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# Discrete and continuous integrable systems possessing the same non-dynamical $r$-matrix

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**Abstract**

We consider two different Lax representations with the same Lax matrix in terms of $2 \times 2$ traceless matrices: one produces the discrete integrable symplectic mapping resulting from the well-known Toda spectral problem under the discrete Bargmann–Garnier (BG) constraint; the other generates the continuous non-linearized integrable system for the c-KdV spectra problem under the corresponding BG constraint. We are surprised to find that the two very different (one is discrete, the other continuous) integrable systems possess the same non-dynamical $r$-matrix. © 1997 Published by Elsevier Science B.V.

## 1. Introduction

In recent years, it has been proven that the well-known non-linearized method (NM) [1] is a powerful tool to construct completely integrable finite-dimensional Hamiltonian systems from the Lax representations of non-linear evolution equations (NLEEs) [2,3]. With the help of this method, many soliton equations or NLEEs have been found to possess the so-called involutive or parameter representations of solutions [4–6]. Besides, the NM can also successively be applied in a discrete context, naturally inducing the integrable symplectic mapping [7–9].

Lately, Ragnisco, Cao and Wu have strictly established the connection of integrable mappings with the stationary Toda flows and with the finite gap sector of the solution manifold for the Toda hierarchy [10]. In a further paper [11] Ragnisco has presented the dynamical $r$-matrices for integrable mappings resulting from the Toda spectral problem under the discrete Bargmann–Garnier (we call this integrable mapping IMTDBG) and discrete Neumann constraints. The conclusion of Ref. [11] concerns an open question: whether one can choose different Lax pairs for the same integrable mapping such that the corresponding $r$-matrices only depend on the spectral parameters, i.e. leading to constant or non-dynamical $r$-matrices. If so, then the calculations and proof (such as the Yang–Baxter equation) related to integrable systems will be much reduced. Thus this open problem is important.

The aims of this paper are twofold. One is to answer the above problem. We find another Lax representation

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for the IMTDBG, which admits a non-dynamical $r$-matrix instead of a dynamical $r$-matrix [11]. Our Lax operator relates to that given by Ragnisco through a gauge transformation. The other is to report a surprising result: we find that IMTDBG and the continuous non-linearized integrable Hamiltonian system for the c-KdV spectra problem [12] possess the same non-dynamical (or constant) $r$-matrix.

The paper is organized as follows. In the next section, we first consider a $2\times 2$ traceless Lax matrix $L(\lambda)$; and then, through introducing two different auxiliary matrices $M_n(\lambda)$, $M(\lambda)$, from the two Lax equations $L_{n+1}M_n = M_nL_n$, $L_x = [L, M]$ (the form of $L_n$ is the same as $L$) we obtain IMTDBG and the continuous non-linearized Hamiltonian system for the c-KdV spectra problem, respectively. Section 3 shows that the above two very different integrable systems possess the same non-dynamical (or constant) $r$-matrix. Also a gauge transformation between our Lax operator and Ragnisco's is given. Furthermore, in Section 4 we have shown that having the same kind of $r$-matrices assures the integrability of the above two non-linearized systems. In the last section, some concluding remarks and discussions are given.

## 2. Discrete and continuous systems

Let $(R^{2N}, \mathrm{d}p\wedge\mathrm{d}q)$ be a standard symplectic structure in the Euclid space $R^{2N} = \{(p,q)|p = (p_1,\dots,p_N),\ q = (q_1,\dots,q_N)\}$, $q_i$, $p_i$ be a pair of canonical variables; $\lambda_1,\dots,\lambda_N$ be $N$ arbitrary given distinct parameters, $\lambda$ be a spectral parameter, and $\langle\ ,\ \rangle$ stand for the standard inner-product in the Euclid space $R^N$. Denote $\Lambda$ by $\Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_N)$.

