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A gauge equivalent pair with two different r-matrices

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Abstract

An interesting fact is found in this Letter that a pair of finite dimensional integrable Hamiltonian systems produced by two gauge equivalent spectral problems possesses the different r-matrices. In addition, an approach is also presented for deriving the finite dimensional integrable systems from the Lax matrix instead of Lax pair. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Lax matrix; r-Matrix; Integrable system

1. Introduction

The r-matrix method plays an important part in the study of integrable systems. The structures of r-matrix and fundamental Poisson bracket include many necessary information of a finite dimensional system, such as the conserved integrals [13,2] etc. Also the classical method for separation of variables to solve the integrable system can be formed in the r-matrix structure [14,5]. Both dynamical and nondynamical r-matrices corresponding to many finite dimensional integrable systems with physics interest have appeared in the literature [7,8]. Recently, we reported an interesting fact [12]: two different finite dimensional systems can share a common r-matrix with a good property of being nondynamical. Those further three examples are still found in a successive paper [11]. Now, we shall have another amazing fact in this Letter: a pair of finite dimensional Hamiltonian systems produced by two *gauge equivalent* spectral problems possesses the *different* r-matrices. Of cause, their Lax matrices and conserved integrals are different, too. In addition, taking this pair of Hamiltonian systems as two examples, we also present an approach for how to derive the finite dimensional integrable systems from the *Lax matrix* instead of *Lax pair*.

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Before displaying our results, let us first give some necessary symbols and notations:

$(R^{2N},dp\wedge dq)$ stands for the standard symplectic structure in Euclid space $R^{2N}=\{(p,q)|\,p=(p_1,\dots,p_N)$, $q=(q_1,\dots,q_N)\}$, p_i,q_i $(i=1,\dots,N)$ are N pairs of canonical coordinates, $\langle\cdot,\cdot\rangle$ is the standard inner product in R^N; in $(R^{2N},dp\wedge dq)$, the Poisson bracket of two Hamiltonian functions F,G is defined by [1]

$$\{F,G\}=\sum_{i=1}^{N}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)=\left\langle\frac{\partial F}{\partial q},\frac{\partial G}{\partial p}\right\rangle-\left\langle\frac{\partial F}{\partial p},\frac{\partial G}{\partial q}\right\rangle. \tag{1}$$

$\lambda_1,\dots,\lambda_N$ are N arbitrary given distinct constants; λ,μ are the two different spectral parameters; $\Lambda=\mathrm{diag}(\lambda_1,\dots,\lambda_N)$. Denote all infinitely times differentiable functions on real field R by $C^\infty(R)$. Let x be the continuous variable of space R or C.

2. Two gauge equivalent spectral problems

In 1992, Geng introduced the following spectral problem [4]

$$\phi_x=M\phi,\quad M=\begin{pmatrix} i\lambda-i\beta uv & u\\ v & -i\lambda+i\beta uv\end{pmatrix},\quad i^2=-1, \tag{2}$$

where u and v are two scalar potentials, λ is a spectral parameter and β is a constant, and discussed its evolution equations, Hamiltonian structure and integrability of the related constrained system. (2) is apparently an extension of the well-known ZS–AKNS spectral problem [16]

$$y_x=\begin{pmatrix}\lambda & u\\ v & -\lambda\end{pmatrix}y. \tag{3}$$

Two years later Qiao considered the following spectral problem [9]

$$\psi_x=\overline{M}\psi,\quad \overline{M}=\begin{pmatrix} -is & \lambda+r+\beta(s^2-r^2)\\ -\lambda+r-\beta(s^2-r^2) & is\end{pmatrix}, \tag{4}$$

where r, s are scalar potentials, the meanings of other signs are the same as ones in Eq. (2), and obtained a completely integrable systems with a set of finite dimensional involutive functions. (4) is actually an extension of the Dirac spectral problem [6]

$$y_x=\begin{pmatrix}-v & \lambda-u\\ -\lambda-u & v\end{pmatrix}y. \tag{5}$$

It is well-known that Eqs. (3) and (5) are gauge equivalent via some transformation. Then, for their extensive spectral problems we have the following further proposition.

Proposition 1 *Let*

$$\psi=G\phi,\ G=\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}. \tag{6}$$

Then,

$$\overline{M}G=GM \tag{7}$$

with $v=i(r-s), u=-i(r+s)$. That is to say, the spectral problems (2) and (4) are gauge equivalent via the transformation (6).

