inside. We can write this in an iterative fashion as

$$s_k = (1 + x)s_{k-1} \tag{13}$$

where $s_0 = 1$. These products correspond to evaluating the scattering operator. In fact, if we wanted to evaluate the geometric series for a square matrix, we could do better using a fast exponentiation method, and a different factorization. The flaw in this route is that we can only do fast multiplication for the scattering operator and not for its squares. We will use Horner's method to compute the multiply scattered data.

3. IMAGE RECONSTRUCTION

Here we briefly derive an image reconstruction method for SAR to test our algorithm against. This is basically the polar format algorithm without the interpolation. The idea in SAR is to move through space and measure the scattered field to solve for the reflectivity function. In this form the problem is still fairly difficult even if we completely know the scattered field. For this reason one typically assumes the higher order terms in the sum are negligible. This is called the Born approximation. The incident field is dependent on the source in the original model. This in turn is dependent on a known input signal. One then uses an imaging operator I on the Born

approximated data and the input signal to estimate the reflectivity function. We use this estimated image to estimate and remove the higher order terms from the collected data. From experimentation it is evident that estimated reflectivity functions tend to be scaled improperly. That is to say, the positions of targets are accurate, and their relative values are reasonable, but the estimated reflectivity values are off. To resolve this we simply scale the estimated high order scattering. In addition, estimates tend to look fairly noisy, so we must perform some denoising/focusing operator $\mathcal{N}$ before using the image estimate. Formally we create a sequence of estimates $\{\hat{V}_k^N\}$ in $\mathcal{V}$ by

$$\hat{V}_{k+1}^N = I\left\{d - \sum_{n=2}^{N} L^n\{\lambda_k \mathcal{N}\{V_k^N\}, u^{in}\}\right\} \tag{14}$$

where λ_k is a scaling parameter. This is the iterative method mentioned in the title. The powers of the scattering operator are linear in the second argument, and homogenous of order n in the first argument, so we can scale by λ_k^n after scatting the data. The order does matter when round off error is taken into account, so we scale before scattering, but after noise reduction. We will discuss the specifics later, but first let's consider when it is appropriate to use this method. We do not establish convergence of this method, but test it computationally. Let u^{sc+} denote the higher order scattered terms in the Neumann series. That is to say, $u = u^{in} + L\{v, u^{in}\} + u^{sc+}$. Let us work in the frequency domain. If we assume a point like, isotropic antenna that is stationary between transmission and reception, the incident field can be found in the closed form

$$u^{in}(x, w) = p(w)G(x - x_0, w) \tag{15}$$

where $p(w)$ is a known input signal, and x_0 is the antenna position. Then, noting that G is even we have that

$$L\{v, u^{in}\} = -w^2 p(w) \int_{\mathbb{R}^3} G(x - z, w)v(z)G(x_0 - z, w)\, dz. \tag{16}$$

Our data d is a set of samples of the scattered field due to different antenna locations, and frequencies. Our antenna is assumed to be essentially stationary during transmissions, so that our data

$$d(x_0, w) = -w^2 p(w) \int_{\mathbb{R}^3} G(x_0 - z, w)^2 v(z)\, dz. \tag{17}$$

One may then make the far field approximation and assume that $\|x\| >> \|z\|$ to have that $\|x - z\| \approx \|x\| - \hat{x} \cdot z$, and $\frac{1}{\|x-z\|} \approx \frac{1}{\|x\|}$ which leads to

$$G(x_0 - z, w) \approx \frac{e^{-iw\frac{\|x\|}{c_0}} e^{iw\frac{\hat{x}\cdot z}{c_0}}}{4\pi\|x\|} \tag{18}$$

so that our data becomes

$$d(x_0, w) \approx -\frac{w^2 p(w) e^{-2iw\frac{\|x_0\|}{c_0}}}{(4\pi)^2\|x_0\|^2} \int_{\mathbb{R}^3} e^{2iw\frac{\hat{x_0}\cdot z}{c_0}} v(z)\, dz. \tag{19}$$

Let

$$S(w, p, x_0) = -\frac{w^2 p(w) e^{-2iw\frac{\|x_0\|}{c_0}}}{(4\pi)^2\|x_0\|^2}. \tag{20}$$

Then

$$d(x_0, w) \approx S(w, p, x_0)\mathcal{F}^{-1}\{v\}(2w\frac{\hat{x_0}}{c_0}). \tag{21}$$

Since v is square integrable, we may write

$$v(x) = \left(\frac{1}{2\pi}\right)^3 \int_{\mathbb{R}^3} e^{-ix\cdot\xi}\mathcal{F}^{-1}\{v\}(\xi)\, d\xi. \tag{22}$$

Then we can perform the Fourier transform of the data divided by S for all frequencies that the input p is supported on. The data we have is not on a uniform grid which is needed for the traditional FFT. Typically one will interpolate to such a grid to speed up computation. We use the Riemann sum directly.

4. COMPUTATION OF THE SCATTERING OPERATOR AND NEUMANN SERIES

There are several efficient methods to evaluate the oscillatory integral that defines the scattering operator.[5,6] Data is often given in the frequency domain, so we consider computation there. In the frequency domain the scattering operator becomes

$$L(v,u) = -w^2 \int_{\mathbb{R}^3} \frac{e^{-iw\frac{\|x-z\|}{c_0}} v(z) u(z,w)}{4\pi\|x-z\|}\, dz \tag{23}$$

which is in tbe form of a Fourier integral operator. We apply the butterfly algorithm to evaluate this integral efficiently. We can efficiently[24] evaluate a truncation of the scattering series by applying Horner's method to calculate the scattered field by evaluating

$$u_N^{sc} = L\{u^{in} + L\{u^{in} + \dots L\{u^{in}\}\}\} = L\{u^{in} + u_{N-1}^{sc}\}. \tag{24}$$

bottom up ($u_1^{sc} = L\{u^{in}\}$) analogous to the geometric series we mentioned earlier. At this point we did not use a noise reduction method. We will pick a fixed scaling parameter λ for each step in the iteration.

5. CONCLUSION

Simulation is still being run. To continue we must employ more accurate image reconstruction methods. We were able to produce high order scattering data, but image formation was a bottle neck. For this reason we are implementing the butterfly algorithm to image as well as scatter data. There is still much work to be done in this direction. We did not have to use the wave model at all for this approach. We could have used more exact computational electromagnetic schemes such as method of moments.[25] Moreover, one might also use a time-frequency method for data generation. This work mostly constituted a numerical experiment. The point of this work was to examine the use of high order terms to clear up images. Convergence of this method still needs to be proven/ disproved. In[26] the authors mention some work which tries to minimize the error created between the data and multiply scattered field. This should be compared to that work. We were successful in producing high order scattering in the frequency domain, and have some idea of when it is appropriate to use a truncated Neumann series to approximate the scattering function. At least one author[27] seems to have found a way around the Born approximation in a MIMO compressive sensing setting.

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