ELSEVIER

Physica A 293 (2001) 143–156

PHYSICA A

www.elsevier.com/locate/physa

L–A–B representation for nonlinear evolution equations and its applications

Zhijun Qiao[a,b,c,*], Walter Strampp[b]

[a] *Institute of Mathematics, Fudan University, Shanghai 200433, People's Republic of China*

[b] *Fachbereich 17 - Mathematik/Informatik, Universität-GH Kassel, Heinrich-Plett-Strasse 40, D-34109 Kassel, Germany*

[c] *Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA*

Received 15 September 2000

Abstract

In this paper we consider the *L–A–B* representation of nonlinear evolution equations (NLEEs) and present an approach to determine its range. The method we adopt is that of Lie quotient algebras. As an example, we consider the Kaup–Newell spectral problem and give a category of NLEEs associated with it. Particularly, we obtain the negative order hierarchies of NLEEs by considering the inverse of recursion operator.

PACS: 03.40.Gc; 03.40.Kf; 47.10.+g

Keywords: *L–A–B* representation; Lie quotient algebra; Category; Kaup–Newell

1. Introduction

It is well known that the nonlinear evolution eqautions (NLEEs) solved by the famous inverse scattering transformation can be understood as compatible conditions of some linear equations [1,2]. Such examples include both the KdV, nonlinear Schrödinger, AKNS system in $(1 + 1)$-dimensional space and the KP, DS system in $(2 + 1)$-dimensional space [3]. These integrable models have attracted attention of mathematician and physists over the past years. This is mainly due to the fascinating properties of mathematics and physics.

* Correspondence address. Fachbereich 17 - Mathematik/Informatik, Universität-GH Kassel, Heinrich-Plett-Strasse 40, D-34109 Kassel, Germany. Tel.: +49-561-8044163; fax: +49-561-8044318.

E-mail addresses: qiaozj@sxx0.math.pku.edu.cn, zhijun@hrz.uni-kassel.de (Z. Qiao), strampp@hrz.uni-kassel.de (W. Strampp).

PII: S0378-4371(00)00593-8

Manakov [4] proved that for some concrete soliton equations the algebraic form of the compatibility conditon is given by the L–A–B triple representation structure. It has been found that some integrable systems possess this kind of representations [4−6].

In recent years, the discrete-time version [7] of integrable systems have already attracted a lot of attention. Capel, Nijhoff and their collaborators have got many developments in this area [8–10]. Their methods have been considered for constructing a new Lax pair from the old Lax pair for the continuous case [11] as well as deriving some new hierarchies of integrable NLEEs from the spectral problems [12]. On the basis of those literatures, this paper will further deal with the hierarchies of NLEEs, especially, the negative order hierarchies. The concept and idea of the negative order hierarchy of NLEEs first appeared in Ref. [13].

Recently, we introduced the category of NLEEs, which is composed of hierarchies of NLEEs and possesses a generalized Lax representation, and presented a constructive approach to the L–A–B triple representations [14]. The present paper aims at determining the range of the generalized Lax representation for a given nonlinear equation. The whole paper is organized as follows. In the next section, we concisely recall the category of NLEEs and the generalized Lax representation. In Sections 3 and 4, we introduce two different Lie quotient algebras for Lax operators, one of them is independent of the hierarchy, while the other one is dependent on the underlying hierarchy. Both of the Lie quotient algebras can be used for determining the range of the generalized Lax representations. In Section 5, as an application we present a new category of nonlinear evolution equations, obtained by the Kaup–Newell (KN) spectral problem. Particularly, we give the formula of the inverse recursion operator of the KN hierarchies, and further discuss the negative order KN hierarchies of NLEEs.

Before displaying our main results, let us first give some necessary notations: $x\in R^l$, $t\in R$, $u=(u_1,\ldots,u_m)^{\mathrm{T}}\in S^m(R^l,R)=\overbrace{S(R^l,R)\times\cdots\times S(R^l,R)}^{m}$, $u_i=u_i(x,t)\in S(R^l,R)$, $i=1,2,\ldots,m$, for arbitrary fixed t, $S(R^l,R)$ stands for the Schwartz function space on R^l. $\mathscr{B}$ stands for all complex (or real) value functions $P(x,t,u)$ of the class C^∞ with respect to x,t, and of the class C^∞ in Gateaux's sense with respect to u. $\mathscr{B}^N=\{(P_1,\ldots,P_N)^{\mathrm{T}}|P_i\in\mathscr{B}\}$, $\mathscr{V}^N$ stands for all linear operators $\phi=\phi(x,t,u):\mathscr{B}^N\to\mathscr{B}^N$ which are of the class C^∞ with respect to x, t, and of the class C^∞ in Gateaux's sense with respect to u.

The Gateaux derivate of a vector function $X\in\mathscr{B}^n$ in the direction $Y\in\mathscr{B}^m$ is defined by

$$X_*(Y)=\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}X(u+\varepsilon Y)\,. \tag{1.1}$$

For two arbitrary vector fields $X,\ Y\in\mathscr{B}^m$, we define the following operation:

$$[X,Y]=X_*(Y)-Y_*(X)\,. \tag{1.2}$$

Then, the set $\mathscr{B}^m$ endowed with the above multiplication operation [15] composes a Lie algebra. For the operator $\phi\in\mathscr{V}^N$, its Gateaux derivate operator $\phi_*:\mathscr{B}^m\to\mathscr{V}^N$ in the direction ξ is defined as follows:

$$\phi_*(\xi) = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \phi(u + \varepsilon\xi), \quad \xi \in \mathscr{B}^m . \tag{1.3}$$

If not otherwise stated, the spectral operators $L = L(u)$ (or the spectral operators $L = L(u, \lambda)$ with the spectral parameter λ) considered in this paper are denoted by $L \in \mathscr{V}^N$, and we assume throughout the paper that $L_* : \mathscr{B}^m \to \mathscr{V}^N$ is an injective homomorphism. An operator H acting on a function f is denoted by $H \cdot f$ and I stands for the $N \times N$ unit operator.