Proof Directly calculate.

We have known that the constrained flows of the ZS–AKNS spectral problem (3) and Dirac spectral problem (5), which are gauge equivalent, share a common standard r-matrix being nondynamical [11]. Thus, it seems to turn out this conclusion: two gauge equivalent spectral problems should have their finite dimensional constrained systems with the same r-matrix. But, unfortunately, it is not the case. The exception is the spectral problems (2) and (4). Please see below.

3. Lax matrix and finite dimensional Hamiltonian flows

Let us consider the following two Lax matrices:

$$L^G = L^G(\lambda) = \begin{pmatrix} 1+2i\beta\langle p,q\rangle & 0 \\ 0 & -1-2i\beta\langle p,q\rangle \end{pmatrix} - iL_0, \tag{8}$$

$$L^Q = L^Q(\lambda) = \begin{pmatrix} 0 & \frac{1}{2} - \beta(\langle p,p\rangle + \langle q,q\rangle) \\ -\frac{1}{2} + \beta(\langle p,p\rangle + \langle q,q\rangle) & 0 \end{pmatrix} + L_0. \tag{9}$$

where the 2×2 matrix L_0 is

$$L_0 = \sum_{j=1}^{N} \frac{1}{\lambda-\lambda_j} \begin{pmatrix} p_j q_j & -q_j^2 \\ p_j^2 & -p_j q_j \end{pmatrix}.$$

Then through calculating their determinants we have:

$$-\lambda^2 \det L^J = \frac{1}{2}\lambda^2 \mathrm{Tr}(L^J)^2 = F_1^J + F_0^J\lambda + H_0^J\lambda^2 + \sum_{j=1}^{N} \frac{\lambda_j^2 E_j^J}{\lambda-\lambda_j},\ J=G,Q, \tag{10}$$

where

$$E_j^J = E_{j,0}^J + \Gamma_j, \tag{11}$$

$$\Gamma_j = \sum_{k=1,k\neq j}^{N} \frac{(p_j q_k - p_k q_j)^2}{\lambda_j - \lambda_k},\ J=G,Q;\ j=1,\dots,N, \quad E_{j,0}^G = -2i(1+2i\beta\langle p,q\rangle)p_j q_j,$$

$$E_{j,0}^Q = \left(\frac{1}{2} - \beta(\langle p,p\rangle + \langle q,q\rangle)\right)\left(p_j^2 + q_j^2\right), H_0^G = (1+2i\beta\langle p,q\rangle)^2, \tag{12}$$

$$H_0^Q = -\left(\frac{1}{2} - \beta(\langle p,p\rangle + \langle q,q\rangle)\right)^2, \tag{13}$$

$$F_k^J = \sum_{j=1}^{N} \lambda_j^k E_j^J,\ k=0,1,\dots;\ J=G,Q. \tag{14}$$

Apparently, Eq. (14) reads the following

$$F_0^G = -2i(1+2i\beta\langle p,q\rangle)\langle p,q\rangle, \tag{15}$$

$$F_1^G = -2(1+2i\beta\langle p,q\rangle)\,H_G - \langle p,q\rangle^2, \tag{16}$$

$$F_0^Q = \left(\frac{1}{2} - \beta(\langle p,p\rangle + \langle q,q\rangle)\right)(\langle p,p\rangle + \langle q,q\rangle), \tag{17}$$

$$F_1^Q = (1-2\beta(\langle p,p\rangle + \langle q,q\rangle))\,H_Q + 4(\langle p,p\rangle + \langle q,q\rangle)^2, \tag{18}$$

where the two key Hamiltonian functions H_G and H_Q are

$$H_G = i\langle \Lambda q,p\rangle - \frac{\langle p,p\rangle\langle q,q\rangle}{2(1+2i\beta\langle p,q\rangle)} \tag{19}$$

and

$$H_Q = \frac{1}{2}\langle \Lambda p,p\rangle + \frac{1}{2}\langle \Lambda q,q\rangle - \frac{4\langle p,q\rangle^2 + (\langle p,p\rangle - \langle q,q\rangle)^2}{4-8\beta(\langle p,p\rangle + \langle q,q\rangle)}. \tag{20}$$

