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Decomposition method for the Camassa–Holm equation

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Abstract

The Adomian decomposition method is applied to the Camassa–Holm equation. Approximate solutions are obtained for three smooth initial values. These solutions are weak solutions with some peaks. We plot those approximate solutions and find that they are very similar to the peaked soliton solutions. Also, one single and two anti-peakon approximate solutions are presented. Compared with the existing method, our procedure just works with the polynomial and algebraic computations for the CH equation.

1. Introduction

The generalized shallow water equation—the Camassa–Holm (CH) equation, which was derived physically as a shallow water wave equation by Camassa and Holm in [10], takes the form

$$m_t + m_x u + 2mu_x = 0, \quad m = u - \frac{1}{4}u_{xx} \tag{1.1}$$

where $u = u(x,t)$ represents the horizontal component of the fluid velocity, and $m = u - \frac{1}{4}u_{xx}$ is the momentum variable. The subscripts x, t of u denote the partial derivatives of the function u w.r.t. x, t, for example, $u_t = \partial u/\partial t$, $u_{xxt} = \partial^3 u/\partial^2 x \partial t$, similar notations will be used frequently later in this paper. This equation was first included in the work of Fuchssteiner and Fokas [15] on their theory of hereditary symmetries of soliton equations. As it was shown by Camassa and Holm, Eq. (1.1) describes the unidirectional propagation of two dimensional waves in shallow water over a flat bottom. The solitary waves of Eq. (1.1) regain their shape and speed after interacting nonlinearly with other solitary waves. The most feature of this equation is peaked soliton (called peakon) solution, which is a weak solution with non-smooth property at some points.

The CH equation possesses the bi-Hamiltonian structure, Lax pair and multi-dimensional peakon solutions, and retains higher order terms of derivatives in a small amplitude expansion of incompressible Euler's equations for unidirectional motion of waves at the free surface under the influence of gravity. In 1995, Calogero [9] extended the class of

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mechanical system of this type. Later, Ragnisco and Bruschi [23] and Suris [24], showed that the CH equation yields the dynamics of the peakons in terms of an N-dimensional completely integrable Hamiltonian system. Such kind of dynamical system has Lax pair and an $N \times N$ r-matrix structure [23].

Recently, the algebro-geometric solution of the CH equation and the CH hierarchy arose much more attraction. This kind of solution for most classical integrable PDEs can be obtained by using the inverse spectral transform theory, see Dubrovin [14], Ablowitz and Segur [4], Novikov et al. [19], Newell [18]. This is done usually by adopting the spectral technique associated with the corresponding PDE. Alber and Fedorov [8] studied the stationary and the time-dependent quasi-periodic solution for the CH equation and Dym type equation through using the method of trace formula [7] and Abel mapping and functional analysis on the Riemann surfaces. Constantin and McKean [11] presented the solution of the CH equation on the circle. Later, Alber, Camassa, Fedorov, Holm and Marsden [6] considered the trace formula under the nonstandard Abel-Jacobi equations and by introducing new parameters presented the so-called weak finite-gap piecewise-smooth solutions of the integrable CH equation and Dym type equations. Very recently, Gesztesy and Holden [16], and Qiao [20] discussed the algebro-geometric solutions for the CH hierarchy using polynomial recursion formalism and the trace formula, and constrained method, respectively. Thereafter, Qiao [21] studied an extension version of the CH equation—the DP equation [13], and presented exact solutions by using the constrained method [22].

The present paper provides a different approach to the solutions of the CH equation. The Adomian decomposition method is implemented to solve the Camassa–Holm equation with smooth initial conditions. Numeric algorithm and graphs are analyzed and plotted, respectively. We also compare our solutions with other existing procedures, and find that our approximate solutions are similar to peaked solitons of the CH equation.