2. Category of NLEEs [14]

Let

$$L \cdot \psi = \lambda\psi, \quad L \in \mathscr{V}^N \tag{2.1}$$

be an $N \times N$ matrix spectral problem, and K, J be a pair of Lenard operators of (2.1), where λ is a spectral parameter, $\psi \in \mathscr{B}^N$. The positive order and negative order generators G_0 and G_{-1} are determined by the following operator equations:

$$L_*(J \cdot G_0) = \bar{M} ,$$

$$L_*(K \cdot G_{-1}) = \tilde{M} ,$$

where $\bar{M} = (\bar{m}_{ij})_{N\times N}$, $\tilde{M} = (\tilde{m}_{ij})_{N\times N}$ are arbitrarily given $(1 + l)$-dimensional linear $N \times N$ matrix operators depending on the variables $(x, t) \in R^l \times R$, $l \geqslant 1$. They yield the category of nonlinear evolution equations of (2.1):

$$u_t = J \cdot G_m(u, G_0, G_{-1}), \quad m \in Z , \tag{2.2}$$

$$G_m = \begin{cases} \mathscr{L}^m \cdot G_0, & j \geqslant 0 , \\ \mathscr{L}^{m+1} \cdot G_{-1}, & j < 0 , \end{cases} \tag{2.3}$$

where $\mathscr{L} = J^{-1}K$ is the recursion operator.

Theorem 2.1 (Qiao et al. [13]). *Let $\bar{M} = (\bar{m}_{ij})_{N\times N}$, $\tilde{M} = (\tilde{m}_{ij})_{N\times N}$ be two arbitrarily given $N\times N$ linear matrix operators. Suppose that for $G = (G^{[1]}, \ldots, G^{[m]})^{\mathrm{T}} \in S^m(R^l, R)$ and $\alpha, \beta \in Z$ the operator equation*

$$[V, L] = L_*(K \cdot G)L^{\beta} - L_*(J \cdot G)L^{\alpha} \tag{2.4}$$

possesses a solution $V = V(G)$, then the vector field $X_m = X_m(u, G_0, G_{-1}) = J \cdot G_m$ satisfy

$$L_*(X_m) = [W_m, L] + ML^{m\eta}, \quad m \in Z, \quad M = \begin{cases} \bar{M}, & m \geqslant 0 , \\ \tilde{M}, & m < 0 , \end{cases} \tag{2.5}$$

where $\eta=\alpha-\beta$ and the operator W_m is given by

$$W_m=\sum V(G_j)L^{(m-j)\eta-\alpha}, \quad \sum=\begin{cases}\sum_{j=0}^{m-1}, & m>0\,,\\ 0, & m=0\,,\\ -\sum_{j=m}^{-1}, & m<0\,.\end{cases} \tag{2.6}$$

Here G_j are determined by (2.3), *and L^{-1} is the inverse of L, i.e., $LL^{-1}=L^{-1}L=I$, and $[\cdot,\cdot]$ stands for the commutator.*

This theorem ensures that category (2.2) has the following formal Lax representation:

$$L_t=[W_m,L]+ML^{m\eta}, \quad m\in Z, \quad M=\begin{cases}\bar{M}, & m\geqslant 0\,,\\ \tilde{M}, & m<0\,.\end{cases} \tag{2.7}$$

which is called the generalized Lax representations (GLR) of (2.2) and W_m is called the generalized Lax operator (GLO).

Apparently, Eq. (2.7) admits the structure of *L–A–B* representations of the category (2.2) in an explicit form.

3. Lie quotient algebra independent of the hierarchy and range of the GLR

In Ref. [14], we defined the Manakov operator pair (A,M) for a given spectral operator L through

$$[A,L]=L_*(X)-M\,, \tag{3.1}$$

where X is the vector field corresponding to (A,M). Denote the set of all Manakov operator pairs (A,M), the set of all vector fields X, and the set of all triples (A,M,X) by $\mathscr{M}_L$, $V(\mathscr{M}_L)$, and $\mathscr{P}_L$, respectively.

As long as Eq. (2.4) has an operator solution for a given $L\in\mathscr{V}^N$, then by Theorem 2.1 and Eq. (2.5) there exists a triple $(A,M,X)\in\mathscr{P}_L$ satisfying (3.1). Apparently, if there is $A,M\in\mathscr{V}^N$ for $X\in\mathscr{B}^m$ such that Eq. (3.1) holds, then $u_t=X$ possesses the GLR $L_t=[A,L]+M$.

We have already proven that under the binary operation

$$(A,M)\odot(B,N)=(A\odot B,M\odot N)\,, \tag{3.2}$$

$$A\odot B=A_*(Y)-B_*(X)+[A,B]\,, \tag{3.3}$$

$$M\odot N=M_*(Y)-N_*(X)+[M,B]-[N,A]\,, \tag{3.4}$$

$\mathscr{M}_L$ forms a Lie algebra (see Ref. [14, Theorem 3.1]). This result shows that there is a universal algebraic structure for the *different* hierarchies of NLEEs within one category, and the GLR of $u_t=[X,Y]$ is produced by GLR of the equations $u_t=X$ and $u_t=Y \quad (X,Y\in\mathscr{B}^m)$.

Set $S_L=\{(A,M)\in\mathscr{V}^N\times\mathscr{V}^N \mid [A,L]=-M\}$. Then S_L corresponds to the stationary systems $X(u)=0$ of the NLEEs $u_t=X(u)$.

Theorem 3.1. *S_L forms a Lie subalgebra as well as an ideal of $\mathscr{M}_L$ with regard to the operation* (3.2).

