Pacific Journal of Applied Mathematics
Volume 1, Number 1, pp. 105–121

ISSN PJAM

CUSP AND SMOOTH SOLITONS OF THE CAMASSA-HOLM EQUATION UNDER AN INHOMOGENEOUS BOUNDARY CONDITION

Guoping Zhang[1,2,*]***, Zhijun Qiao***[1†]***and Fengshan Liu***[2 ‡]
[1]Department of Mathematics, The University of Texas-Pan American
1201 W University Drive, Edinburg, TX 78541
[2]Applied Mathematics Research Center, Delaware State University
1200 North Dupont Highway, Dover, DE 19901

Abstract

This paper is contributed to explore all possible single peak soliton solutions for the Camassa-Holm (CH) equation $m_t + m_x u + 2mu_x = 0,\ m = u - u_{xx}$ under the boundary condition $u \to A$ (A is a constant) as $x \to \pm\infty$. Our procedure shows that the CH equation *either* has cusp soliton and smooth soliton solutions only under the inhomogeneous boundary condition $\lim_{|x|\to\infty} u = A \neq 0$, *or* possesses the regular peakon solutions $ce^{-|x-ct|}$ (c is the wave speed) only when $\lim_{|x|\to\infty} u = 0$ (see Theorem 4.1). We also prove that the constructed cusp soliton and smooth soliton are weak solutions in distribution sense. Moreover we present new cusp soliton and smooth soliton solutions in an explicit form. Asymptotic analysis and numerical simulations are provided for smooth solitons and cusp solitons of the CH equation.

2000 Mathematics Subject Classification: 35Q35, 35Q58 and 45G15.

Keywords: Camassa-Holm equation, traveling wave, smooth soliton, cusp soliton, inhomogeneous boundary condition, weak solution, asymptotic analysis

1. Introduction

The Camassa-Holm (CH) equation [4] is written in the following form

$$m_t + m_x u + 2mu_x = 0,\ m = u - \alpha^2 u_{xx}, \tag{1.1}$$

which has excited much interest in recent years. Here $u = u(x,t)$ represents the horizontal component of the fluid velocity, and $m = u - \alpha^2 u_{xx}$ is the momentum variable with the parameter $\alpha > 0$ introducing nonlocality. The subscripts x, t of u denote the partial derivatives

*E-mail address: gzhang@utpa.edu
†E-mail address: qiao@utpa.edu
‡E-mail address: fliu@desu.edu

of the function u w.r.t. x,t. For example, $u_t = \partial u/\partial t$, $u_{xxt} = \partial^3 u/\partial^2 x\partial t$. Similar notations will be used frequently later in this paper. The existence of the CH equation was implied by the work of Fokas and Fuchssteiner (1981) on hereditary symmetries [11]. It came to be remarkable in the work of Camassa and Holm (1993) where a new type of soliton solution (called peakon), was described [4]. A peakon is a weak solution with non-smooth property at some points.

As it was shown by Camassa and Holm, equation (1.1) describes the unidirectional propagation of two dimensional waves in shallow water over a flat bottom. The solitary waves of equation (1.1) regain their shape and speed after interacting nonlinearly with other solitary waves. A discussion of the integrability of the CH equation and a method of solution can be found in Camassa and Holm (1993) [4], and more mathematical detail is given in Beals et al. (1998) [2]. A further exploration, opening the way to the construction of solutions, is given by Contantin, Echer, and McKean (1998,1999) [5, 6, 7], Alber et al (2001) [1], Johnson (2002) [13], and Qiao (2003) [15]. Thereafter, Qiao [16] dealt with an extension version of the CH equation - the DP equation [10].

Because the parameter α can be scaled to unity, without any loss of generality, we set $\alpha = 1$ and consider the standard CH equation

$$m_t + m_x u + 2mu_x = 0,\ m = u - u_{xx},\ x \in \mathbb{R}. \tag{1.2}$$

In literature, several authors [3, 9, 14] studied the traveling wave solutions of CH equation decaying at both infinities by using the asymptotic analysis theory. Most solutions were given in an implicit form.

The present paper provides an approach to construct explicit solitary wave solutions of the CH equation (1.2) under inhomogeneous boundary condition. We give new solutions of the CH equation through setting the traveling wave solution under the only boundary condition $u \to A$ (A is a constant) as $x \to \pm\infty$. All possible exact single peak soliton solutions of the CH equation (1.2) are obtained both in explicit and in implicit form, particularly regular peakon solutions of the CH equation correspond to the homogeneous boundary condition $A = 0$. For the inhomogeneous boundary condition $A \neq 0$ both smooth solitons and cuspons are obtained in our paper. We will analyze in detail the cases of exact solutions, and classify the cases we obtain an explicit solutions and the cases we obtain implicit solutions showing in numeric graph. Our main results will be summarized in Theorem 4.1 after introducing some notations and definitions.

2. Traveling Wave Setting

Let us consider the traveling wave solution of the CH equation (1.2) through a generic setting $u(x,t) = U(x-ct)$, where c is the wave speed. Let $\xi = x - ct$, then $u(x,t) = U(\xi)$. Substituting it into the CH equation (1.2) yields

$$(U-c)(U-U'')' + 2U'(U-U'') = 0, \tag{2.1}$$

where $U' = U_\xi$, $U'' = U_{\xi\xi}$, $U''' = U_{\xi\xi\xi}$.

If $U - U'' = 0$, then equation (2.1) has general solutions of $U(\xi) = c_1 e^{\xi} + c_2 e^{-\xi}$ with any real constants c_1, c_2. Of course, they are the solutions of the CH equation (1.2). This result is not so interesting to us.

On the other hand, the CH equation has the peakon solution [4] $u(x,t) = U(\xi) = ce^{-|x-ct-\xi_0|}$ $(\xi_0 = x_0 - ct_0)$ with the following properties

$$U(\xi_0) = c, \qquad U(\pm\infty) = 0, \qquad U'(\xi_0-) = c, \qquad U'(\xi_0+) = -c, \tag{2.2}$$

where $U'(\xi_0-)$ and $U'(\xi_0+)$ represent the left-derivative and the right-derivative at ξ_0, respectively.