Under the standard symplectic structure $(R^{2N}, \mathrm{d}p\wedge\mathrm{d}p)$ the Poisson bracket of two Hamiltonian functions $F$, $G$ is defined by [13]

$$\{F,G\} = \sum_{i=1}^{N}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right). \tag{1}$$

Now let us consider a $2\times 2$ traceless matrix (called the Lax matrix)

$$L = \begin{pmatrix} -\frac{1}{2}\lambda & \langle p,q\rangle \\ -1 & \frac{1}{2}\lambda \end{pmatrix} + \sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\begin{pmatrix} -p_jq_j & p_j^2 \\ -q_j^2 & p_jq_j \end{pmatrix} \equiv \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}, \tag{2}$$

then we have

*Theorem* 1. Let $n$ be the discrete variable, $p\to p_n$, $q\to q_n$ (i.e. $p_j\to p_{nj}$, $q_j\to q_{nj}$) and $L\to L_n$. Choose a $2\times 2$ matrix $M_n$ as follows,

$$M_n = \begin{pmatrix} 0 & \pm(\langle\Lambda q_n,q_n\rangle - \langle p_n,q_n\rangle - \langle q_n,q_n\rangle^2)^{1/2} \\ m_{31} & m_{32} \end{pmatrix} \tag{3}$$

where

$$m_{31} = \mp(\langle\Lambda q_n,q_n\rangle - \langle p_n,q_n\rangle - \langle q_n,q_n\rangle^2)^{-1/2},$$
$$m_{32} = \pm(\lambda - \langle q_n,q_n\rangle)\times(\langle\Lambda q_n,q_n\rangle - \langle p_n,q_n\rangle - \langle q_n,q_n\rangle^2)^{-1/2}.$$

Then, the discrete Lax equation

$$L_{n+1}M_n = M_nL_n \tag{4}$$

is equivalent to a symplectic mapping (discrete system (DS)),

$$p_{n+1} = \pm(\langle\Lambda q_n,q_n\rangle - \langle p_n,q_n\rangle - \langle q_n,q_n\rangle^2)^{1/2}q_n,$$
$$q_{n+1} = \pm(\langle\Lambda q_n,q_n\rangle - \langle p_n,q_n\rangle - \langle q_n,q_n\rangle^2)^{-1/2}\times(\Lambda q_n - p_n - \langle q_n,q_n\rangle q_n), \tag{5}$$

which can be simply written as a mapping form,

$$H: \begin{pmatrix} p_n \\ q_n \end{pmatrix} \to \begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix}. \tag{6}$$

*Proof.* Through a lengthy but direct calculation, we know (3)⇔(4). The mapping $H$ defined by (6) is symplectic by virtue of $\mathrm{d}p_{n+1}\wedge\mathrm{d}q_{n+1} = \mathrm{d}p_n\wedge\mathrm{d}q_n$.

Set

$$u_n = \pm(\langle\Lambda q_n,q_n\rangle - \langle p_n,q_n\rangle - \langle q_n,q_n\rangle^2)^{1/2},$$
$$v_n = \langle q_n,q_n\rangle, \tag{7}$$

then (5) naturally becomes

$$u_n q_{n+1j} + v_n q_{nj} + u_{n-1} q_{n-1j} = \lambda_j q_{nj}, \quad j = 1, \dots, N, \tag{8}$$

which is just the well-known Toda spectral problem (TSP)

$$(E^{-1} u_n + v_n + u_n E)\Psi = \lambda \Psi,$$
$$E f_n = f_{n+1}, \quad E^{-1} f_n = f_{n-1} \tag{9}$$

with $\lambda = \lambda_j$, $\Psi = q_{nj}$. Simultaneously, the potentials $u_n$, $v_n$ defined by (7) are exactly the Bargmann–Garnier constraint [10]

$$G_0 = (2u_n, v_n)^{\mathrm{T}} = \sum_{j=1}^{N} \left( \frac{\delta \lambda_j}{\delta u_n}, \frac{\delta \lambda_j}{\delta v_n} \right)^{\mathrm{T}} \tag{10}$$

of TSP (9), where $\delta\lambda_j/\delta u_n$, $\delta\lambda_j/\delta v_n$ are the two spectral gradients of the spectral parameter $\lambda_j$ with respect to the potentials $u_n$ and $v_n$, respectively. Therefore, Eq. (5) is IMTDBG.

