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AN INTEGRABLE EQUATION WITH NONSMOOTH SOLITONS

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We present the bi-Hamiltonian structure and Lax pair of the equation $\rho_t = bu_x + (1/2)[(u^2 - u_x^2)\rho]_x$*, where* $\rho = u - u_{xx}$ *and* $b = \text{const}$*, which guarantees its integrability in the Lax pair sense. We study nonsmooth soliton solutions of this equation and show that under the vanishing boundary condition* $u \to 0$ *at the space and time infinities, the equation has both "W/M-shape" peaked soliton (peakon) and cusped soliton (cuspon) solutions.*

Keywords: integrable equation, Lax pair, peakon, cuspon

1. Introduction

Soliton theory and integrable systems play an important role in studying nonlinear wave equations. They have many significant applications in fluid mechanics, nonlinear optics, classical and quantum fields theories, etc. In particular, much attention has recently been given to integrable systems with nonsmooth solitons such as peakons and cuspons since the study of the well-known Camassa–Holm (CH) equation with peakon solutions [1]. Much progress in studying shallow water waves has recently been achieved [2]–[13].

Here, we study the bi-Hamiltonian structure and Lax pair of the equation

$$\rho_t = bu_x + \frac{1}{2}[(u^2 - u_x^2)\rho]_x, \qquad \rho = u - u_{xx}, \quad b = \text{const}, \quad x \in \mathbb{R}. \tag{1}$$

With some suitable rescaling, this equation can be a particular case of the generalizations of the modified Korteweg–de Vries (KdV) equation that was discussed in [14] and can also be derived by "reshuffling" the Hamiltonian operators of the bi-Hamiltonian structure of the modified KdV (mKdV) equation and adding one more linear term bu_x [15]. The equation without the linear term and its Lax pair were later constructed using the spectral problem technique in [16], where the M/W-shape soliton solution and a new type of cusp soliton solution were also first proposed in explicit formulas. The equation can be extended to an integrable hierarchy using a recursion operator and solving a crucial matrix equation [17]. Although Eq. (1) and the equation in [16] could be obtained from generalizations of the mKdV equation [14] via some transformations, they are nonequivalent under rescaling and other transformations because both of them include nonlinear cubic terms. This case is very different from the case of the quadratic CH equation and equations from the b-family [18].

The conditions for the existence of a peakon solution of Eq. (1) were discussed in [15], but explicit peakon and cuspon solutions were not given. Here, we show that Eq. (1) has both "W/M-shape" peakon and cuspon solutions under the vanishing boundary condition at the space and time infinities. We present graphs of those peakon and cuspon solutions.

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2. Bi-Hamiltonian structure and Lax pair

2.1. Bi-Hamiltonian structure. Wave equation (1) can be rewritten as

$$\rho_t = \left[bu + \frac{1}{2}(u^2 - u_x^2)\rho\right]_x = J\frac{\delta H_1}{\delta \rho} = K\frac{\delta H_2}{\delta \rho}, \tag{2}$$

where

$$J = \partial\rho\partial^{-1}\rho\partial + b\partial, \qquad K = \partial - \partial^3, \tag{3}$$

$$H_1 = \int_\Omega \rho u\, dx, \qquad H_2 = \frac{1}{8}\int_\Omega \left(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4 + 4bu^2\right) dx \tag{4}$$

and $\Omega = (x_0, x_0{+}T)$ or $\Omega = (-\infty, +\infty)$, which makes u periodic in T or asymptotically constant at infinities. Obviously, J and K are Hamiltonian operators, and H_1 and H_2 are therefore Hamiltonians. Hence, Eq. (1) has a bi-Hamiltonian structure. In verifying the bi-Hamiltonian structure, we use $(1 - \partial^2)(\delta H_2/\delta\rho) = (\delta H_2/\delta u)$.

2.2. Lax pair. We show that Eq. (1) is integrable in the Lax pair sense. We consider the spectral problem

$$\psi_x = \begin{pmatrix} -\dfrac{\sqrt{1-\lambda^2 b}}{2} & \dfrac{1}{2}\lambda\rho \\ -\dfrac{1}{2}\lambda\rho & \dfrac{\sqrt{1-\lambda^2 b}}{2} \end{pmatrix}\psi, \tag{5}$$

where λ is a spectral parameter, ρ is a scalar potential function periodic or decaying at infinities, and $\psi = (\psi_1, \psi_2)^{\mathrm{T}}$ is the spectral function corresponding to the spectral parameter λ. We then have

$$K\nabla\lambda = \lambda^2 J\nabla\lambda, \tag{6}$$

where $\nabla\lambda = \lambda(\psi_1^2 + \psi_2^2)$.