Let us now assume that U is neither a constant function nor satisfies $U - U'' = 0$. Then equation (2.1) can be changed to

$$\frac{(U-U'')'}{U-U''} = \frac{2U'}{c-U}. \tag{2.3}$$

Taking the integration on both sides we obtain

$$U'^2 = U^2 + \frac{2C_2}{c-U} + C_1, \tag{2.4}$$

where $C_1, C_2 \in \mathbb{R}$ are two integration constants. Let us solve equation (2.4) with the following boundary condition

$$\lim_{\xi\to\pm\infty} U = A, \tag{2.5}$$

thus the ODE (2.4) becomes

$$U'^2 = (U-A)^2 \frac{U-c+2A}{U-c}. \tag{2.6}$$

The fact that both sides of (2.6) are nonnegative implies

$$\frac{U-c+2A}{U-c} \geq 0. \tag{2.7}$$

Since we only care about the case $A \neq 0$ and $A \neq c$, we introduce the ratio $\alpha = c/A$. After applying the boundary condition (2.5), we obtain the following lemma.

Lemma 2.1. *Let U be a solution of* (2.6) *with boundary condition* (2.5), *then*

$$\alpha < 1 \quad \text{or} \quad \alpha \geq 3.$$

3. Smooth Solution and Weak Solution

From now on we focus on seeking single soliton solutions for CH equation (1.2). Let $C^k(\Omega)$ denote the set of all k times continuously differentiable functions on the open set Ω. We denote $L^p_{loc}(\mathbb{R})$ be the set of all functions whose restriction on any compact subset is L^p integrable. $H^1_{loc}(\mathbb{R}) = \{u \in L^2_{loc}(\mathbb{R}) : u' \in L^2_{loc}(\mathbb{R})\}$.

Definition 3.1. *A function $u(x,t) = U(x-ct)$ is said to be a single peak soliton solution for CH equation (1.2) if U satisfies the following conditions*
(C1) $U(\xi)$ is continuous on $\mathbb{R}$ and has a unique peak point, denoted by ξ_0, where $U(\xi)$ attains its local maximum or minimum value;
(C2) $U(\xi)$ is $C^3(\mathbb{R}-\{\xi_0\})$ and satisfies the equation (2.1) on $\mathbb{R}-\{\xi_0\}$;
(C3) $U(\xi)$ satisfies the boundary condition (2.5).

Without losing the generality we assume that $\xi_0 = 0$.

Lemma 3.2. *If $u(x,t) = U(x-ct)$ is a single peak soliton solution for CH equation (1.2) with the only peak point 0, then $U(0) = c$ or $U(0) = c-2A$. Moreover, we have*
(i) if $U(0) = c-2A$, then $U(\xi) \in C^\infty(\mathbb{R})$, in this case u is said to be a smooth soliton solution for (CH) equation (1.2);
(ii) if $U(0) = c$, then U has the following asymptotic behavior

$$U(\xi) - c = \lambda \xi^{2/3} + O(\xi^{4/3}), \quad \xi \to 0;$$

$$U'(\xi) = \frac{2}{3}\lambda \xi^{-1/3} + O(\xi^{1/3}), \quad \xi \to 0;$$

where $\lambda = (\frac{9(c-A)^2 A}{2})^{1/3}$. Thus $U(\xi) \in H^1_{loc}(\mathbb{R})$.

Proof. If $U(0) \neq c$, by virtue of (2.6) we know $U'(0)$ exists. Thus $U'(0) = 0$ since 0 is a peak point. By (2.6) again we obtain $U(0) = c-2A$ since $U(0) = A$ contradicts the fact that 0 is the unique peak point.
(i) If $U(0) = c-2A$, then $U(\xi) \neq c$ for any $\xi \in \mathbb{R}$ since $U(\xi) \in C^3(\mathbb{R}-\{0\})$. By differentiating both sides of (2.6) we conclude that $U \in C^\infty(\mathbb{R})$.
(ii) It follows from the standard asymptotic analysis. We refer readers to [14]. For readers' convenience we give the simple proof here.
By (2.6) and phase analysis, we know that

$$U' = -sign(\xi)(U-A)\sqrt{\frac{U-c+2A}{U-c}} \tag{3.1}$$

Since $U(0) = c$ is the local maximum or minimum value, we distinguish two cases.
(1) c is minimum value, thus as $\xi \to 0$, $U(\xi) > c$. Let $h(U) = \frac{1}{(A-U)\sqrt{U-c+2A}}$, then $h(c) = \frac{1}{(A-c)\sqrt{2A}}$ and (3.1) becomes

$$\int h(U)\sqrt{U-c}\,dU = \int sign(\xi)\,d\xi. \tag{3.2}$$

Inserting $h(U) = h(c) + O(U-c)$ into (3.2) and using the initial condition $U(0) = c$ we obtain

$$\frac{2}{3}h(c)(U-c)^{3/2}(1+O(U-c)) = |\xi| \tag{3.3}$$

thus

$$U-c = (\frac{3}{2h(c)})^{2/3}\xi^{2/3}(1+O(U-c))^{-2/3} = (\frac{3}{2h(c)})^{2/3}\xi^{2/3}(1+O(U-c)) \tag{3.4}$$

which implies $U-c=O(\xi^{2/3})$. Therefore we obtain

$$U-c=(\frac{3}{2h(c)})^{2/3}\xi^{2/3}+O(\xi^{4/3})=\lambda\xi^{2/3}+O(\xi^{4/3}). \quad (3.5)$$

(2) c is maximum value, thus as $\xi\to 0$, $U(\xi)<c$. Let $h(U)=\frac{1}{(A-U)\sqrt{c-2A-U}}$, then $h(c)=\frac{1}{(A-c)\sqrt{-2A}}$ and (3.1) becomes

$$\int h(U)\sqrt{c-U}dU=\int sign(\xi)d\xi. \quad (3.6)$$

By a completely similar analysis we obtain

$$U-c=-(\frac{-3}{2h(c)})^{2/3}\xi^{2/3}+O(\xi^{4/3})=\lambda\xi^{2/3}+O(\xi^{4/3}). \quad (3.7)$$

By the equation (3.1) we may obtain the asymptotic estimate for U' easily.
$U\in H^1_{loc}(\mathbb{R})$ follows from the asymptotic estimates.∎

Proposition 3.3. *If $u(x,t)=U(x-ct)$ is a single peak soliton solution for (CH) equation (1.2), then U must be a weak solution in the distributional sense of the equation* (2.6). *In this sense we say u is a weak solution for (CH) equation (1.2).*