Now we turn to the Lax matrix (2). Analogous to the calculations of Theorem 1, it is not difficult to obtain

*Theorem* 2. Let $x$ be the continuous spatial variable, and the $2 \times 2$ traceless matrix $M$ is chosen as

$$M = \begin{pmatrix} -\frac{1}{2}\lambda + \frac{1}{2}\langle q, q \rangle & \langle p, q \rangle \\ -1 & \frac{1}{2}\lambda - \frac{1}{2}\langle q, q \rangle \end{pmatrix}. \tag{11}$$

Then the continuous Lax equation

$$L_x = [M, L] \equiv ML - LM, \quad L_x = \frac{\partial L}{\partial x} \tag{12}$$

is equivalent to a finite-dimensional Hamiltonian system (continuous system (CS)), $H$:

$$p_x = -\tfrac{1}{2}\Lambda p + \tfrac{1}{2}\langle q, q \rangle p + \langle p, q \rangle q = -\frac{\partial H}{\partial q},$$
$$q_x = -p + \tfrac{1}{2}\Lambda q - \tfrac{1}{2}\langle q, q \rangle q = \frac{\partial H}{\partial p} \tag{13}$$

with the Hamiltonian function

$$H = -\tfrac{1}{2}\langle p, p \rangle + \tfrac{1}{2}\langle \Lambda q, p \rangle - \tfrac{1}{2}\langle q, q \rangle \langle p, q \rangle. \tag{14}$$

Set

$$u = \langle q, q \rangle, \quad v = \langle p, q \rangle \tag{15}$$

then (13) actually reads

$$p_{jx} = -\tfrac{1}{2}\lambda_j p_j + \tfrac{1}{2} u p_j + v q_j,$$
$$q_{jx} = -p_j + \tfrac{1}{2}\lambda_j q_j - \tfrac{1}{2} u q_j, \tag{16}$$

which is non-other than the c-KdV spectral problem (CSP) [12]

$$\Psi_x = \begin{pmatrix} -\frac{1}{2}\lambda + \frac{1}{2}u & v \\ -1 & \frac{1}{2}\lambda - \frac{1}{2}u \end{pmatrix} \Psi, \tag{17}$$

with $\lambda = \lambda_j$, $\Psi = (p_j, q_j)^{\mathrm{T}}$. Simultaneously, the potentials $u, v$ determined by (15) exactly present the BG constraint [1]

$$G_0 = (v, u)^{\mathrm{T}} = \sum_{j=1}^{N} \left( \frac{\delta \lambda_j}{\delta u}, \frac{\delta \lambda_j}{\delta v} \right)^{\mathrm{T}}$$

of CSP (17). In Ref. [1], Cao and Geng studied the non-linearized Bargmann–Garnier system and Neumann system, but they did not give the Lax representation of (13).

## 3. The same r-matrix for DS (5) and CS (13)

In Section 2, we have already seen that the two very different systems DS (5) and CS (13) have the same form of Lax matrix $L$ defined by (2). Thus, they should possess the same $r$-matrices.