Then, the above two Hamiltonians give the following finite dimensional Hamiltonian flows:

$$\begin{cases} q_x = \dfrac{\partial H_G}{\partial p} = \Lambda q + i\beta\dfrac{\langle p,p\rangle\langle q,q\rangle}{(1+2i\beta\langle p,q\rangle)^2}q - \dfrac{\langle q,q\rangle}{1+2i\beta}p, \\ p_x = -\dfrac{\partial H_G}{\partial q} = -\Lambda p - i\beta\dfrac{\langle p,p\rangle\langle q,q\rangle}{(1+2i\beta\langle p,q\rangle)^2}p + \dfrac{\langle p,p\rangle}{1+2i\beta}q, \end{cases} \tag{21}$$

and

$$\begin{cases} q_x = \dfrac{\partial H_Q}{\partial p} = \Lambda p - \beta\dfrac{4\langle p,q\rangle^2 + (\langle p,p\rangle - \langle q,q\rangle)^2}{(1-2\beta(\langle p,p\rangle + \langle q,q\rangle))^2}p - \dfrac{2\langle p,q\rangle q + (\langle p,p\rangle - \langle q,q\rangle)p}{1-2\beta(\langle q,q\rangle + \langle p,p\rangle)}, \\ p_x = -\dfrac{\partial H_{QZ}}{\partial q} = -\Lambda q + \beta\dfrac{4\langle p,q\rangle^2 + (\langle p,p\rangle - \langle q,q\rangle)^2}{(1-2\beta(\langle p,p\rangle + \langle q,q\rangle))^2}q + \dfrac{2\langle p,q\rangle p - (\langle p,p\rangle - \langle q,q\rangle)q}{1-2\beta(\langle q,q\rangle + \langle p,p\rangle)}. \end{cases} \tag{22}$$

Obviously, Eqs. (21) and (22) can become Eqs. (2) and (4) with the constraints

$$u = -\frac{\langle q,q\rangle}{1+2i\beta\langle p,q\rangle},\, v = \frac{\langle p,p\rangle}{1+2i\beta\langle p,q\rangle}, \tag{23}$$

$\lambda = \lambda_j$, $\phi = (q_j,p_j)^T$, $j = 1,\dots,N$; and the constraints

$$s = \frac{-2i\langle p,q\rangle}{1-2\beta(\langle q,q\rangle + \langle p,p\rangle)},\, r = \frac{\langle q,q\rangle - \langle p,p\rangle}{1-2\beta(\langle q,q\rangle + \langle p,p\rangle)}, \tag{24}$$

$\lambda = \lambda_j$, $\psi = (q_j,p_j)^T$, $j = 1,\dots,N$, respectively. Therefore, the finite dimensional Hamiltonian systems (21) and (22) coincide with the constrained flows of the spectral problems (2) and (4), respectively. For their integrability, we need to discuss their own r-matrix structure.

4. Different *r*-matrices and integrability

Let $L_1^J(\lambda) = L^J(\lambda) \otimes I$, $L_2^J(\mu) = I \otimes L^J(\mu)$, $J = G, Q$, where I is the 2×2 unit matrix, $\otimes$ is the tensor product of matrix. We shall search for a 4×4 r-matrix structure $r_{12}(\lambda,\mu)$ satisfying the fundamental Poisson bracket [3]:

$$\left\{L^J(\lambda) \overset{\otimes}{,} L^J(\mu)\right\} = \left[r_{12}(\lambda,\mu), L_1^J(\lambda)\right] - \left[r_{21}(\mu,\lambda), L_2^J(\mu)\right],\ J = G, Q, \tag{25}$$

where $\{L^J(\lambda) \overset{\otimes}{,} L^J(\mu)\}_{kl,mn} = \{L^J(\lambda)_{km}, L^J(\mu)_{ln}\}$, $r_{21}(\lambda,\mu) = P r_{12}(\lambda,\mu) P$, $P = \frac{1}{2}\sum_{i=0}^{3} \sigma_i \otimes \sigma_i$, σ_i is the standard Pauli matrices, and $[\cdot,\cdot]$ stands for the usual commutator of matrix.