Proof. If $U(0)=c-2A$, by Lemma 3.2 we know that U is a smooth solution, thus it is a weak solution in the distributional sense.
If $U(0)=c$, then $U\in H^1_{loc}$ which implies $U'^2\in L^1_{loc}$, thus the left hand side of (2.6) does make sense. Notice that U is bounded, we know that the right hand side of (2.6) is also in L^1_{loc} due to the asymptotic estimate of U in Lemma 3.2. Thus we may define the distribution function $L(U)=U'^2-(U-A)^2\frac{U-c+2A}{U-c}$. By the definition condition (C2)we know that suppL$(U)\subset\{0\}$. Thus $L(U)$ must be a linear combination of Dirac function $\delta(\xi)$ and its derivatives. However the previous analysis shows that $L(U)\in L^1_{loc}(\mathbb{R})$. Therefore we must have $L(U)=0$.∎

For a traveling wave $u(x,t)=U(x-ct)$, Lenells rewrote (2.1) into the following equation

$$U'^2+3U^2-2cU=((U-c)^2)''+a \quad (3.8)$$

where a is integration constant. By using (3.8) Lenells defined that U is a traveling wave of the CH equation (1.2) if U satisfies (3.8) in distribution sense (see definition 1 in [14]).

Proposition 3.4. *If $u(x,t)=U(x-ct)$ is a single peak soliton solution for (CH) equation (1.2), then U must be a a weak solution of* (3.8) *in distribution sense and thus it is a traveling wave of CH equation according to Lenells' definition in [14].*

Proof. By virtue of Lemma 3.2 we know that $U \in H^1_{loc}$, the both sides of the equation (3.8) make sense and belong to L^1_{loc}. By the definition of single peak soliton, the equation (3.8) is satisfied for any nonzero point. By a similar argument with the proof of the proposition Proposition 3.3 we can conclude that U is a weak solution of (3.8) in distribution sense. ∎

In [8] Constantin and Strauss considered the following equation

$$\tilde{m}_t + \tilde{m}_x\tilde{u} + 2\tilde{m}\tilde{u}_x + 2k\tilde{u}_x = 0,\ \tilde{m} = \tilde{u} - \tilde{u}_{xx},\ x \in \mathbb{R}. \tag{3.9}$$

It is easy to check that the solution $\tilde{u}(t,x)$ of (3.9) with the homogeneous boundary condition $\lim_{|x|\to\infty} u = 0$ corresponds to the solution $u(t,x)$ of the solution of (1.2) with the inhomogeneous boundary condition $\lim_{|x|\to\infty} u = k$ under the following tranformation

$$\tilde{u}(t,x) = u(t,x+kt) - k; \qquad \tilde{m}(t,x) = m(t,x+kt) - k.$$

Constantin and Strauss recast equation (3.9) in the following nonlocal conservation law form

$$u_t + uu_x + \partial_x(1-\partial_x^2)^{-1}(u^2 + \frac{1}{2}u_x^2) = 0 \tag{3.10}$$

and defined that $u(x,t) = U(x-ct)$ with $U \in H^1_{loc}$ is a traveling wave of CH equation (1.2) if U satisfies (3.10) in distribution sense (see the definition 1 in [8], the other conditions in this definition are satisfied automatically for traveling wave).

In [14] Lenells indicated that the equations (3.8) and (3.10) are equivalent for traveling wave $u(x,t) = U(x-ct)$ with $U \in H^1_{loc}$. Therefore we obtain

Proposition 3.5. *If $u(x,t) = U(x-ct)$ is a single peak soliton solution for (CH) equation (1.2), then U must be a a weak solution of* (3.10) *in distribution sense and thus it is a traveling wave solution of CH equation according to Constantin and Strauss' definition in [8].*

4. Construction of Single Peak Soliton Solutions

Now we construct the single peak soliton solutions for (2.6). We rewrite (2.6) as

$$U' = -(U-A)\sqrt{\frac{U-c+2A}{U-c}}sign(\xi). \tag{4.1}$$

By virtue of Lemma 3.2 we only need to seek the single peak soliton solution satisfying the initial condition

$$U(0) = c, \quad \text{or} \quad U(0) = c - 2A. \tag{4.2}$$

Let us assume that $A \neq 0$ (if $A = 0$, the traveling wave solutions are peakon solutions, which were already dealt with by Camassa and Holm [4]). Let $X = \sqrt{\frac{U-c}{U-c+2A}}$, then

$$U = (c-2A) + \frac{2A}{1-X^2}, \tag{4.3}$$

$$dU = \frac{4AX}{(X^2-1)^2}dX, \tag{4.4}$$

and equation (4.1) is converted to

$$\frac{4AX^2}{(X^2-1)[(3A-c)X^2-(A-c)]}dX = d\xi sign(\xi). \tag{4.5}$$

By virtue of Lemma 2.1 we only need to discuss the following three cases.

Case I: $\alpha < 1$ **(i.e.** $A > 0, c < A$ **or** $A < 0, c > A$)
1. If $A > 0, c < A$, then by (2.7), we have $U > c$ and $0 < X < 1$. Let

$$a = \sqrt{\frac{A-c}{3A-c}},$$

then $0 < a < 1$, and equation (4.5) is changed to

$$f(X)dX \equiv \frac{4A}{(A-c)}\frac{a^2X^2}{(X^2-1)(X^2-a^2)}dX = d\xi sign(\xi). \tag{4.6}$$

Integrating this equation we obtain the following implicit solutions

$$F(X) \equiv a \ln\left|\frac{X+a}{X-a}\right| - \ln\frac{1+X}{1-X} = |\xi| + K, \;\; K = F(X(0)). \tag{4.7}$$

Since $F'(X) = f(X)$, we know that $F(X)$ strictly increases on the interval $0 < X < a$ with $F(0) = 0, F(a-) = \infty$, and strictly decreases on the interval $a < X < 1$ with $F(1-) = -\infty, F(a+) = \infty$.