2. Adomian decomposition method for Camassa–Holm equation

The Camassa–Holm equation (1.1) for real $u(x,t)$

$$u_t - \frac{1}{4}u_{xxt} + \frac{3}{2}(u^2)_x - \frac{1}{8}(u_x^2)_x - \frac{1}{4}(uu_{xx})_x = 0 \tag{2.2}$$

is written as

$$L_t\left(u - \frac{1}{4}u_{xx}\right) = L_x\left(-\frac{3}{2}(u^2)_x + \frac{1}{8}u_x^2 + \frac{1}{4}uu_{xx}\right) \tag{2.3}$$

where $L_t = \frac{\partial}{\partial t}$ and $L_x = \frac{\partial}{\partial x}$. Then $L_x^{-1}(\cdot) = \int_0^x (\cdot)\,\mathrm{d}x$ and $L_t^{-1}(\cdot) = \int_0^t (\cdot)\,\mathrm{d}t$. After operating the two sides of Eq. (2.3) with L_t^{-1}, we have

$$\begin{aligned} u(x,t) &= u(x,0) - \frac{1}{4}u_{xx}(x,0) + \frac{1}{4}u_{xx} + L_t^{-1}L_x\left(-\frac{3}{2}(u^2)_x + \frac{1}{8}u_x^2 + \frac{1}{4}uu_{xx}\right) \\ &= u(x,0) - \frac{1}{4}u_{xx}(x,0) + \frac{1}{4}u_{xx} + L_t^{-1}(h(u)) = u(x,0) - \frac{1}{4}u_{xx}(x,0) + \frac{1}{4}u_{xx} + \int_0^t h(u(x,s))\,\mathrm{d}s \end{aligned} \tag{2.4}$$

where $h(u)$ denote the differential operator

$$h(u) := L_x\left(-\frac{3}{2}(u^2)_x + \frac{1}{8}u_x^2 + \frac{1}{4}uu_{xx}\right) \tag{2.5}$$

The Adomian decomposition method consists of calculating the solution of Eq. (2.4) in a series form

$$u = \sum_{n=0}^{\infty} u_n \tag{2.6}$$

and the nonlinear term becomes

$$h(u) = \sum_{n=0}^{\infty} A_n \tag{2.7}$$

where A_n are polynomials of $u_0, u_1, \ldots, u_n$ called Adomian's polynomials and are given by

$$\begin{cases} A_0(u_0) = h(u_0) \quad n = 0, \\ A_n(u_0, u_1, \ldots, u_n) = \sum\limits_{\beta_1 + \cdots + \beta_n = n} h^{(\beta_1)}(u_0) \frac{u_1^{(\beta_1 - \beta_2)}}{(\beta_1 - \beta_2)!} \cdots \frac{u_{n-1}^{(\beta_{n-1} - \beta_n)}}{(\beta_{n-1} - \beta_n)!} \frac{u_n^{\beta_n}}{\beta_n!} \quad \text{if } n \neq 0 \end{cases} \tag{2.8}$$

where h is a real function. (See for instance [5,1,2] for more details about the preceded procedure.)

By the use of the relationships shown in the paper of Abbaoui and Cherruault [1], the A_n are determined as follows:

$$\begin{cases} A_0 = h(u_0) \\ A_1 = h^{(1)}(u_0)u_1 \\ A_2 = h^{(1)}(u_0)u_2 + \frac{1}{2}h^{(2)}(u_0)u_1^2 \\ A_3 = h^{(1)}(u_0)u_3 + h^{(2)}(u_0)u_1u_2 + \frac{1}{6}h^{(3)}(u_0)u_1^2 \\ A_4 = h^{(1)}(u_0)u_4 + h^{(2)}(u_0)\left(u_1u_3 + \frac{1}{2}u_2^2\right) + \frac{1}{2}h^{(3)}(u_0)u_1^2u_2 + \frac{1}{24}h^{(4)}(u_0)u_1^4 \\ \vdots \end{cases} \tag{2.9}$$

which recursively generates the formula of u_n:

$$\begin{cases} u_0 = u(x,0) - \frac{1}{4}u_{xx}(x,0) \quad n = 0 \\ u_{n+1} = \frac{1}{4}u_{nxx} + \int_0^t A_n\,\mathrm{d}s \quad \text{if } n \neq 0 \end{cases} \tag{2.10}$$

Following Adomian decomposition methods, we consider the following functional equation:

$$u - w = NL(u) + L(u) \tag{2.11}$$

where u is to be determined approximately in some appropriate functional space S, w is a given element of S, NL and L are a nonlinear operator and a linear operator from a subset X of the functional space S onto itself, respectively. Here, we seek a solution of Eq. (2.11) in the form $u = \sum_{n=0}^{\infty}u_n$. To do so, we approximate the nonlinear operator NL with

$$NL(u) = h(u) = \sum_{n=0}^{\infty} A_n\{u\} \tag{2.12}$$

where the functions A_n's ($n = 0,1,2,\ldots$) are the so-called Adomian's polynomials and determined by