*Proposition* 1. Let $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ be defined in (2), and $\lambda$, $\mu$ be two different parameters. Then

$$\{A(\lambda), A(\mu)\} = \{C(\lambda), C(\mu)\} = 0,$$
$$\{B(\lambda), B(\mu)\} = 2(B(\mu) - B(\lambda)),$$
$$\{A(\lambda), B(\mu)\} = \frac{2}{\mu - \lambda} \times (B(\mu) - B(\lambda)),$$
$$\{A(\lambda), C(\mu)\} = \frac{2}{\mu - \lambda}(C(\lambda) - C(\mu)),$$
$$\{B(\lambda), C(\mu)\} = \frac{4}{\mu - \lambda}(A(\mu) - A(\lambda)) - 2C(\mu). \tag{18}$$

Let $L_1(\lambda) = L(\lambda) \otimes I$ and $L_2(\mu) = I \otimes L(\mu)$, here $I$ is the $2 \times 2$ unit matrix. Then from the above proposition one obtains the following theorem.

*Theorem* 3. The Lax matrix $L$ defined by (2) satisfies the fundamental Poisson bracket

$$\{L_1, L_2\} = [r_{12}(\lambda,\mu), L_1] - [r_{21}(\mu,\lambda), L_2], \tag{19}$$

where $\{L_1(\lambda), L_2(\mu)\}$ is a $4 \times 4$ matrix [14] consisting of various possible Poisson brackets of the elements for $L(\lambda)$ and $L(\mu)$, and the $r$-matrices $r_{12}(\lambda,\mu)$, $r_{21}(\mu,\lambda)$ are given by

$$r_{12}(\lambda,\mu) = \frac{2}{\mu-\lambda}P - S,$$
$$r_{21}(\lambda,\mu) = Pr_{12}(\lambda,\mu)P, \tag{20}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{21}$$

where [ , ] is the usual matrix commutator.

We readily verify the classical Yang–Baxter equation (YBE),

$$[r_{ij}, r_{ik}] + [r_{ij}, r_{jk}] + [r_{kj}, r_{ik}] = 0, \quad i,j,k = 1,2,3. \tag{22}$$

Obviously, the $r$-matrix formula (20) only depends on two constant spectral parameters $\lambda, \mu$ and has nothing to do with the dynamical variables $p_j$, $q_j$ for the continuous case (or $p_{nj}$, $q_{nj}$ for the discrete case ($j = 1,\ldots,N$)). Thus, the symplectic mapping (5) for the discrete Toda lattice and the Hamiltonian system (13) for the continuous c-KdV hierarchy possess the same non-dynamical (or constant) $r$-matrices formula (20).

*Remark.* In fact, since the $r$-matrix relation is concerned only with the commutator, the matrix $S$ given by (21) can be chosen as

$$S = \begin{pmatrix} 0 & 1 & d & 0 \\ c & 0 & 0 & d \\ 0 & 0 & 0 & -1 \\ 0 & 0 & c & 0 \end{pmatrix}. \tag{23}$$

The elements $c, d$ in the $r$-matrix formula (20) which make (19) hold can be chosen as arbitrary functions $c(\lambda,\mu,p,q)$, $d(\lambda,\mu,p,q)$ or $c(\lambda,\mu,p_n,q_n)$, $d(\lambda,\mu,p_n,q_n)$ with respect to the spectral parameters $\lambda, \mu$ and the dynamical variables $p, q$ or $p_n, q_n$. This shows that given a Lax operator, the associated $r$-matrix is not uniquely defined (even infinitely many). Here we take the simplest case $c = d = 0$.

## 4. Integrability of DS (5) and CS (13)

Now, from the Lax matrix (2), let us introduce two $N$-involutive sets, which guarantee the integrability of DS (5) and CS (13).