Let

$$F_1(X) = F|_{(0,a)}(X) = a \ln\frac{a+X}{a-X} - \ln\frac{1+X}{1-X}, \tag{4.8}$$

and

$$F_2(X) = F|_{(a,1)}(X) = a \ln\frac{X+a}{X-a} - \ln\frac{1+X}{1-X}. \tag{4.9}$$

Then $F_1(X),\ F_2(X)$ are two strictly monotone functions, hence their inverses exist on the interval $(0,a)$ and $(a,1)$, respectively. So the equation (4.7) can be solved uniquely for X on the interval $(0,a)$ and $(a,1)$. Therefore, we can define their inverses as

$$X_1(\xi) = F_1^{-1}(|\xi| + K_1), \quad \text{and} \quad X_2(\xi) = F_2^{-1}(|\xi| + K_2),$$

which generate the following candidate solutions

$$U_1(\xi) = (c-2A) + \frac{2A}{1-(F_1^{-1}(|\xi|+K_1))^2}, \qquad K_1 \geq 0; \tag{4.10}$$

$$U_2(\xi) = (c-2A) + \frac{2A}{1-(F_2^{-1}(|\xi|+K_2))^2}, \qquad K_2 \in \mathbb{R}. \tag{4.11}$$

It is easy to check:

- $U_1(0) = (c-2A) + \frac{2A}{1-(F_1^{-1}(K_1))^2} \in [c, A), \quad U_1(\pm\infty) = A$, and U_1 strictly increases on $[0,\infty)$ and strictly decreases on $(-\infty, 0]$;

- $U_2(0) = (c-2A) + \frac{2A}{1-(F_2^{-1}(K_2))^2} \in (A,\infty)$, $U_2(\pm\infty) = A$, and U_2 strictly decreases on $[0,\infty)$ and strictly increases on $(-\infty,0]$.

By virtue of Lemma 3.2 we know the only possible single peak soliton solution for (2.1) is $U_1(\xi)$ with $U_1(0) = c$ (correspondingly, $K_1 = 0$), defined by

$$U_1(\xi) = (c-2A) + \frac{2A}{1-(F_1^{-1}(|\xi|))^2}. \tag{4.12}$$

2. If $A < 0, c > A$, then by (2.7), we have $U < c$, $0 < X < 1$ and $a < 1$. This case is completely similar to the case of $A > 0, c < A$. Thus we can conclude that the function $U_1(\xi)$, defined by equation (4.12), is the uinque single peak soliton solution of (2.1) for the case of $A < 0, c > A$.

Case II: $\alpha > 3$ **(i.e.** $c > 3A > 0$ **or** $c < 3A < 0$)
1. If $c > 3A > 0$, then by (2.7) we know that

$$U < c-2A, \qquad X > 1, \qquad a > 1.$$

On the interval $a < X < \infty$, $U > A$ and equation (2.6) is equivalent to

$$\frac{1}{U-A}\sqrt{\frac{c-U}{c-2A-U}}dU = -sign(\xi)d\xi. \tag{4.13}$$

In a similar way, we can obtain the implicit solution of the ODE (2.6)

$$F_3(X) \equiv a\ln\frac{X+a}{X-a} - \ln\frac{X+1}{X-1} = |\xi| + K_3, \;\; K_3 = F_3(X(0))). \tag{4.14}$$

By $F_3'(X) = f(X)$, we know that $F_3(X)$ is strictly decreasing on the interval $a < X < \infty$ with $F_3(a+) = \infty, F_3(\infty) = 0$.

On the interval $1 < X < a$, $U < A$ and equation (2.6) is equivalent to

$$\frac{1}{U-A}\sqrt{\frac{c-U}{c-2A-U}}dU = -sign(\xi)d\xi. \tag{4.15}$$

In this case the implicit solution of the ODE (2.6) is

$$F_4(X) \equiv a\ln\frac{a+X}{a-X} - \ln\frac{X+1}{X-1} = |\xi| + K_4, \;\; K_4 = F_4(X(0))). \tag{4.16}$$

By $F_4'(X) = f(X)$, we know that $F_4(X)$ is strictly increasing on the interval $1 < X < a$ with $F_4(a-) = \infty, F(1+) = -\infty$.

Therefore, $F_3(X), F_4(X)$ have inverses on intervals $a < X < \infty$, and $1 < X < a$, respectively. By a similar analysis we obtain the following candidate solutions

$$U_3(\xi) = (c-2A) + \frac{2A}{1-(F_3^{-1}(|\xi|+K_3))^2}, \qquad K_3 \geq 0; \tag{4.17}$$

and

$$U_4(\xi) = (c-2A) + \frac{2A}{1-(F_4^{-1}(|\xi|+K_4))^2}, \qquad K_4 \in \mathbb{R}; \tag{4.18}$$

It is easy to check:

- $U_3(0) = (c-2A) + \frac{2A}{1-(F_3^{-1}(K_3))^2} \in (A, c-2A]$, $U_3(\pm\infty) = A$, and U_3 strictly decreases on $[0,\infty)$ and strictly increases on $(-\infty, 0]$;
- $U_4(0) = (c-2A) + \frac{2A}{1-(F_4^{-1}(K_4))^2} \in (-\infty, A)$, $U_4(\pm\infty) = A$, and U_4 strictly increases on $[0,\infty)$ and strictly decreases on $(-\infty, 0]$.

By virtue of Lemma 3.2 we know the only possible single peak soliton solution for (2.1) is $U_3(\xi)$ with $U_3(0) = c-2A$ (correspondingly, $K_3 = 0$), defined by

$$U_3(\xi) = (c-2A) + \frac{2A}{1-(F_3^{-1}(|\xi|))^2}. \tag{4.19}$$

2. If $c < 3A < 0$,then by (2.7), we have $U > c-2A$, $X > 1$ and $a > 1$. This case is completely analogous to the case of $A > 0, c > 3A$. Thus we can conclude that the function $U_3(\xi)$, defined by equation (4.19), is the uinque single peak soliton solution of (2.1) for the case of $A < 0, c < 3A$.