Theorem 1. *Eq. (25) is satisfied with the following two different r-matrix structures*

$$r_{12}^J(\lambda,\mu) = \frac{2}{\mu - \lambda} P + S^J,\ J = G, Q, \tag{26}$$

where

$$S^G = 4i\beta \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ S^Q = 2\beta \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{27}$$

Evidently, (26) is nondynamical both for $J = G$ and for $J = Q$.

Proof. We denote $L^J(\lambda)$ by

$$L^J(\lambda) = \begin{pmatrix} A_J(\lambda) & B_J(\lambda) \\ C_J(\lambda) & -A_J(\lambda) \end{pmatrix},\ J = G, Q, \tag{28}$$

where

$$A_G(\lambda) = 1 + 2i\beta\langle p,q\rangle - i\sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} p_j q_j, \tag{29}$$

$$B_G(\lambda) = i\sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} q_j^2, \tag{30}$$

$$C_G(\lambda) = -i\sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} p_j^2; \tag{31}$$

$$A_Q(\lambda) = \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} p_j q_j, \tag{32}$$

$$B_Q(\lambda) = \frac{1}{2} - \beta(\langle p,p\rangle + \langle q,q\rangle) - \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} q_j^2, \tag{33}$$

$$C_Q(\lambda) = -\frac{1}{2} + \beta(\langle p,p\rangle + \langle q,q\rangle) + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} p_j^2. \tag{34}$$

Then under the standard Poisson bracket (1), it is not difficult to calculate the following equalities

$$\begin{cases} \{A_G(\lambda),A_G(\mu)\} = \{B_G(\lambda),B_G(\mu)\} = \{C_G(\lambda),C_G(\mu)\} = 0, \\ \{A_G(\lambda),B_G(\mu)\} = -4i\beta B_G(\mu) + \dfrac{2}{\mu-\lambda}(B_G(\mu)-B_G(\lambda)), \\ \{A_G(\lambda),C_G(\mu)\} = 4i\beta C_G(\mu) + \dfrac{2}{\mu-\lambda}(C_G(\lambda)-C_G(\mu)), \\ \{B_G(\lambda),C_G(\mu)\} = \dfrac{4}{\mu-\lambda}(A_G(\mu)-A_G(\lambda)); \\ \{A_Q(\lambda),A_Q(\mu)\} = 0, \\ \{B_Q(\lambda),B_Q(\mu)\} = -4\beta(A_Q(\mu)-A_Q(\lambda)), \\ \{C_Q(\lambda),C_Q(\mu)\} = 4\beta(A_Q(\mu)-A_Q(\lambda)), \\ \{A_Q(\lambda),B_Q(\mu)\} = -2\beta(C_Q(\lambda)-B_Q(\lambda)) + \dfrac{2}{\mu-\lambda}(B_Q(\mu)-B_Q(\lambda)), \\ \{A_Q(\lambda),C_Q(\mu)\} = 2\beta(C_Q(\lambda)-B_Q(\lambda)) - \dfrac{2}{\mu-\lambda}(C_Q(\mu)-C_Q(\lambda)), \\ \{B_Q(\lambda),C_Q(\mu)\} = -4\beta(A_Q(\lambda)+A_Q(\mu)) + \dfrac{4}{\mu-\lambda}(A_Q(\mu)-A_Q(\lambda)). \end{cases} \tag{35}$$

After substituting the above equalities into Eq. (25), we can obtain Eqs. (26) and (27). □

An immediate consequence of Eq. (25) is

$$\left\{(L^J(\lambda))^2 \overset{\otimes}{,} (L^J(\mu))^2\right\} = \left[\bar{r}_{12}(\lambda,\mu),L_1^J(\lambda)\right] - \left[\bar{r}_{21}(\mu,\lambda),L_2^J(\mu)\right], \tag{36}$$

where

$$\bar{r}_{ij}(\lambda,\mu) = \sum_{k=0}^{1}\sum_{l=0}^{1}\left(L_1^J(\lambda)\right)^{1-k}\left(L_2^J(\mu)\right)^{1-l}\cdot r_{ij}(\lambda,\mu)\cdot\left(L_1^J(\lambda)\right)^{k}\left(L_2^J(\mu)\right)^{l}, \quad ij=12,21. \tag{37}$$