Proof. Let (A,M,X), $(B,N,Y) \in \mathscr{P}_L$, then

$$(A,M),\ (B,N) \in S_L \Leftrightarrow X = Y = 0\ .$$

Thus, S_L is a subalgebra of $\mathscr{M}_L$.

Let $(A,M) \in S_L$ and arbitrarily choose $(B,N,Y) \in \mathscr{P}_L$, then we have $(A \odot B, M \odot N, [0,Y]) \in \mathscr{P}_L$, which implies $(A \odot B, M \odot N) \in S_L$. □

For (A,M), $(B,N) \in S_L$, operation (3.2) reads:

$$A \odot B = [A,B], \quad M \odot N = [M,B] + [A,N]\ . \tag{3.5}$$

Definition 3.1. In $\mathscr{V}^N \times \mathscr{V}^N$, two pairs of operators (A,M) and (B,N) are called equivalent $(A,M) \sim (B,N)$, iff $[A,L] + M = [B,L] + N$.

Obviously, $\sim$ is an equivalence relation in $\mathscr{V}^N \times \mathscr{V}^N$. Denote the equivalent class of (A,M) with regard to S_L by $\overline{(A,M)}$. Set $E(\mathscr{M}_L) = \{\overline{(A,M)} \mid (A,M) \in \mathscr{M}_L\}$, then by Theorem 3.1, $E(\mathscr{M}_L) = \mathscr{M}_L / S_L$ forms a quotient algebra with the operation

$$\overline{(A,M)} \odot \overline{(B,N)} = \overline{(A \odot B, M \odot N)}, \quad (A,M),(B,N) \in \mathscr{M}_L\ . \tag{3.6}$$

This algebra is independent of the hierarchy of NLEEs.

Theorem 3.2. *$(E(\mathscr{M}_L), \odot)$ is isomorphic to the Lie algebra $(V(\mathscr{M}_L), [\cdot,\cdot])$ and forms a Lie algebra with operation* (3.6), *called the Lie quotient algebra.*

Proof. For any $(A,M,X) \in \mathscr{P}_L$, we construct the following map:

$$\sigma : E(\mathscr{M}_L) \to V(\mathscr{M}_L), \quad \overline{(A,M)} \mapsto X\ .$$

It is easy to know σ is a linear isomorphism.

For any (A,M,X), $(B,N,Y) \in \mathscr{P}_L$, since

$$\sigma(\overline{(A,M)} \odot \overline{(B,N)}) = \sigma(\overline{(A \odot B, M \odot N)}) = [X,Y] = [\sigma(\overline{(A,M)}), \sigma(\overline{(B,N)})]\ ,$$

σ is a Lie isomorphism. □

This Theorem shows that if the spectral operator L is given, then all possible Manakov operator pairs (A,M) for every hierarchy $u_t = X(u)$ $(X \in V(\mathscr{M}_L))$ are exactly $\overline{(A,M)}$. Thus, we determine the range of the GLR $L_t = [A,L] + M$, which by setting $M = BL$ can be evidently transformed into the L–A–B representation $L_t = [A,L] + BL$.

4. Lie quotient algebra dependent on the underlying hierarchy and range of the related GLR

Let $L \in \mathscr{V}^N$ and M be a given spectral operator and an invertible $N \times N$ matrix operator, respectively. Then from Ref. [14], we know that $\mathscr{L}_L^M$ forms an algebra under the operation

$$(A,P) \ominus (B,Q) = (A \ominus B, P \ominus Q) \tag{4.1}$$

with

$$A \ominus B = A_*(Y) - B_*(X) + [A,B]\,, \tag{4.2}$$

$$\begin{aligned} P \ominus Q = {} & P_*(Y) - Q_*(X) + [A,Q] - [B,P] \\ & + M^{-1}(M_*(Y) - [B,M])P - M^{-1}(M_*(X) - [A,M])Q\,, \end{aligned} \tag{4.3}$$

where $\mathscr{L}_L^M$ stands for the set of all LM operator pairs (A,P) of L satisfying the following property [14]:

$$[A,L] + MP = L_*(X)\,. \tag{4.4}$$

This result reveals that there exists an algebraic structure available for all equations in the *same* hierarchy. If for given operators L, M there exist $A, P \in \mathscr{V}_L^N$ such that (4.4) holds, then obviously the evolution equation $u_t = X$ has the following Lax form (also called generalized Lax representation (GLR))

$$L_t = [A,L] + MP\,. \tag{4.5}$$

And if $u_t = X$, $u_t = Y$ $(X, Y \in \mathscr{B}^m)$ have the GLR (4.5), then the evolution equation $u_t = [X, Y]$ is still in the same hierarchy, and possesses the GLR (4.5), too.

In general, $\mathscr{L}_L^M$ is not forming a Lie algebra under operation (4.1), because the Jacobi identity cannot be guaranteed. Nevertheless, the subset $S_L^M \subset \mathscr{L}_L^M$ considered below is an exception.

Set $S_L^M = \{(A,P) \in \mathscr{V}^N \times \mathscr{V}^N \mid [A,L] = -MP\}$, then S_L^M is corresponding to the stationary system $X(u) = 0$ of evolution equation $u_t = X(u)$.

Theorem 4.1. *Under operation* (4.1) S_L^M *forms an ideal of* $\mathscr{L}_L^M$.