Case III: $\alpha = 3$ **(i.e.** $c = 3A$)
1. If $c = 3A > 0$, then equations (4.3) and (4.5) become

$$U = A + \frac{2A}{1-X^2} \tag{4.20}$$

and

$$\frac{2X^2}{X^2-1}dX = sign(\xi)d\xi. \tag{4.21}$$

Taking the integration, we have

$$F_5(X) \equiv 2X - \ln\left|\frac{X+1}{X-1}\right| = |\xi| + K_5, \quad K_5 = F_5(X(0)). \tag{4.22}$$

Noticing that $X = \sqrt{\frac{3A-U}{A-U}}$ and $A > 0$ imply

$$U < A, \qquad X > 1,$$

and $\operatorname{arctanh} y = \frac{1}{2}\ln\frac{1+y}{1-y}$, $(0 < y < 1)$, therefore, we can reduce (4.22) to the following implicit solution

$$F_5(X) = 2X - 2\operatorname{arctanh}\frac{1}{X} = |\xi| + K_5, \;\; \xi = x - 3At. \tag{4.23}$$

Thus we obtain the following candidate solution

$$U_5(\xi) = A + \frac{2A}{1-(F_5^{-1}(|\xi|+K_5))^2}, \qquad K_5 \in \mathbb{R}; \tag{4.24}$$

It is easy to check that
$U_5(0) = (c-2A) + \frac{2A}{1-(F_5^{-1}(K_5))^2} \in (-\infty, A)$, $U_5(\pm\infty) = A$, and U_5 strictly increases on $[0,\infty)$ and strictly decreases on $(-\infty, 0]$. Since neither c nor $c-2A$ belong to the range of U_5, by virtue of Lemma 3.2 we know there is no single peak soliton solution for this case.
2. If $c = 3A < 0$, similarly we can conclude that there is no single peak soliton solution.

Therefore we get our main theorem.

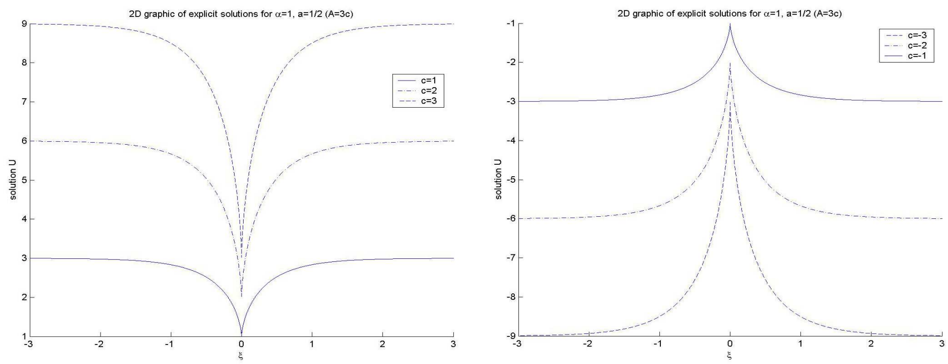

Figure 1. 2D graphics for single peak soliton solutions with $a = 1/2$.

Theorem 4.1. *Assume that the single peak soliton solution* $u(x,t) = U(x-ct)$ *(*0 *be the unique peak point of* U*) of the CH equation (1.2) satisfies the boundary condition* (2.5)*. Then we have*

1. *if* $A = 0$*, the only single peak soliton solution* $u(x,t)$ *is the following peakon solution*

$$u(x,t) = U(x-ct) = ce^{-|x-ct|},$$

with the properties:

$$U(0) = c,\ U(\pm\infty) = 0,\ U'(0+) = -c,\ U'(0-) = c;$$

Let $\alpha = c/A$ *if* $A \neq 0$*.*

2. *if* $1 \leq \alpha \leq 3$*, there is no single peak soliton solution for the CH equation (1.2) ;*

3. *if* $\alpha < 1$*, the only single peak soliton solution (see Figures 1, 2 and 3) of the CH equation (1.2) can be expressed as*

$$u(x,t) = U(x-ct) = (c-2A) + \frac{2A}{1-(F_1^{-1}(|x-ct|))^2},$$

with the properties:

$$U(0) = c,\ U(\pm\infty) = A,\ U'(0+) = sign(A)\infty,\ U'(0-) = -sign(A)\infty,$$

where F_1 *is defined by equation* (4.8)*; This is a cusp soliton solution.*

4. *if* $\alpha > 3$*, the only single peak soliton solution of the CH equation (1.2) is of the following form (see Figures 4, 5 and 6)*

$$u(x,t) = U(x-ct) = (c-2A) + \frac{2A}{1-(F_3^{-1}(|x-ct|))^2}$$

with the properties:

$$U(0) = c-2A, \qquad U(\pm\infty) = A, \qquad U'(0) = 0,$$

where F_3 *is defined by equation* (4.14)*. This is a smooth soliton solution.*

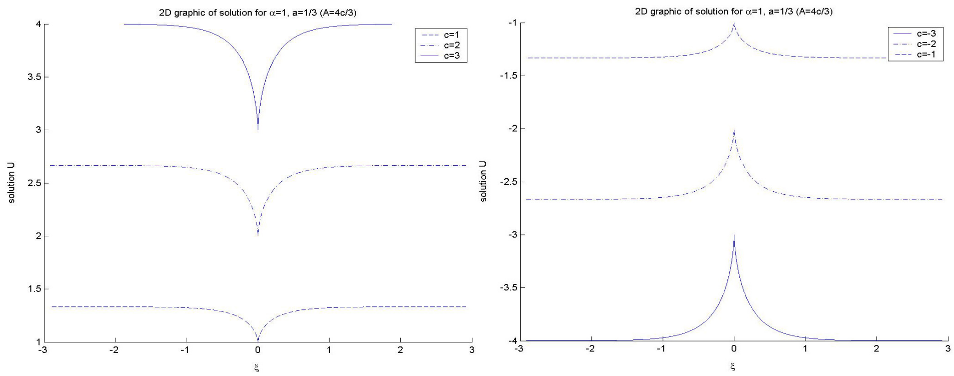

Figure 2. 2D graphics for single peak soliton solutions with $a = 1/3$.

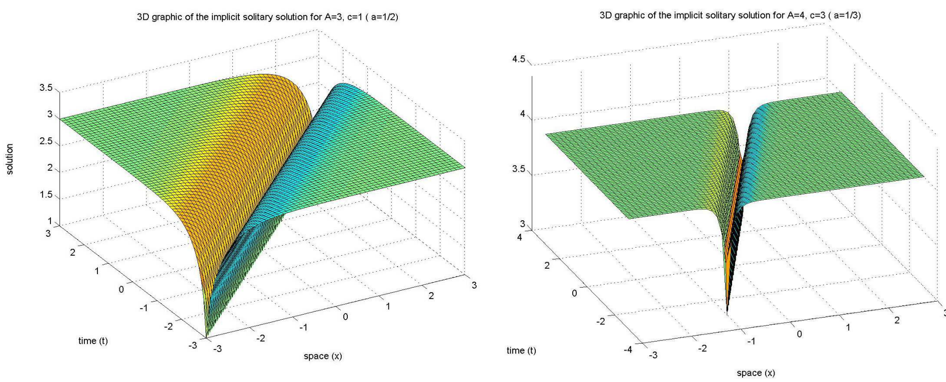

Figure 3. 3D images for the implicit single peak soliton solutions with $a = 1/2$ and $1/3$.