Thus, Eq. (36) leads to

$$4\left\{\mathrm{Tr}\left(L^J(\lambda)\right)^2,\mathrm{Tr}\left(L^J(\mu)\right)^2\right\} = \mathrm{Tr}\left\{\left(L^J(\lambda)\right)^2 \overset{\otimes}{,} \left(L^J(\mu)\right)^2\right\} = \mathrm{Tr}\left\{\left(L_1^J(\lambda)\right)^2 \overset{\otimes}{,} \left(L_2^J(\mu)\right)^2\right\} = 0.$$

which guarantees the involutivity of those integrals of motion obtained in Eq. (10). Therefore, we have

$$\left\{E_i^J,E_j^J\right\} = \left\{H_0^J,E_j^J\right\} = \left\{F_m^J,E_j^J\right\} = 0,\ J=G,Q, \quad i,j=1,2,\dots,N, \quad m=0,1,2,\dots. \tag{38}$$

In addition, noticing Eqs. (16) and (18), the following equalities

$$\{H_G,\langle p,q\rangle\} = 0, \left\{E_j^G,\langle p,q\rangle\right\} = 0,\ j=1,\dots,N; \tag{39}$$

$$\{H_Q,\langle p,p\rangle + \langle q,q\rangle\} = 0, \left\{E_j^Q,\langle p,p\rangle + \langle q,q\rangle\right\} = 0\ j=1,\dots,N, \tag{40}$$

and a further property: $E_1^J,E_2^J,\dots,E_N^J$ $(J=G,Q)$ are functionally independent on certain region of R^{2N}, we obtain the following theorem.

Theorem 2. *The constrained flows (21) and (22) are two completely integrable systems in Liouville's sense.*

Remark 1. *Usually, the Hamiltonian H is one or at least the functional combinations of the N-involutive set* $\{F_j\}$*. But, here is not the case (see Eqs. (16) and (18)). Thus, we must verify Eqs. (39) and (40). The present calculation is not simple but guessed and skilled.*

5. Conclusions

In this Letter it is revealed that there exists such an example that two spectral problems can be gauge equivalent, but the associated r-matrices determined by their own finite dimensional constrained system are different. Why does this phenomenon occur? Evidently, it does not depend upon the equivalence transformation (6), but upon the Lax matrices (8), (9) which directly effect on the actual form of the r-matrices (see Theorem 1). The two integrable Hamiltonian systems (21) and (22), produced by the Lax matrices (8), (9), respectively read the spectral problems (2) and (4) via the constraint conditions (23) and (24) which are generally determined by the functional gradient of spectral parameter λ with respect to the potentials (u,v) and (s,r). For a given spectral problem, its functional gradient can be uniquely calculated [15]. Although Eqs. (2) and (4) are equivalent, obviously their constraints are not. So, the form of constraints or functional gradient of spectral parameter has the actual effect on the choice of Lax matrices and the further calculation of r-matrices.

The present Letter along with the previous papers [12,11] bring us the amazing consequence: a pair of different finite dimensional constrained integrable flow can share a common r-matrix, even the same Lax matrix and involutive conserved integrals; but a pair of gauge equivalent spectral problem yields different r-matrices. The latter tells us that r-matrix structure is an innate property of finite dimensional integrable Hamiltonian flow, in the meantime also implies a fact: a pair of gauge equivalent spectral problems indeed produces two different finite dimensional integrable flows via some constraints.

In our previous papers [8,11,12], it was from the Lax pair that we obtained the finite dimensional constrained integrable systems. Now, the starting point of this Letter is the *Lax matrix* instead of the Lax pair. This is a different and more terse way. We do not need to construct the auxiliary matrix. As we see in Sections 3 and 4, the Lax matrix is sufficient enough to generate the finite dimensional integrable Hamiltonian systems (like the r-matrix, involutive set, etc), especially to constrained flows. Simultaneously, we have also got a procedure about how to induce a spectral problem starting from a given *Lax matrix*. The Hamiltonian system corresponding to the induced spectral problem is namely the usual constrained flow. Moreover, in this way the scope of finite dimensional integrable systems will be greatly enlarged [10]. Another aim of studying the Lax matrix and r-matrix structure is to classify the finite dimensional integrable systems (including constrained and restricted systems) from the viewpoint of r-matrix. We think that this can be realized. In a future step, we shall consider promoting the r-matrix structure of finite dimensional constrained flows to the nonlinear evolution equations of infinite dimensional systems [10].

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