It is easy to prove that Eq. (1) has the Lax pair

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x = U(\rho,\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{7}$$

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_t = V(\rho,u,\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{8}$$

where

$$U = \begin{pmatrix} -\dfrac{\sqrt{1-\lambda^2 b}}{2} & \dfrac{1}{2}\lambda\rho \\ -\dfrac{1}{2}\lambda\rho & \dfrac{\sqrt{1-\lambda^2 b}}{2} \end{pmatrix}, \qquad \rho = u - u_{xx},$$

$$V = -\frac{1}{2}\begin{pmatrix} \begin{array}{l}\lambda^{-2}\sqrt{1-\lambda^2 b} + \\ \quad + \dfrac{\sqrt{1-\lambda^2 b}}{2}(u^2 - u_x^2)\end{array} & \begin{array}{l}\lambda^{-1}(\sqrt{1-\lambda^2 b}\,u_x - u_{xx}) - \\ \quad - \dfrac{1}{2}\lambda\rho(u^2 - u_x^2)\end{array} \\ \begin{array}{l}\lambda^{-1}(\sqrt{1-\lambda^2 b}\,u_x + u_{xx}) + \\ \quad + \dfrac{1}{2}\lambda\rho(u^2 - u_x^2)\end{array} & \begin{array}{l}-\lambda^{-2}\sqrt{1-\lambda^2 b} - \\ \quad - \dfrac{\sqrt{1-\lambda^2 b}}{2}(u^2 - u_x^2)\end{array} \end{pmatrix}.$$

We can easily verify that the zero-curvature equation

$$U_t - V_x + [U, V] = 0 \tag{9}$$

gives precisely Eq. (1), which is therefore completely integrable in the Lax pair sense.

Remark 1. Equation (6) plays an important role in finding finite-dimensional constrained integrable systems [19]. In what follows, we obtain parametric solutions of Eq. (1) using the constraint idea in [19].

3. Nonsmooth soliton solutions

3.1. Traveling-wave setting. We consider the traveling-wave solution of Eq. (1), assuming that $u(x,t) = U(x - ct)$ in the general case, where c is the wave speed. Let $\xi = x - ct$. Then $u(x,t) = U(\xi)$. Substituting this expression in (1) yields

$$c(U - U'')' + bU' + \frac{1}{2}\big((U^2 - U'^2)(U - U'')\big)' = 0, \tag{10}$$

where $U' = U_\xi$, $U'' = U_{\xi\xi}$, and $U''' = U_{\xi\xi\xi}$.

Integrating both sides of (10) twice and using the vanishing-at-infinities condition, we obtain

$$U'^2 = U^2 + 2c \mp 2\sqrt{c^2 - bU^2}, \tag{11}$$

where the minus sign is chosen for $c > 0$ and the plus sign is chosen for $c < 0$ because U decays at infinities.

Remark 2. In the case $c = 0$, Eq. (11) has no traveling-wave solution satisfying the zero boundary conditions at infinities. Even for negative b, Eq. (11) becomes $U'^2 = U^2 \mp 2\sqrt{-b}U$, which obviously has no solution decaying at infinities.

3.2. "W/M-shape" peaked solitons. Let $X = \sqrt{c^2 - bU^2}$ in Eq. (11). Then $U^2 = (c^2 - X^2)/b$, and

$$U' = \pm\sqrt{\frac{c^2 - X^2}{b} + 2c \mp 2X}. \tag{12}$$

Substituting U yields

$$\frac{X\,dX}{\sqrt{bc^2 - bX^2}} = \pm\sqrt{\frac{c^2 - X^2}{b} + 2c \mp 2X\,d\xi} \tag{13}$$

or, equivalently,

$$\frac{X\,dX}{|c - \sigma X|\sqrt{(b + c + \sigma X)^2 - b^2}} = \pm d\xi, \quad \sigma = \pm 1. \tag{14}$$

We now consider the cases $\sigma = 1$ ($c > 0$) and $\sigma = -1$ ($c < 0$).

Case 1: $\sigma = 1$ ($c > 0$).

1.1. Let $b < 0$ and $c > 0$. If $b < 0$, $b \le -c$, and $c > 0$, then we can verify that no soliton solution exists. But if $-c < b < 0$ and $c > 0$, then Eq. (14) can be integrated as

$$\big(X + \alpha + \sqrt{(X+\alpha)^2 - b^2}\big)\left(\frac{X - c}{8c\alpha + (4c + 2b)(X - c) + 4\sqrt{c\alpha}\sqrt{(X+\alpha)^2 - b^2}}\right)^{1/2\theta} = e^{-|\xi|}, \tag{15}$$

where $\theta = \sqrt{\alpha/c}$, $0 < \theta < 1$, and $\alpha = b + c$. In the general case, Eq. (15) cannot be solved for X explicitly, but this is possible for some special θ. For example, we take $\theta = 1/2$. Then $b = -3c/4$. Under this condition, Eq. (15) becomes