5. Explicit Single Peak Soliton Solutions

In this section, we discuss the exact single peak soliton solutions of the CH equation (1.2). However, F_1^{-1} and F_3^{-1} are given in an implicit form (see Theorem 4.1). In general, we can not get an explicit form from F_1^{-1} and F_3^{-1} because of an arbitrary constant a $(0 < a < 1)$ involved. But, for some special a's, we do have explicit solutions, which are discussed below.

Let us work on the inverse functions of F_1 and F_3. $F_1(X) = |\xi|$ implies that

$$\ln \frac{1+X}{1-X} + a \ln \frac{a-X}{a+X} = -|\xi|;\ 0 < X < a.$$

Taking the exponential operation on both sides gives

$$\frac{1+X}{1-X}\left(\frac{a-X}{a+X}\right)^a = e^{-|\xi|}.$$

Generally, this equation is hard to solve. However, if we select some specific constants $c,\ A$, we are able to express X in terms of ξ. Let us list these cases below.

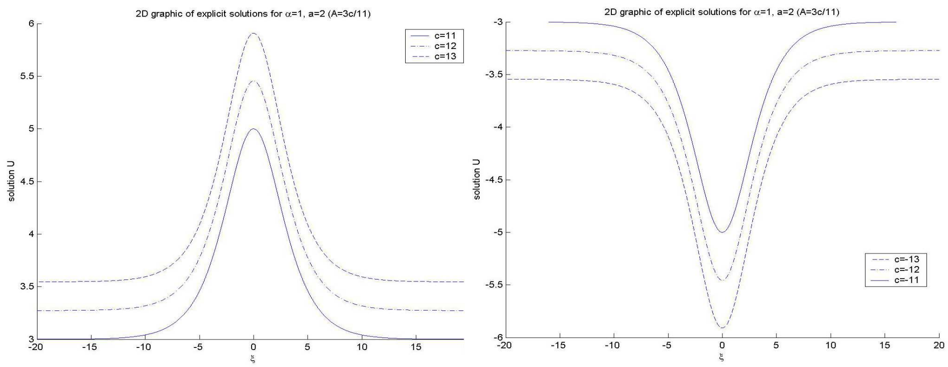

Figure 4. 2D graphics for single peak soliton solutions with $a = 2$.

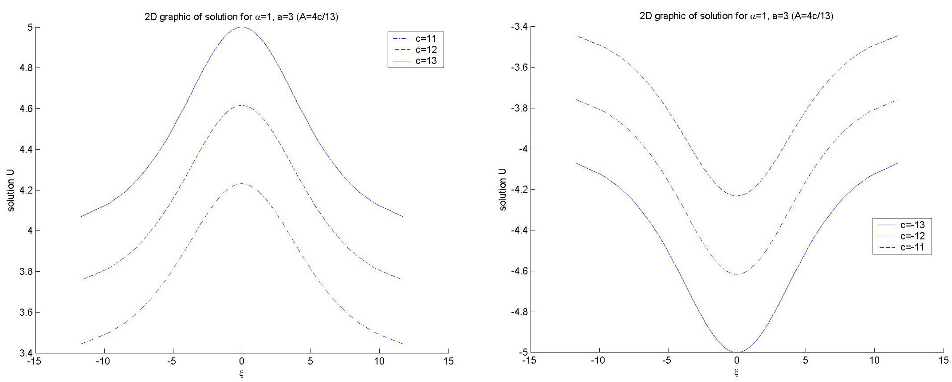

Figure 5. 2D graphics for single peak soliton solutions with $a = 3$.

Case $A = 3c$.
In this case, we have $a = \sqrt{\frac{A-c}{3A-c}} = \frac{1}{2}$ and

$$\left(\frac{1+X}{1-X}\right)^2 \frac{1-2X}{1+2X} = e^{-2|\xi|},$$

which can be reduced to

$$X^3 + 3bX^2 - b = 0, \tag{5.1}$$

where $b = \frac{1}{2}\tanh|\xi|$.

$$X = -b + b\left(\frac{1-\sqrt{1-4b^2}}{1+\sqrt{1-4b^2}}\right)^{1/3} + b\left(\frac{1+\sqrt{1-4b^2}}{1-\sqrt{1-4b^2}}\right)^{1/3}.$$

Substituting $b = \frac{1}{2}\tanh(|\xi|)$ into the above equation and computing it, we obtain a simplified form of X

$$X(\xi) = \frac{1}{2}\tanh|\xi| \left[\tanh^{2/3}\frac{|\xi|}{2} + \coth^{2/3}\frac{|\xi|}{2} - 1\right], \tag{5.2}$$

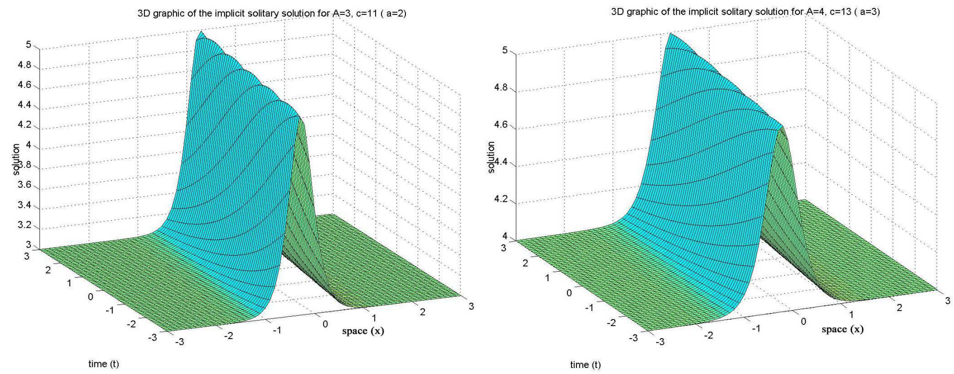

Figure 6. 3D images for the implicit single peak soliton solutions with $a = 2$ and $a = 3$.