Proof. The proof is analogous to Theorem 3.1. □

Now, we define the equivalence relation "$\sim$" as follows: if

$$[A,L] + MP = [B,L] + MQ\,, \tag{4.6}$$

then (A,P) is said to be equivalent to (B,Q), and write $(A,P) \sim (B,Q)$. Denote the equivalent class of (A,P) by $\overline{(A,P)}$. Set $E(\mathscr{L}_L^M) = \{\overline{(A,P)} \mid (A,P) \in \mathscr{L}_L^M\}$, then $E(\mathscr{L}_L^M) = \mathscr{L}_L^M / S_L^M$ is a quotient algebra, whose operation is

$$\overline{(A,P)} \ominus \overline{(B,Q)} = \overline{(A \ominus B, P \ominus Q)}, \quad (A,P),(B,Q) \in \mathscr{L}_L^M\,. \tag{4.7}$$

Theorem 4.2. *Quotient algebra $(E(\mathscr{L}_L^M), \ominus)$ is isomorphic to the Lie algebra $(V(\mathscr{L}_L^M), [\cdot, \cdot])$. Thus, $E(\mathscr{L}_L^M)$ is also a Lie algebra.*

Proof. The proof is similar to Theorem 3.2. $\square$

This Theorem assures that the range of generalized Lax-operators for Eq. (4.5) is $\overline{(A,P)}$.

5. An example

In this section we consider the Kaup–Newell spectral problem as an example. For convenience, we make the following conventions: $m \in Z$,

$$f^{(m)} = \begin{cases} \frac{\partial^m}{\partial x^m} f = f_{mx}, & m \geqslant 0, \\ \underbrace{\int \dots \int}_{-m} f \, \mathrm{d}x, & m < 0, \end{cases} \qquad \sum = \begin{cases} \sum_{j=0}^{m-1}, & m > 0 \ , \\ 0, & m = 0 \ , \\ -\sum_{j=m}^{-1}, & m < 0 \ , \end{cases}$$

$f_t = \partial f/\partial t$, $f_{mxt} = \partial^{m+1} f/\partial t \partial x^m (m \geqslant 0)$, $\partial = \partial/\partial x$, ∂^{-1} is the inverse of ∂, i.e., $\partial\partial^{-1} = \partial^{-1}\partial = 1$, $\partial^m f$ means the operator $\partial^m f$ acts on some function g, i.e., $\partial^m f \cdot g = \partial^m (fg)$. The imaginary unit i is satisfying $\mathrm{i}^2 = -1$, and $I_{2\times 2}$ stands for the 2×2 unit matrix.

In the following, the functions q, r stand for potentials, and λ is assumed to be a spectral parameter, and the domain of the spatial variable x is Ω which becomes equal to $(-\infty, +\infty)$ or $(0, T)$, while the domain of the time variable t is the positive time axis $R^+ = \{t | t \in R, \ t \geqslant 0\}$. In the case $\Omega = (-\infty, +\infty)$, the decaying condition at infinity and in the case $\Omega = (0, T)$, the periodicity condition for the potential function is imposed.

Consider the Kaup–Newell (KN) spectral problem [16]

$$y_x = \begin{pmatrix} -\mathrm{i}\lambda^2 & \lambda q \\ \lambda r & \mathrm{i}\lambda^2 \end{pmatrix} y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{5.1}$$

which is equivalent to

$$L \cdot y = \lambda^2 y, \quad L = L(q, r, \lambda) = \begin{pmatrix} \mathrm{i}\partial & -\mathrm{i}\lambda q \\ -\lambda^{-1} r \partial & -\mathrm{i}\partial + qr \end{pmatrix}. \tag{5.2}$$

It is easy to obtain the spectral gradient $\delta\lambda/\delta q = \lambda y_2^2$, $\delta\lambda/\delta r = -\lambda y_1^2$, satisfying the Lenard eigenvalue problem

$$K \cdot \frac{\delta\lambda}{\delta u} = \lambda^2 J \cdot \frac{\delta\lambda}{\delta u}, \quad \frac{\delta\lambda}{\delta u} = \left(\frac{\delta\lambda}{\delta q}, \frac{\delta\lambda}{\delta r}\right)^{\mathrm{T}}, \tag{5.3}$$

with the Lenard's operators pair

$$K = \frac{1}{2} \begin{pmatrix} \partial q \partial^{-1} q \partial & \mathrm{i}\partial^2 + \partial q \partial^{-1} r \partial \\ -\mathrm{i}\partial^2 + \partial r \partial^{-1} q \partial & \partial r \partial^{-1} r \partial \end{pmatrix}, \quad J = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \tag{5.4}$$

which yields the recursion operator

$$\mathscr{L} = J^{-1}K = \frac{1}{2}\begin{pmatrix} -\mathrm{i}\partial + r\partial^{-1}q\partial & r\partial^{-1}r\partial \\ q\partial^{-1}q\partial & \mathrm{i}\partial + q\partial^{-1}r\partial \end{pmatrix} . \tag{5.5}$$

Apparently, the Gateaux derivative operator $L_*(\xi)$ of the spectral operator L given by (5.2) in the direction $\xi = (\xi_1, \xi_2)^{\mathrm{T}} \in \mathscr{B}^2$ is

$$L_*(\xi) = \begin{pmatrix} 0 & -\mathrm{i}\lambda\xi_1 \\ -\lambda^{-1}\xi_2\partial & r\xi_1 + q\xi_2 \end{pmatrix} \tag{5.6}$$

which is an injective homomorphism.