First, for the discrete version (i.e. $L \to L_n$) of (2), one gets

$$\det L_n(\lambda) = -\tfrac{1}{2}\operatorname{Tr} L_n^2(\lambda)$$
$$= -\tfrac{1}{4}\lambda^2 - \sum_{i=1}^{N} \frac{E_{ni}}{\lambda - \lambda_i}, \tag{24}$$

where

$$E_{ni} = \lambda_i p_{ni} q_{ni} - p_{ni}^2 - \langle p_n, q_n\rangle q_{ni}^2 - \sum_{i\neq j, j=1}^{N} \frac{(q_{ni}p_{nj} - p_{ni}q_{nj})^2}{\lambda_i - \lambda_j},$$
$$i = 1,\ldots,N. \tag{25}$$

By the $r$-matrix relation (19), we can immediately obtain

$$\{L_{n1}^2(\lambda), L_{n2}^2(\mu)\} = [\bar{r}_{12}(\lambda,\mu), L_{n1}(\lambda)] - [\bar{r}_{21}(\mu,\lambda), L_{n2}(\mu)], \tag{26}$$

where [16]

$$\bar{r}_{ij}(\lambda,\mu) = \sum_{k=0}^{1}\sum_{l=0}^{1} L_{n1}^{1-k}(\lambda) L_{n2}^{1-l}(\mu) \times r_{ij}(\lambda,\mu) L_{n1}^{k}(\lambda) L_{n2}^{l}(\mu),$$
$$i,j = 12, 21, \tag{27}$$

and

$$L_{n1}(\lambda) = L_n(\lambda) \otimes I, \quad L_{n2}(\mu) = I \otimes L_n(\mu).$$

Then, it follows from (26) that

$$4\{\mathrm{Tr}\, L_n^2(\lambda), \mathrm{Tr}\, L_n^2(\mu)\} = \mathrm{Tr}\{\lambda_n^2(\lambda) \otimes L_n^2(\mu)\}$$
$$= \mathrm{Tr}\{L_{n1}^2(\lambda), L_{n2}^2(\mu)\} = 0. \tag{28}$$

Substituting (24) into (28), we obtain

$$\{E_{ni}, E_{nj}\} = 0, \quad i,j, = 1, \ldots, N. \tag{29}$$

In addition, $E_{(n+1)i} = E_{ni}$ and $dE_{n1}, \ldots, dE_{nN}$ are linearly independent. So, according to the principle of integrable symplectic mapping (8), from the view point of the non-dynamical $r$-matrix we have shown

*Theorem* 4. The symplectic mapping $H$ defined by (6) for the Toda lattice is completely integrable, and its $N$-involutive systems (invariant and functionally independent) are $\{E_{ni}\}_{i=1}^N$ defined by (25).

Second, let $p_{nj} \to p_j$, $q_{nj} \to q_j$ in (25); for the continuous version of (2) we obtain the $N$-involutive systems $E_1, \ldots, E_n$ of CS (13)

$$E_k = -p_k^2 - \langle p, q\rangle q_k^2 + \lambda_k p_k q_k$$
$$- \sum_{l=1, l\neq k} \frac{(p_l q_k - p_k q_l)^2}{\lambda_l - \lambda_k}. \tag{30}$$

Since for the $H$ given by (14) we have $H = \frac{1}{2}\sum_{j=1}^N E_j$, we obtain the following theorem.

*Theorem* 5. The finite-dimensional Hamiltonian system $(H)$ defined by (13) for the c-KdV hierarchy is completely integrable in the Liouville sense, and its independent $N$-involutive systems are $\{E_k\}_{k=1}^N$ defined by (30).

## 5. Concluding remarks and discussions

In the present paper, we have shown that, starting from the Lax matrix (2), two very different finite-dimensional integrable systems (one is DS (5), the other CS (13)) possess the same kind of non-dynamical $r$-matrices. This is surprising and interesting. Are there, for other various finite-dimensional integrable systems (including discrete and continuous integrable systems), also two (or at least two) such systems like DS (5) and CS (13) that possess the same $r$-matrix (it would be best if the $r$-matrix is non-dynamical)? Besides, can we regard the discrete system as an exact discretization of some flow in the corresponding family of continuous systems? This problem is still open.