$$X^2 - vX + cv = 0, \tag{16}$$

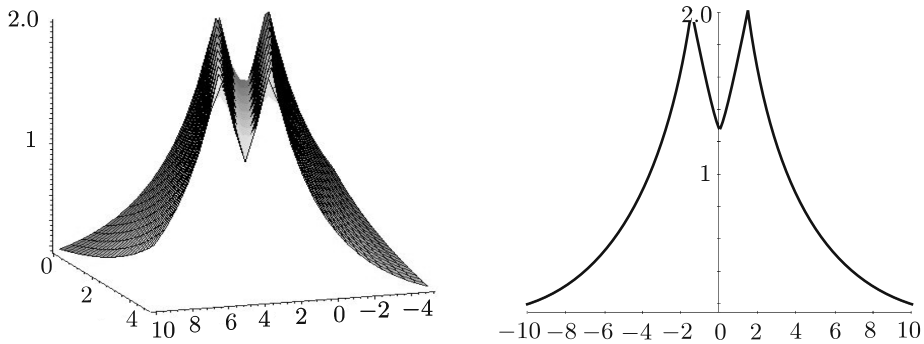

Fig. 1. Three- and two-dimensional graphs of the M-shape-peak soliton solution with $\theta = 1/2$.

where

$$v = c\left(\frac{9}{4}e^{-|\xi|} + \frac{9}{16}e^{|\xi|} + \frac{7}{4}\right)$$

or

$$v = \frac{9}{4}\varrho c + 2c \quad \text{with } \varrho = \cosh\left(\frac{|\xi|}{2} - \log 2\right) - \frac{1}{9}.$$

Solving Eq. (16) yields

$$X = \frac{v - \sqrt{v^2 - 4cv}}{2} = c + \frac{9c}{8}\left(\varrho - \sqrt{\varrho^2 - \frac{64}{81}}\right).$$

Because $U^2(\xi) = (c^2 - X^2)/b$, we obtain

$$U^2(\xi) = c\left(3 + \frac{27}{8}\varrho\right)\left(\varrho - \sqrt{\varrho^2 - \frac{64}{81}} - \frac{4}{3}\right), \tag{17}$$

where $\varrho = \cosh Y - 1/9$ and $Y = |x - ct|/2 - \log 2$. If we take the very special value $b = -3/4$ ($c = 1$), then the solution can be expressed as

$$U(\xi) = \pm\sqrt{\frac{1}{8}(7 + 9\cosh Y)\left(3\cosh Y - \frac{1}{3} - \sqrt{(9\cosh Y + 7)(\cosh Y - 1)}\right) - \frac{4}{3}}, \tag{18}$$

where $Y = |x - t|/2 - \log 2$.

Remark 3. Graphs for the "plus" case of solution (18) are shown in Fig. 1. It is an M-shape-peak soliton solution. If we select the "minus" case of solution (18), then we can get the W-shape-peak soliton solution. Obviously, three peaks occur at $-2\log 2$, 0, and $2\log 2$ in Fig. 1. In fact, we have calculated both left and right derivatives at those three points: $U'(-2\log 2^-) = \sqrt{2}/2$, $U'(-2\log 2^+) = -\sqrt{2}/2$; $U'(0^-) \approx -0.396$, $U'(0^+) \approx 0.396$; and $U'(2\log 2^-) = \sqrt{2}/2$, $U'(2\log 2^+) = -\sqrt{2}/2$.

If we do not consider the absolute value involved in the solution form during the solution procedure for the case $-c < b < 0$ and $c > 0$, then we find another new solution

$$U(\xi) = \pm\sqrt{\frac{1}{8}(7 + 9\cosh Y)\left(3\cosh Y - \frac{1}{3} - \sqrt{(9\cosh Y + 7)(\cosh Y - 1)}\right) - \frac{4}{3}}, \tag{19}$$

where $Y = (x - ct)/2 = \xi/2$.

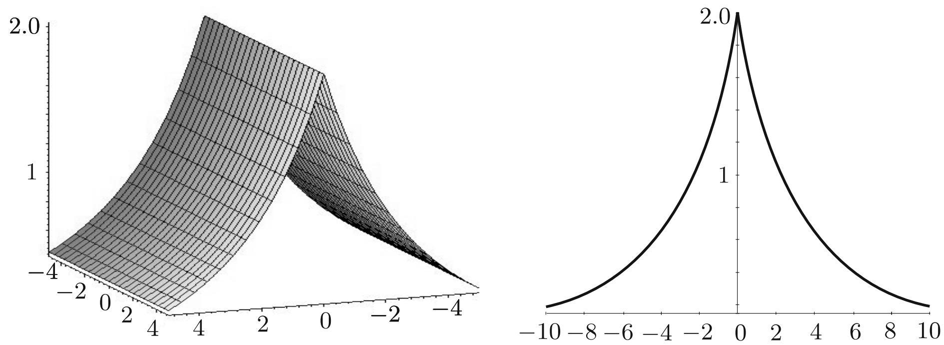

Fig. 2. Three- and two-dimensional graphs for the positive cuspon solution with $\theta = 1/2$.