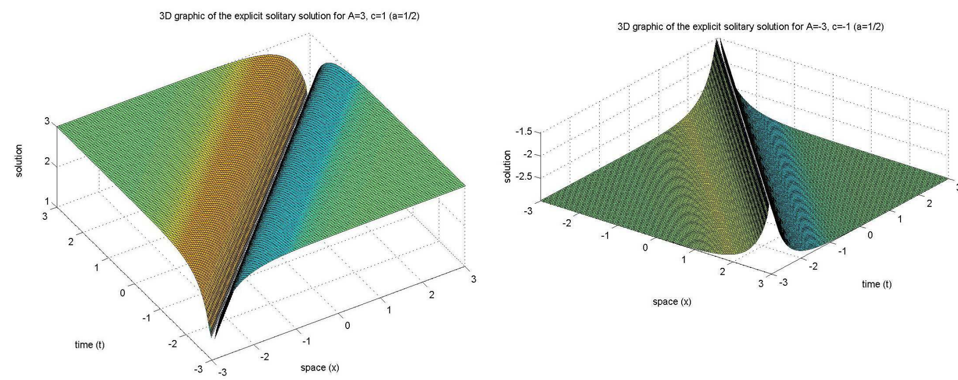

Figure 7. 3D images for the explicit single peak soliton solution with $a = 1/2$.

which gives a family of explicit solutions of equation (2.6)

$$U(\xi) = -5c + \frac{6c}{1 - X(\xi)^2} = cV_1(\xi), \tag{5.3}$$

where c is a wave speed being an arbitrary nonzero constant, and

$$V_1(\xi) = 1 + 6\frac{\left[\tanh^{2/3}\frac{|\xi|}{2} + \coth^{2/3}\frac{|\xi|}{2} - 1\right]^2}{4\coth^2|\xi| - \left[\tanh^{2/3}\frac{|\xi|}{2} + \coth^{2/3}\frac{|\xi|}{2} - 1\right]^2}$$

is the solution corresponding to $c = 1$ (See Figure 7).

Case $A = 3c/11$.

In this case $a = 2$, and from

$$F_3(X) = |\xi|,$$

we obtain

$$\frac{X+1}{X-1}\left(\frac{X-2}{X+2}\right)^2 = e^{-|\xi|}$$

which is able to be simplified to

$$X^3 - 3bX^2 + 4b = 0, \tag{5.4}$$

where $b = \coth\frac{|\xi|}{2}$.

An analogous procedure is applied to solve equation (5.4). On the interval $(2,\infty)$, we obtain the unique real root of equation (5.4)

$$X = b + b\Big(\frac{\sqrt{1-b^2}-1}{\sqrt{1-b^2}+1}\Big)^{1/3} + b\Big(\frac{\sqrt{1-b^2}+1}{\sqrt{1-b^2}-1}\Big)^{1/3}.$$

By the substitution of $b = \coth\frac{|\xi|}{2}$ into the above formula, X is simplified as follows

$$X(\xi) = \coth\frac{|\xi|}{2}\Big[1 + 2\mathrm{Re}\Big(\mathrm{sech}\frac{|\xi|}{2} + i\tanh\frac{|\xi|}{2}\Big)^{2/3}\Big]. \tag{5.5}$$

Noticing the identity $\tanh^2(x) + \mathrm{sech}^2(x) = 1$, we have the existence of $\theta(\xi)$ such that

$$\mathrm{sech}\frac{|\xi|}{2} + i\tanh\frac{|\xi|}{2} = e^{i\,\theta(\xi)}, \qquad 0 < \theta(\xi) < \pi/2,$$

which implies that

$$\mathrm{Re}\Big(\mathrm{sech}\frac{|\xi|}{2} + i\tanh\frac{|\xi|}{2}\Big)^{2/3} = \cos\frac{2}{3}\theta(\xi) > 0.$$

Therefore we can rewrite equation (5.5) as

$$\begin{aligned} X(\xi) &= \frac{1}{\sin\theta(\xi)}\Big[1 + 2\cos\frac{2}{3}\theta(\xi)\Big] \\ &= \coth\frac{|\xi|}{2}\Big[1 + 2\cos\Big(\frac{2}{3}\arccos\mathrm{sech}\frac{|\xi|}{2}\Big)\Big]. \end{aligned} \tag{5.6}$$

Apparently, $X(\xi) > \frac{2}{\sin\theta(\xi)} > 2$. Thus we obtain a family of explicit solutions of equation (2.6)

$$\begin{aligned} U(\xi) &= \frac{5c}{11} + \frac{6c}{11}\frac{1}{1 - X(\xi)^2} \\ &= \frac{c}{11}\Big(5 - 6\tanh^2\frac{\theta(\xi)}{3}\Big) \\ &\equiv cV_2(\xi), \end{aligned} \tag{5.7}$$

where the wave speed c can be an arbitrary nonzero constant, $X(\xi)$ is defined by equation (5.6), and

$$V_2(\xi) = \frac{\cos\Big(\frac{2}{3}\arccos\mathrm{sech}\frac{|\xi|}{2}\Big) - \frac{1}{11}}{\cos\Big(\frac{2}{3}\arccos\mathrm{sech}\frac{|\xi|}{2}\Big) + 1}$$

is the solution of equation (2.6) corresponding to $c = 11$ (See Figure 8).

The case of $a = 3$ ($A = 4c/13$) is complicated, but we can also obtain the explicit solutions through repeating the above procedures. Here, we ignore detailed computations, but give their graphs (See Figure 9).

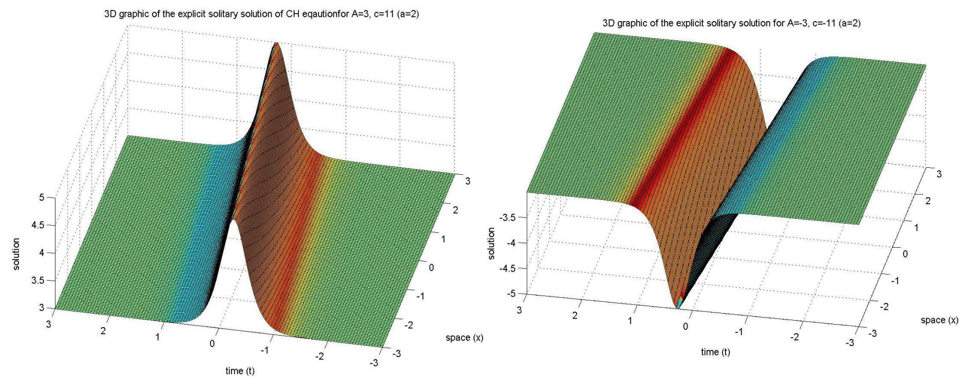

Figure 8. 3D images for the explicit single peak soliton solution with $a = 2$.