The results on the Toda symplectic mapping in this paper give a definite solution for the open problem (see also Section 1) stated by Ragnisco in Ref. [11]. On the other hand, we have found some other finite-dimensional integrable systems which have a dynamical $r$-matrix [18]. Do they have a non-dynamical $r$-matrix? Furthermore, are there any gauge transformations to relate corresponding Lax operators? To the authors' knowledge, these problems do not seem to have any solutions.

The Lax matrix (2) plays a key role in this paper. Beginning with it, we first obtained the discrete integrable system (5) and the continuous integrable system (13) through introducing the two auxiliary $2\times 2$ matrices $M_n$, $M$; then we constructed the non-dynamical $r$-matrix formula (20) for two different integrable systems. Finally, by the use of the determinant of $L_n$ and $L$ (the forms of $L_n$ and $L$ are the same), we have established two involutive sets $\{E_{ni}\}_{i=1}^N$ for DS (5), and $\{E_k\}_{k=1}^N$ for CS (13), which assure the integrability of (5) and (13). In addition, using the separation of variables [17], we have found $N$ pairs of Darboux canonical coordinates $\pi_i$, $\mu_i$ and thus separated the variables of DS (5) and CS (13). Eventually we managed to develop an approach from the nonlinearized method to obtain the exact expressions of the algebraic geometry solution for the soliton equationn (19). This result will be published elsewhere.

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## References

[1] C.W. Cao, Chin. Quart. J. Math. 3 (1988) 90; Sci. Sin. A 7 (1989) 701;
C.W. Cao and X.G. Geng, J. Phys. A 23 (1990) 4117.
[2] Y.B. Zeng and Y.S. Li, J. Math. Phys. 30 (1989) 1679.
[3] M. Antonowicz and S. Rauch-Wojciechowski, Phys. Lett. A 147 (1990) 455.
[4] C.W. Cao, Acta Math. Sin. New Ser. 7 (1991) 216; Phys. Lett. A 184 (1994) 333.
[5] X.G. Geng, Acta Math. Sin. New Ser. 8 (1992) 78.
[6] Z.J. Qiao, J. Phys. A 26 (1993) 4407.
[7] G.P.W. Quispel, J.A.G. Roberts and G.J. Thompson, Phys. Lett. A 126 (1988) 419;
H.W. Capel, F.W. Nijhoff and V.G. Papageorgou, Phys. Lett. A 147 (1990) 106.
[8] M. Bruschi, O. Ragnisco, P.M. Santini and G.Z. Tu, Physica D 49 (1991) 273.
[9] O. Ragnisco, Phys. Lett. A 167 (1992) 165.
[10] O. Ragnisco, C.W. Cao and Y.T. Wu, J. Phys. A 28 (1995) 573.
[11] O. Ragnisco, Phys. Lett. A 198 (1995) 295.
[12] D. Levi, A. Sym and S. Rauch-Wojciechowski, J. Phys. A 16 (1983) 2423.
[13] V.I. Arnold, Mathematical Methods of Classical Mechanics (Springer, Berlin, 1978).
[14] L.D. Faddeev and L.A. Takhtajan, Hamiltonian Methods in the Theory of Solitons (Springer, Berlin, 1986).
[15] J.C. Eilbeck, V.Z. Eznol'skii, V.B. Kuznetzov and A.V. Tsinganov, J. Phys. A 27 (1994) 567.
[16] O. Babelon and C.M. Viallet, Phys. Lett. B 237 (1990) 411.
[17] E.K. Sklyanin, Prog. Theor. Phys. Suppl. 118 (1995) 35.
[18] R.G. Zhou, Phys. Lett. A 220 (1996) 320;
Y.B. Zeng and J. Hietarinta, J. Phys. A 29 (1996) 5241.
[19] Z.J. Qiao, Algebraic geometry solutions and $r$-matrices for the integrable simplectic mapping, 1996, preprint;
R.G. Zhou, finite-band solution of Jaulent-Miodek equation, to appear in J. Math. Phys. (1997).