Through guesswork and calculations, we can obtain the inverse operators of L, J, K and $\mathscr{L}$:

$$L^{-1} = \begin{pmatrix} -\mathrm{i}\partial^{-1} + \partial^{-1}q\partial^{-1}r & \mathrm{i}\lambda\partial^{-1}q\partial^{-1} \\ \lambda^{-1}\partial^{-1}r & \mathrm{i}\partial^{-1} \end{pmatrix} , \tag{5.7}$$

$$J^{-1} = \begin{pmatrix} 0 & \partial^{-1} \\ \partial^{-1} & 0 \end{pmatrix} , \tag{5.8}$$

$$K^{-1} = 2\begin{pmatrix} -\partial^{-1}r\partial^{-1}r\partial^{-1} & \mathrm{i}\partial^{-2} + \partial^{-1}r\partial^{-1}q\partial^{-1} \\ -\mathrm{i}\partial^{-2} + \partial^{-1}q\partial^{-1}r\partial^{-1} & -\partial^{-1}q\partial^{-1}q\partial^{-1} \end{pmatrix} , \tag{5.9}$$

$$\mathscr{L}^{-1} = 2\begin{pmatrix} \mathrm{i}\partial^{-1} + \partial^{-1}r\partial^{-1}q & -\partial^{-1}r\partial^{-1}r \\ -\partial^{-1}q\partial^{-1}q & -\mathrm{i}\partial^{-1} + \partial^{-1}q\partial^{-1}r \end{pmatrix} . \tag{5.10}$$

Let $A_i = A_i(x,t,q(x,t),r(x,t))$, $B_i = B_i(x,t,q(x,t),r(x,t))$ $(i = 1,2)$ be four arbitrarily given C^∞-functions, then iff

$$\bar{M} = \begin{pmatrix} 0 & -\mathrm{i}\lambda A_2 \\ -\lambda^{-1}A_1\partial & rA_2 + qA_1 \end{pmatrix}, \quad \tilde{M} = \begin{pmatrix} 0 & -\mathrm{i}\lambda B_2 \\ -\lambda^{-1}B_1\partial & rB_2 + qB_1 \end{pmatrix} , \tag{5.11}$$

the operator equations $L_*(J \cdot G_0) = \bar{M}$, $L_*(K \cdot G_{-1}) = \tilde{M}$ have the following functions as solutions:

$$G_0 = \begin{pmatrix} A_1^{(-1)} \\ A_2^{(-1)} \end{pmatrix} , \tag{5.12}$$

$$G_{-1} = 2\begin{pmatrix} \mathrm{i}B_1^{(-2)} + \partial^{-1}r\partial^{-1} \cdot (qB_1^{(-1)} - rB_2^{(-1)}) \\ -\mathrm{i}B_2^{(-2)} - \partial^{-1}q\partial^{-1} \cdot (qB_1^{(-1)} - rB_2^{(-1)}) \end{pmatrix} , \tag{5.13}$$

which directly yields the KN category of NLEEs:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = J \cdot G_m, \quad m \in Z , \tag{5.14}$$

$$G_m = \begin{cases} \mathscr{L}^m \cdot G_0, & m=0,1,2\dots\,, \\ \mathscr{L}^{m+1} \cdot G_{-1}, & m=-1,-2,\dots\,, \end{cases} \tag{5.15}$$

where J, $\mathscr{L}$ and $\mathscr{L}^{-1}$ are defined by (5.4), (5.5) and (5.10), respectively.

For any given $G=(G^{[1]},G^{[2]})^{\mathrm{T}} \in \mathscr{B}^2$, the equation $[V,L]=L_*(K\cdot G)-L_*(J\cdot G)L$ has the operator solution

$$V=V(G) = \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i}\lambda G_x^{[2]} \\ 0 & -qG_x^{[1]} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} (qG_x^{[1]}+rG_x^{[2]})^{(-1)} & 0 \\ \lambda^{-1}G_x^{[1]} & (qG_x^{[1]}+rG_x^{[2]})^{(-1)} \end{pmatrix}\partial\,. \tag{5.16}$$

Therefore, the KN category (5.14) has the GLR:

$$L_t=[W_m,L]+ML^m, \quad m\in Z, \quad M=\begin{cases} \bar{M}, & m\geqslant 0\,, \\ \tilde{M}, & m<0 \end{cases} \tag{5.17}$$

with the GLO

$$W_m=\sum V(G_j)L^{m-j-1}, \quad m\in Z\,, \tag{5.18}$$

where L, L^{-1} and $V(G_j)$ are given by (5.2), (5.7) and (5.16) with $G=G_j$ defined by (5.15), respectively.

Let us discuss now reductions of (5.14) below.

I. Positive Case $(m=0,1,2,\dots)$

- With $A_1=A_2=0$, $G_0=(a_1,\ a_2)^{\mathrm{T}}\in Ker\,J$, $a_i=a_i(t)\in C^\infty(R)$, $i=1,2$, the positive order category of (5.14) reads as the Kaup–Newell isospectral $(\lambda_t=0)$ hierarchy

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = J\mathscr{L}^m\cdot\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad m=0,1,2,\dots\,, \tag{5.19}$$

which has the following representative equations:

$$\begin{matrix} q_t=q_x, \\ r_t=r_x, \end{matrix} \quad m=1\,, \tag{5.20}$$

$$\begin{matrix} q_t=\frac{1}{2}\mathrm{i}q_{xx}+\frac{1}{2}(q^2r)_x, \\ r_t=-\frac{1}{2}\mathrm{i}r_{xx}+\frac{1}{2}(r^2q)_x, \end{matrix} \quad m=2\,. \tag{5.21}$$

They possess the standard Lax operators

$$W_1=I_{2\times 2}\partial, \tag{5.22}$$

$$W_2=\begin{pmatrix} \mathrm{i}\partial^2+\frac{1}{2}qr & -\mathrm{i}\lambda q\partial-\frac{1}{2}\mathrm{i}\lambda q_x \\ -\lambda^{-1}r\partial^2-\frac{1}{2}\lambda^{-1}r_x\partial & -\mathrm{i}\partial^2+\frac{3}{2}qr\partial+\frac{1}{2}q_xr \end{pmatrix}, \tag{5.23}$$

respectively. With $r=q^*$ Eq. (5.21) becomes the well-known derivative Schrödinger equation

$$q_t=\tfrac{1}{2}\mathrm{i}q_{xx}+\tfrac{1}{2}(q|q|^2)_x\,, \tag{5.24}$$

with the standard Lax operator W_2 by substitution of $r=q^*$ in Eq. (5.23).