$$\begin{aligned} A_n\{u\} = \frac{1}{n!}\left[\frac{\mathrm{d}}{\mathrm{d}\lambda^n}h(u_\lambda)\right]_{\lambda=0} &= \frac{1}{n!}\left[\frac{\mathrm{d}}{\mathrm{d}\lambda^n}\left[\left(a\left(\sum_{n=0}^{\infty}\lambda^j u_j\right)_x^2 + b\left(\sum_{n=0}^{\infty}\lambda^j u_{jx}\right)^2 + c\left(\sum_{n=0}^{\infty}\lambda^j u_j\right)\left(\sum_{n=0}^{\infty}\lambda^j u_{jxx}\right)\right)_x\right]\right]_{\lambda=0} \\ &= \frac{1}{n!}\sum_{j=0}^{n}\left[\binom{n}{j}j!(n-j)!\left(a(u_{jx}u_{n-j} + u_{(n-j)x}u_j\right)_x + b(u_{jx}u_{(n-j)x})_x + c(u_ju_{(n-j)xx})_x\right)\right] \\ &= a\sum_{j=0}^{n}[u_{jxx}u_{n-j} + 2u_{jx}u_{(n-j)x} + u_{(n-j)xx}u_j] + b\sum_{j=0}^{n}[u_{jxx}u_{(n-j)x} + u_{jx}u_{(n-j)xx}] \\ &\quad + c\sum_{j=0}^{n}[u_{jx}u_{(n-j)xx} + u_ju_{(n-j)xxx}] \end{aligned}$$

where $a = -3/2$, $b = 1/8$, $c = 1/4$ and $u_\lambda = \sum_{i=0}^{\infty}\lambda^i u_i$.

The expected solution $u = \sum_{n=0}^{\infty}u_n$ is approximated by the following m term's sum:

$$\phi_m[u] = \sum_{n=0}^{m-1} u_n \tag{2.13}$$

which rapidly converges u. In this sense, m is able to be chosen as a small number so that this series is convergent to u. This method has been investigated in several authors' work (see [12,1,2] for more details).

As we see, it is not hard to write a program for generating the Adomian polynomials. We summarize the entire procedure in the following algorithm:

Algorithm

- Input: $J(x)$—initial conditions, i.e: $u(x,0) - \frac{1}{4}u_{xx}(x,0) = J(x).k$—number of terms in the approximation
- Output: $u_{\text{approx}}(x,t)$: the approximate solution
  - Step 1: Set $u_0 = J(x)$ and $u_{\text{approx}}(x,t) = u_0$.
  - Step 2: For $k = 0$ to $n-1$, do Step 3, Step 4, and Step 5.

– Step 3: Compute

$$A_k = a\sum_{j=0}^{k}[u_{jxx}u_{k-j} + 2u_{jx}u_{(k-j)x} + u_{(k-j)xx}u_j] + b\sum_{j=0}^{k}[u_{jxx}u_{(k-j)x} + u_{jx}u_{(k-j)xx}] + c\sum_{j=0}^{k}[u_{jx}u_{(k-j)xx} + u_j u_{(k-j)xxx}].$$

– Step 4: Compute

$$u_{k+1} = \frac{1}{4}u_{kxx} + \int_0^t A_k \,\mathrm{d}s \quad \text{if } k \neq 0.$$

– Step 5: Compute $u_{\text{approx}} = u_{\text{approx}} + u_{k+1}$.
– Stop

Remark 2.1. It is not hard to see that the above procedure also works for the following general equation:

$$u_t + au_{xxt} + b(u^2)_x + c(u_x^2)_x + d(uu_{xx})_x = \gamma(x,t) \tag{2.14}$$

where a, b, c, d are real constants and the function γ is sufficiently smooth.