Remark 4. Graphs for the "plus" case of solution (19) are shown in Fig. 2. It is a Λ-shape cuspon solution because $U'(0^+) = -\infty$ and $U'(0^-) = \infty$. If we select the "minus" case of solution (19), then we can obtain the V-shape (anti-Λ-shape) cuspon solution.

1.2. If $b > 0$ and $c > 0$, then Eq. (14) can be integrated the same as (15) but with $\theta > 1$. In this case, we can obtain implicit peakon and cuspon solutions, whose graphs have profiles similar to those in Figs. 1 and 2.

Case 2: $\sigma = -1$ ($c < 0$). Here, we only present our results because the analysis and computation procedure is the same as in case 1.

2.1. We consider $b > 0$ and $c < 0$. If $b > 0$, $b \geq -c$, and $c < 0$, then we can verify that no soliton solution exists. But if $0 < b < -c$ and $c < 0$, then Eq. (14) can be integrated as

$$(X - \alpha + \sqrt{(X-\alpha)^2 - b^2})\left(\frac{X + c}{8c\alpha - (4c + 2b)(X + c) + 4\sqrt{c\alpha}\sqrt{(X-\alpha)^2 - b^2}}\right)^{1/2\theta} = e^{-|\xi|}, \tag{20}$$

where $\theta = \sqrt{\alpha/c}$, $0 < \theta < 1$, and $\alpha = b + c$. In the general case, Eq. (20) cannot be solved for X explicitly, but for some special θ, $\theta = 1/2$ for example, we can obtain explicit solutions of the equation using a procedure similar to that in the case $-c < b < 0$ and $c > 0$ (see 1.1). The explicit solution has form (18), but the traveling-wave speed c is negative, i.e., the wave propagates backward.

2.2. If $b < 0$ and $c < 0$, then Eq. (14) can be integrated the same as (15) but with $\theta > 1$. In this case, we can obtain implicit peakon and cuspon solutions, whose graphs are similar to those in Figs. 1 and 2.

4. Conclusions

Using the idea in [16], [20], we have provided the Lax representation, bi-Hamiltonian structure, and M/W-shape peakons and also new cuspons for Eq. (1). We gave the M/W-shape peakon and cuspon solutions in explicit form, which confirms the peakon existence for Eq. (1) in [15]. Considering differential form (14), we can conjecture that Eq. (1) can have smooth periodic solutions in the form of a Jacobian elliptic function if the step analysis procedure [10], [11] is used.

On the other hand, we have tried to find multicuspon or multipeakon solutions of Eq. (1) because it has the Lax pair and Bi-Hamiltonian structures, but because of the cubic nonlinearity of Eq. (1), we have so far found neither its Hirota bilinear form nor Darboux transforms. Multicuspon or multipeakon solutions still need further study.

Equation (1) has a physical sense as shown in [15]. On the other hand, we believe that it could be derived from the two-dimensional Euler equation. In the case of the traveling wave with zero boundary conditions, it is transformed into the nonlinear equation $U'^2 = P(U)$, which is actually the result of integrating the Newton equation of a particle, $U'' = U \pm (bU)/(c^2 - bU^2)$. Here $P(U) = U^2 + 2c \mp 2\sqrt{c^2 - bU^2}$ is a new potential function (we take the minus sign if $c > 0$ and the plus sign if $c < 0$). We here successfully solved this Newton equation with new cuspons and M/W-shape peakons. These peak and cusp solutions may be applicable in neuroscience for providing a mathematical model and explaining electrophysiological responses of visceral nociceptive neurons and sensitization of dorsal root reflexes [21].

As far as we know, smooth decaying solitons have not yet been found for Eq. (1), although it is completely integrable. Furthermore, a more general case $\rho_t = bu_x + \gamma u_{xxx} + (1/2)[(u^2 - u_x^2)\rho]_x$, $\rho = u - u_{xx}$, where b and γ are constants, can be considered. When $\gamma = -b$, it can be transformed into the equation in [16], [20] because $bu_x + \gamma u_{xxx}$ is incorporated with ρ_t in the traveling-wave case. In the general case $\gamma \neq -b$, we cannot integrate to obtain an explicit solution as we did here, but we can seek a numerical solution based on an implicit formula, which is beyond the topic of this paper.

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