Apparently, hierarchy (5.19) has the standard Lax representation $L_t=[W_m,L]$, $W_m=\sum_{j=0}^{m-1}V(G_j)L^{m-j-1}$, where $V(G_j)$ is given by (5.16) with $G=G_j=\mathscr{L}^j\cdot(a_1,a_2)^{\mathrm{T}}$, $j\geqslant 0$.

- Let

$$\begin{pmatrix}A_1\\A_2\end{pmatrix}=\begin{pmatrix}2a_1\\2a_2\end{pmatrix},\quad\begin{pmatrix}2\mathrm{i}r\\-2\mathrm{i}q\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}2q_x\\2r_x\end{pmatrix},$$

respectively, where $a_i=a_i(t)\in C^\infty(R)$ $(i=1,2)$ are arbitrarily given. Then the positive order category of (5.14) reads as the following hierarchies:

$$\begin{pmatrix}q\\r\end{pmatrix}_t=J\mathscr{L}^{m-1}\cdot\begin{pmatrix}-\mathrm{i}a_1+r(a_1q^{(-1)}+a_2r^{(-1)})\\\mathrm{i}a_2+q(a_1q^{(-1)}+a_2r^{(-1)})\end{pmatrix}, \tag{5.25}$$

$$\begin{pmatrix}q\\r\end{pmatrix}_t=2\mathrm{i}J\mathscr{L}^{m}\cdot\begin{pmatrix}r^{(-1)}\\-q^{(-1)}\end{pmatrix}, \tag{5.26}$$

$$\begin{pmatrix}q\\r\end{pmatrix}_t=2J\mathscr{L}^{m}\cdot\begin{pmatrix}q\\r\end{pmatrix}. \tag{5.27}$$

They, respectively, have the following representative equations:

$$\begin{aligned}q_t&=(a_1qq^{(-1)}+a_2qr^{(-1)})_x,\\r_t&=(a_1rq^{(-1)}+a_2rr^{(-1)})_x,\end{aligned}\qquad m=1\,, \tag{5.28}$$

$$\begin{aligned}q_t&=\tfrac{1}{2}\mathrm{i}q_{xx}+\tfrac{1}{2}(q^2r)_x,\\r_t&=-\tfrac{1}{2}\mathrm{i}r_{xx}+\tfrac{1}{2}(r^2q)_x,\end{aligned}\qquad m=2\,, \tag{5.29}$$

$$\begin{aligned}q_t&=\mathrm{i}r_{xx}+\tfrac{3}{2}q^2q_x+\tfrac{1}{2}(qr^2)_x,\\r_t&=-\mathrm{i}q_{xx}+\tfrac{3}{2}r^2r_x+\tfrac{1}{2}(rq^2)_x,\end{aligned}\qquad m=1\,. \tag{5.30}$$

Their GLOs are, respectively,

$$W_1=\begin{pmatrix}0&\mathrm{i}\lambda a_2\\0&-qa_1\end{pmatrix}+\begin{pmatrix}a_1q^{(-1)}+a_2r^{(-1)}&0\\\lambda^{-1}a_1&a_1q^{(-1)}+a_2r^{(-1)}\end{pmatrix}\partial\,,$$

$$\begin{aligned}W_2=&\begin{pmatrix}0&\tfrac{1}{2}\lambda q(\mathrm{i}+2qr)\\0&-\tfrac{1}{2}qr-q^2r^2+q_xr\end{pmatrix}+\begin{pmatrix}-\tfrac{1}{2}qr&-\mathrm{i}\lambda q\\\tfrac{1}{2}\lambda^{-1}r(1+2\mathrm{i}qr)&\tfrac{1}{2}qr\end{pmatrix}\partial\\&+\begin{pmatrix}0&0\\-\lambda^{-1}r&0\end{pmatrix}\partial^2\,,\end{aligned}$$

$$W_1 = \begin{pmatrix} 0 & \mathrm{i}\lambda r_x \\ 0 & -qq_x \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(q^2+r^2) & 0 \\ \lambda^{-1} q_x & \frac{1}{2}(q^2+r^2) \end{pmatrix} \partial \, .$$

Obviously, Eq. (5.29) coincides with Eq. (5.21). Therefore, hierarchy (5.26) again reads as the KN hierarchy (5.19), but now it has the GLR $L_t = [W_m, L] + ML^m$ with

$$W_m = \sum_{j=0}^{m-1} V(G_j) L^{m-j-1} \quad \text{and} \quad M = \begin{pmatrix} 0 & -2\lambda q \\ -2\mathrm{i}\lambda^{-1} r\partial & 0 \end{pmatrix} .$$

Here $V(G_j)$ is given by (5.16) with

$$G = G_j = \mathscr{L}^j \cdot \begin{pmatrix} 2\mathrm{i} r^{(-1)} \\ -2\mathrm{i} q^{(-1)} \end{pmatrix}, \quad j \geqslant 0, \quad j \in Z \, .$$

II. Negative Case ($m = -1, -2, \ldots$)

• Let $B_1 = B_2 = 0$, then the negative order generator G_{-1} has the following two seed functions:

$$G^1_{-1} = \tfrac{1}{2} \begin{pmatrix} \mathrm{i} + r^{(-1)} \\ -\mathrm{i} - q^{(-1)} \end{pmatrix} , \tag{5.31}$$

$$G^2_{-1} = \begin{pmatrix} \mathrm{i} a_1 x + \left(r\left(a_1 q^{(-1)} - a_2 r^{(-1)}\right)\right)^{(-1)} \\ -\mathrm{i} a_2 x - \left(q\left(a_1 q^{(-1)} - a_2 r^{(-1)}\right)\right)^{(-1)} \end{pmatrix} , \tag{5.32}$$

where $a_i = a_i(t)$, $i = 1,2$ are two arbitrarily given C^∞-functions. They generate two isospectral ($\lambda_t = 0$) negative order hierarchies of (5.2)

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = J\mathscr{L}^{m+1} \cdot G^k_{-1}, \quad m < 0, \quad m \in Z, \quad k = 1,2 \, , \tag{5.33}$$

which have the standard Lax representation $L_t = [W^k_m, L]$ with $W^k_m = -\sum_{j=m}^{-1} V(G^k_j) L^{m-j-1}$, $k = 1,2$, where $V(G^k_j)$ and L^{-1} are given by (5.16) with $G = G^k_j = \mathscr{L}^{j+1} \cdot G^k_{-1}$ and by (5.7), respectively. Thus, hierarchies (5.33) are integrable.