3. Convergence analysis

In this section, we discuss the convergence property of the approximated solution for the CH equation.
Let us consider the CH equation in the Hilbert space $H = L^2((\alpha,\beta)\times[0,T])$:

$$H = \left\{ v : (\alpha,\beta)\times[0,T] \text{ with } \int_{(\alpha,\beta)\times[0,T]} v^2(x,s)\,\mathrm{d}s\,\mathrm{d}\tau < +\infty \right\} \tag{3.15}$$

Then the operator is of the form

$$T(u) = L_t(u + au_{xx}) = -b(u^2)_x - c(u_x^2)_x - d(uu_{xx})_x + \gamma(x,t) \tag{3.16}$$

The Adomian decomposition method is convergent if the following two hypotheses are satisfied:[1]

- (Hyp1): There exists a constant $k > 0$ such that the following inner product holds in H:
$$(T(u) - T(v), u - v) \geqslant k\|u - v\|, \quad \forall u, v \in H \tag{3.17}$$
- (Hyp2): As long as both $u \in H$ and $v \in H$ are bounded (i.e. there is a positive number M such that $\|u\| \leqslant M$, $\|v\| \leqslant M$), there exists a constant $\theta(M) > 0$ such that
$$(T(u) - T(v), u - v) \leqslant \theta(M)\|u - v\|\|w\|, \quad \forall w \in H \tag{3.18}$$

Theorem 3.1 (**Sufficient conditions of convergence for the CH equation**). *Let*

$$T(u) = L_t(u + au_{xx}) = -b(u^2)_x - c(u_x^2)_x - d(uu_{xx})_x + \gamma(x,t), \quad \textit{with } d - c > 0,\ L_t = \frac{\partial}{\partial t}$$

and consider the free initial and boundary conditions for the CH equation. Then the Adomian decomposition method leads to a special solution of the CH equation.

Proof. To prove the theorem, we just verify the conditions (Hyp1) and (Hyp2). For $\forall u, v \in H$, let us calculate:

[1] See Abbaoui and Cherruault [1,2] and some references therein for more details.

$$\begin{aligned}
T(u)-T(v) &= -b(u^2-v^2)_x - c(u_x^2-v_x^2)_x - d(uu_{xx}-vv_{xx})_x \\
&= -b(u^2-v^2)_x - (2c+d)(u_xu_{xx}-v_xv_{xx}) - d(uu_{xxx}-vv_{xxx}) \\
&= -b\frac{\partial}{\partial x}(u^2-v^2) - (2c+d)(u_xu_{xx}-v_xv_{xx}) - \frac{d}{2}\left(\frac{\partial^3}{\partial x^3}(u^2-v^2) - 3\frac{\partial}{\partial x}(u_x^2-v_x^2)\right) \\
&= -b\frac{\partial}{\partial x}(u^2-v^2) - (c-d)\frac{\partial}{\partial x}(u_x^2-v_x^2) - \frac{d}{2}\left(\frac{\partial^3}{\partial x^3}(u^2-v^2)\right)
\end{aligned}$$

Therefore, we have the inner product

$$\begin{aligned}
(T(u)-T(v),u-v) &= b\left(-\frac{\partial}{\partial x}(u^2-v^2),u-v\right) + (c-d)\left(-\frac{\partial}{\partial x}(u_x^2-v_x^2),u-v\right) \\
&\quad + \frac{d}{2}\left(-\frac{\partial^3}{\partial x^3}(u^2-v^2),u-v\right)
\end{aligned} \tag{3.19}$$

Let us assume that u, v are bounded and there is a constant $M>0$ such that (u,u), $(v,v)<M^2$. By using Schwartz inequality

$$\left(\frac{\partial}{\partial x}(u^2-v^2),u-v\right) \leqslant \|(u^2-v^2)_x\|\|u-v\| \tag{3.20}$$

and since there exist θ_1 and θ_2 such that $\|(u-v)_x\| \leqslant \theta_1\|u-v\|$, $\|(u+v)_x\| \leqslant \theta_2\|u-v\|$ and $\|u+v\| \leqslant 2M$, we have

$$\begin{aligned}
&\left(\frac{\partial}{\partial x}(u^2-v^2),u-v\right) \leqslant 2M\theta_1\theta_2\|u-v\|^2 \\
&\iff \\
&\left(-\frac{\partial}{\partial x}(u^2-v^2),u-v\right) \geqslant 2M\theta_1\theta_2\|u-v\|^2
\end{aligned} \tag{3.21}$$

Following the preceding procedure, we can calculate:

$$\begin{aligned}
\left(\frac{\partial}{\partial x}(u_x^2-v_x^2),u-v\right) &\leqslant \|(u_x^2-v_x^2)_x\|\|u-v\| \\
&\leqslant \theta_3\|u_x+v_x\|\|u_x-v_x\|\|u-v\| \\
&\leqslant 2M\theta_3\theta_4\theta_5\|u-v\|^2 \\
\iff & \\
\left(-\frac{\partial}{\partial x}(u_x^2-v_x^2),u-v\right) &\geqslant 2M\theta_3\theta_4\theta_5\|u-v\|^2
\end{aligned} \tag{3.22}$$

where θ_i $(i=3,4,5)$ are positive constants.