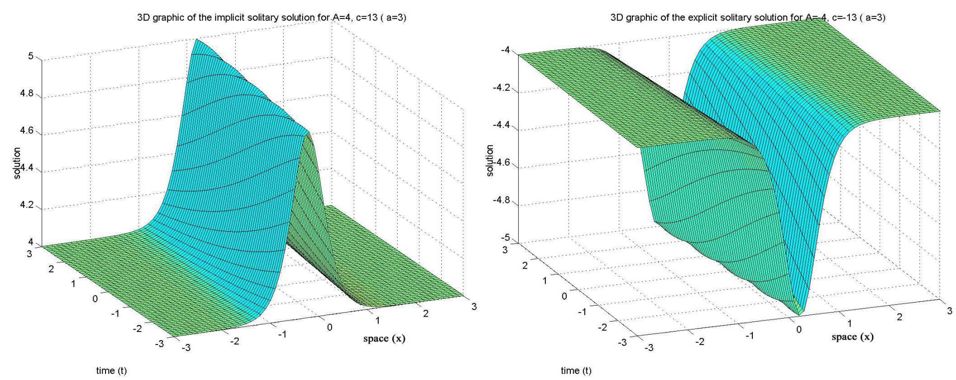

Figure 9. 3D images for the implicit single peak soliton solution with $a = 3$.

6. Conclusions

In this paper we provide an approach to obtain explicit solutions of the Camassa-Holm (CH) equation $m_t + m_x u + 2mu_x = 0,\ m = u - u_{xx}$ under the inhomogeneous boundary condition $u \to A$ (A is a constant) as $x \to \pm\infty$. Actually, this approach can be also applied to other types of nonlinear PDEs. What we are interested in is to find new solutions of nonlinear equations regardless of integrability. We are applying this method to b-equation: $m_t + m_x u + bmu_x = 0,\ m = u - u_{xx}$ and already got some new solutions, which we will do this in near future.

Another aspect of this method is compare with regular peakon solutions of the CH equation. Regular peakon solutions are continuous, but not smooth because left derivative equals 1 and right derivative equals -1 at peak points. The cusp soliton solutions we get in this paper are different from the regular peakons since booth left derivative and right derivative do not exist at peak points (see Theorem 4.1). In addition, under the inhomogeneous boundary condition $A \neq 0$, we obtain smooth soliton solutions for the CH equation (1.2) as well as new cusp soliton solutions in our paper. Mathematical analysis and numeric graphs are also provided for those smooth soliton and peakon solutions.

Acknowledgment

The authors thank the financial support of DoD through a grant DAAD19-03-1-0375. Qiao did his partial work during he visited the ICTP, Trieste, Italy, University of Kassel, Germany, and Delaware State University in the summer 2005. Qiao specially expresses his sincere thanks to Prof. Strampp (University of Kassel), Prof. Fengshan Liu (DSU), Prof. Xiquan Shi (DSU), and Dr. Guoping Zhang (DSU) for their friendly hospitality. Qiao's work was partially supported by the UTPA-FDP grant, and the AvH, Germany.

References

[1] Mark S. Alber, Roberto Camassa, Yuri N. Fedorov, Darryl D. Holm and Jerrold E. Marsden, The complex geometry of weak piecewise smooth solutions of integrable nonlinear PDE's of shallow water and Dym type, *Commun. Math. Phys*. **221**(2001), 197-227.

[2] R. Beals, D. Sattinger, and J. Szmigielski, Acoustic scattering and the extended Korteweg-de Vries hierarchy, *Adv. Math.* **140**(1998), 190-206.

[3] C. Cao, D. D. Holm and E.S. Titi, Traveling wave solutions for a class of one-dimensional nonlinear shallow water wave models, *Journal of Dynamics and Differential Equations*, **16**:1 (2004), 167–178.

[4] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* **71**(1993), 1661-1664.

[5] A. Constantin, On the inverse spectral problem for the Camassa-Holm equation, *J. Funct. Anal.* **155**(1998), 352-363.

[6] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.* **181**(1998), 229-243.

[7] A. Constantin and H P McKean, A shallow water equation on the circle, *Comm. Pure Appl. Math.* **52**(1999), 949-982.

[8] A. Constantin and W. A. Strauss, Stability of the Camassa-Holm Solitons, *J. Nonlinear Sci.* Vol. 12: pp. 415-422 (2002).

[9] H.-H. Dai and Y. Huo, Solitary Shock Waves and Other Travelling Waves in a General Compressible Hyperelastic Rod, Proceedings: Mathematical, *Physical and Engineering Sciences,* Vol. 456, No. 1994 (Feb. 8, 2000) , pp. 331-363.

[10] A. Degasperis and M. Procesi, Asymptotic integrability, in A. Degasperis and G. Gaeta (eds), *Symmetry and Perturbation Theory,* World Scientific, 1999, pp. 23-37.

[11] B. Fuchssteiner and A. S. Fokas, Symplectic structures, their Baecklund transformations and hereditaries, *Physica D* **4**(1981), 47-66.

[12] F. Gesztesy and H. Holden, Algebraic-geometric solutions of the Camassa-Holm hierarchy, *Rev. Mat. Iberoamericana* **19**(2003), 73-142.

[13] R. Johnson, Camassa-Holm, Korteweg-de Vries and related models for water waves, *J. Fluid Mech*. **455** (2002), 63-82.

[14] J. Lenells, Traveling wave solutions of the Camassa-Holm equation, *J. Differential Equations* **217** (2005) 393-430.

[15] Z. J. Qiao, The Camassa-Holm hierarchy, N-dimensional integrable systems, and algebro-geometric solution on a symplectic submanifold, *Communications in Mathematical Physics* **239**(2003), 309-341.

[16] Z. J. Qiao, Integrable hierarchy, 3×3 constrained systems, and parametric and stationary solutions, *Acta Applicandae Mathematicae* **83**(2004), 199-220.

[17] Z. J. Qiao, Generalized *r*-matrix structure and algebro-geometric solutions for integrable systems, *Rev. Math. Phys.* **13**(2001), 545-586.