Eq. (5.33) has the following representative equations

$$\begin{aligned} q_t &= -1 + \mathrm{i} q^{(-1)} - \mathrm{i} q\left(q^{(-1)} + r^{(-1)}\right) - qq^{(-1)} r^{(-1)}, \\ r_t &= -1 + \mathrm{i} r^{(-1)} + \mathrm{i} r\left(q^{(-1)} + r^{(-1)}\right) + rq^{(-1)} r^{(-1)}, \end{aligned} \quad m = -2, \; k = 1 \, , \tag{5.34}$$

and

$$\begin{aligned} q_t &= -\mathrm{i} a_2 - q\left(a_1 q^{(-1)} - a_2 r^{(-1)}\right), \\ r_t &= \mathrm{i} a_1 + r\left(a_1 q^{(-1)} - a_2 r^{(-1)}\right), \end{aligned} \quad m = -1, \; k = 2 \, , \tag{5.35}$$

which can be, respectively, changed to

$$\begin{aligned} Q_{xt} &= -1 + \mathrm{i} Q - \mathrm{i}(Q+R) Q_x - RQQ_x \, , \\ R_{xt} &= -1 + \mathrm{i} R + \mathrm{i}(Q+R) R_x - RQR_x \end{aligned} \tag{5.36}$$

and

$$\begin{aligned}Q_{xt} &= -\mathrm{i}a_2 - Q_x(a_1Q - a_2R)\,,\\ R_{xt} &= \mathrm{i}a_1 + R_x(a_1Q - a_2R)\,,\end{aligned}\tag{5.37}$$

via the transformations $q^{(-1)} = Q$, $r^{(-1)} = R$. Eqs. (5.36) and (5.37) possess the standard Lax operators

$$\begin{aligned}W_{-2} &= -V(G^1_{-2})L^{-1} - V(G^1_{-1})L^{-2}\\ &= \frac{1}{2}\begin{pmatrix}(\mathrm{i}+Q)\partial^{-1}R_x - \mathrm{i}(Q+R) - QR & \mathrm{i}\lambda(\mathrm{i}+Q)\partial^{-1}\\ -\lambda^{-1}(\mathrm{i}+R) & \mathrm{i}(Q+R)+QR\end{pmatrix}\\ &\quad+\frac{1}{4}\begin{pmatrix}\mathrm{i}\partial^{-1} - \partial^{-1}Q_x\partial^{-1}R_x & -\mathrm{i}\lambda\partial^{-1}Q_x\partial^{-1}\\ \lambda^{-1}\partial^{-1}R_x & \mathrm{i}\partial^{-1}\end{pmatrix},\end{aligned}\tag{5.38}$$

$$\begin{aligned}W_{-1} &= -V(G^2_{-1})L^{-1}\\ &= -\frac{1}{2}\begin{pmatrix}a_2\partial^{-1}R_x + a_1Q - a_2R & \mathrm{i}\lambda a_2\partial^{-1}\\ \lambda^{-1}a_1 & -a_1Q + a_2R\end{pmatrix},\end{aligned}\tag{5.39}$$

where L, L^{-1} are given by (5.2) and (5.7) with $q = Q_x$, $r = R_x$, respectively.

- Let $B_1, B_2 \in \mathscr{B}$ $(B_1 \neq 0,\ B_2 \neq 0)$ be two arbitrarily given C^∞-functions, then the negative order generator G_{-1} defined by (5.13) generates the negative order category of (5.2)

$$\begin{pmatrix}q\\ r\end{pmatrix}_t = J\mathscr{L}^{m+1}\cdot G_{-1}, \quad m < 0,\ m \in Z\,,\tag{5.40}$$

where J, $\mathscr{L}^{-1}$ are given by (5.4) and (5.10), respectively. With $m = -1$, Eq. (5.40) reads

$$\begin{aligned}q_t &= -2\mathrm{i}B_2^{(-1)} - 2q\left(qB_1^{(-1)} - rB_2^{(-1)}\right)^{(-1)},\\ r_t &= 2\mathrm{i}B_1^{(-1)} + 2r\left(qB_1^{(-1)} - rB_2^{(-1)}\right)^{(-1)},\end{aligned}\tag{5.41}$$

which becomes a pair of differential equations

$$\begin{aligned}\left(\frac{Q_x}{B_1^{(-1)}}\right)_t &= -2\mathrm{i}B_2^{(-1)} - 2\frac{Q_x}{B_1^{(-1)}}(Q-R)\,,\\ \left(\frac{R_x}{B_2^{(-1)}}\right)_t &= 2\mathrm{i}B_1^{(-1)} + 2\frac{R_x}{B_2^{(-1)}}(Q-R)\,,\end{aligned}\tag{5.42}$$

via the transformations $q = Q_x/B_1^{(-1)}$, $r = R_x/B_2^{(-1)}$. Eq. (5.42) has the GLR $L_t = [W_{-1}, L] + M$ with

$$W_{-1} = \begin{pmatrix}R - Q - B_2^{(-1)}\partial^{-1}\frac{R_x}{B_2^{(-1)}} & -\mathrm{i}\lambda B_2^{(-1)}\partial^{-1}\\ -\lambda^{-1}B_1^{(-1)} & Q - R\end{pmatrix},\tag{5.43}$$

$$M = \begin{pmatrix} -\mathrm{i}B_2\partial^{-1}\frac{R_x}{B_2^{(-1)}} & \lambda B_2\partial^{-1} \\ \lambda^{-1}\left(\mathrm{i}B_1 + \frac{B_2R_x}{B_2^{(-1)}}\partial^{-1}\frac{R_x}{B_2^{(-1)}}\right) & \mathrm{i}\frac{B_2R_x}{B_2^{(-1)}}\partial^{-1} \end{pmatrix}, \tag{5.44}$$

where L is given by (5.2) with $q = Q_x/B_1^{(-1)},\ r = R_xB_2^{(H)}$.

In particular, with $B_1,\ B_2 = const$. Eq. (5.42) becomes

$$\begin{aligned} Q_{xt} &= -2Q_x(Q-R) - 2\mathrm{i}B_1B_2x^2\,, \\ R_{xt} &= 2R_x(Q-R) + 2\mathrm{i}B_1B_2x^2\,. \end{aligned} \tag{5.45}$$

With $B_1B_2 = 1,\ Q = R^*$ ($*$ stands for the complex conjugate representation), Eq. (5.45) reads as a simple nonlinear equation

$$R_{xt} = -4\mathrm{i}R_x ImR + 2\mathrm{i}x^2 \tag{5.46}$$

which has the GLR $L_t = [W_{-1}, L] + M$ with

$$W_{-1} = \begin{pmatrix} -x\partial^{-1}\frac{R_x}{x} + 2\mathrm{i}ImR & -\mathrm{i}\lambda B_2x\partial^{-1} \\ -\lambda^{-1}B_1x & -2\mathrm{i}ImR \end{pmatrix},$$

$$M = \begin{pmatrix} -\mathrm{i}\partial^{-1}\frac{R_x}{x} & \lambda B_2\partial^{-1} \\ \lambda^{-1}B_1(\mathrm{i} + \frac{R_x}{x}\partial^{-1}\frac{R_x}{x}) & \mathrm{i}\frac{R_x}{x}\partial^{-1} \end{pmatrix},$$

where L is given by (5.2) with $q = R_x^*/B_1x,\ r = R_x/B_2x,\ ImR = \frac{1}{2}\mathrm{i}(R^* - R)$. Of course, if we choose different functions $B_1,\ B_2$, then we shall still have other negative order hierarchies of NLEEs.

By Section 3, the set of all Manakov operator pairs for every hierarchy produced with different M in the KN category (5.14) is $\overline{(W_m, ML^m)}$, which is defined according to Definition 3.1. Here, $L,\ W_m$, and M are given by (5.2), (5.18), and (5.17), respectively.

By Section 4, the set of all LM operator pairs for every equation in a fixed hierarchy (i.e., produced with a fixed M) of the KN category (5.15) is $\overline{(W_m, L^m)}$, which is defined through (4.6). Here, L and W_m are given by (5.2) and (5.18), respectively.

Acknowledgements

The first author (Zhijun Qiao) would like to express his sincere thanks to the Fachbereich 17 of the University-GH Kassel, in particular to Prof. Strampp and Prof. Varnhorn for their warm invitation and hospitality. This work has been supported by the Alexander von Humboldt Foundation, Germany; the Special Grant of Chinese National Excellent Ph.D. Dissertation; the Chinese National Basic Research Project "Nonlinear Science".

References

[1] V.E. Zakharov, S.V. Manakov, S.P. Novikov, L.P. Pitaevsky, Soliton Theory: The Method of Inverse Problem, Nauka, Moscow, 1980.

[2] M.J. Ablowitz, H. Segur, Solitons and Inverse Scattering Transform, SIAM, Philadelphia, PA, 1981.
[3] V.E. Zakharov, A.B. Shabat, Funct. Anal. Pril. 8 (1974) 43.
[4] S.V. Manakov, Usp. Mat. Nauk. 31 (1976) 245.
[5] V.E. Zakharov, Lecture Notes in Physics, Springer, Berlin, New York, Vol. 153, 1982, p. 190.
[6] B.G. Konopelchenko, Inv. Prob. 4 (1988) 151.
[7] H.W. Capel, F.W. Nijhoff, Integrable lattice equation, in: A.S. Fokas, V.E. Zakharov (Eds.), Important Developments in Soliton Theory, Springer Lecture Notes in Nonlinear Dynamica, Springer, Berlin, 1993, p. 38.
[8] F.W. Nijhoff, H.W. Capel, G.L. Wiersma, G.R.W. Quispel, Phys. Lett. A 105 (1984) 267.
[9] F.W. Nijhoff, V.G. Papageorgious, H.W. Capel, Integrable time-discrete systems: lattices and mappings, in: P.P. Kulish (Ed.), Quantum Groups, Springer LNM, Vol. 1510, Springer, Berlin, 1992, p. 312.
[10] F.W. Nijhoff, O. Rangnisco, V.B. Kuznetsov, Comm. Math. Phys. 176 (1996) 681.
[11] Z. Qiao, Physica A 243 (1997) 141.
[12] Z. Qiao, Physica A 252 (1998) 377.
[13] Z. Qiao, Acta Math. Appl. Sin. 18 (1995) 287 (preprint 1992).
[14] Z. Qiao, C. Cao, W. Strampp, Category of nonlinear evolution equations, algebraic structure, and r-matrix, preprint, 1999, submitted for publication.
[15] S. Bowman, Math. Proc. Cambridge Philos. Soc. 102 (1987) 173.
[16] J.D. Kaup, A.C. Newell, J. Math. Phys. 19 (1978) 789.