Moreover, the Cauchy–Schwartz–Buniakowski inequality yields

$$\left(\frac{\partial^3}{\partial x^3}(u^2-v^2),u-v\right) \leqslant \|(u^2-v^2)_{xxx}\|\|u-v\| \tag{3.23}$$

then by using the mean value theorem, we have

$$\begin{aligned}
\left(\frac{\partial^3}{\partial x^3}(u^2-v^2),u-v\right) &\leqslant \theta_6\theta_7\theta_8\|u^2-v^2\|\|u-v\| \\
&\leqslant 2M\theta_6\theta_7\theta_8\|u-v\|^2 \\
\iff & \\
\left(-\frac{\partial^3}{\partial x^3}(u^2-v^2),u-v\right) &\geqslant 2M\theta_6\theta_7\theta_8\|u-v\|^2
\end{aligned} \tag{3.24}$$

where θ_j $(j=6,7,8)$ are three positive constants, and $\|(u^2-v^2)_{xxx}\| \leqslant \theta_6\|(u^2-v^2)_xx\|$, $\|(u+v)_{xx}\| \leqslant \theta_7\|(u+v)_x\|$ and $\|(u+v)_x\| \leqslant \theta_8\|u+v\|$.

Substituting (3.21), (3.22), (3.24) into (3.19) generates the following inner product:

$$(T(u)-T(v),u-v)=\left(-b\frac{\partial}{\partial x}(u^2-v^2),u-v\right)-(c-d)\left(\frac{\partial}{\partial x}(u_x^2-v_x^2),u-v\right)-\frac{d}{2}\left(\frac{\partial^3}{\partial x^3}(u^2-v^2),u-v\right)$$
$$\geqslant k\|u-v\|^2$$

where $k=(2b\theta_1\theta_2+2(c-d)\theta_3\theta_4\theta_5+d\theta_6\theta_7\theta_8)M$. So, (Hyp1) is true for the CH equation.

Let us now verify the hypotheses (Hyp2) for the operator $T(u)$. We directly compute:

$$(T(u)-T(v),w)=\left(-b\frac{\partial}{\partial x}[u^2-v^2],w\right)-(c-d)\left(\frac{\partial}{\partial x}[u_x^2-v_x^2],w\right)-\frac{d}{2}\left(\left[\frac{\partial^3}{\partial x^3}(u^2-v^2)\right],w\right)\leqslant\theta(M)\|u-v\|\|w\|$$

where $\theta(M)=(-2b+d-2c)M$. Therefore, (Hyp2) is correct as well. □

Remark 3.2. Choice of $b=-3/2$, $c=1/8$, $d=1/4$, $\gamma(x,t)\equiv 0$ corresponds to the CH equation. So, the Adomian decomposition method works for the CH equation.

4. Implementation of the method and approximate solutions

In this section, we take some examples to show the procedure and present some approximate solutions for the CH equation.

Example 4.1

$$\begin{cases} u_t-\frac{1}{4}u_{xxt}+\frac{3}{2}(u^2)_x-\frac{1}{8}(u_x^2)_x-\frac{1}{4}(uu_{xx})_x=0\\ u_0=u(x,0)-\frac{1}{4}u_{xx}(x,0)=c\sinh(x)\end{cases}\tag{4.25}$$

In this case, one straightforwardly gets $u_{0xx}=u_0$, $u_{0x}=c\cosh(x)$, $u_{0x}^2-u_0^2=c^2$ and $h^{(n+1)}(u_0)=(h^{(n)}(u_0))_x/u_{0x}$ where $u_{0x}\neq 0$, $h^{(0)}=h$ and $h^{(n)}$ denotes the nth derivative of h. Since the formula (2.13) implies the formula (2.9), we need the explicit expression of the nth derivative of h. Through direct calculations, we obtain the following formulas:

$$A_0=h(u_0)=-3\left(u_{0x}^2-u_0^2-\frac{1}{4}u_0u_{0x}\right)_x=-\frac{3c^2}{4}(\cosh^2(x)+\sinh^2(x))$$

$$u_1(x,t)=\frac{1}{4}u_{0xx}+\int_0^t h(u_0)\,ds=\frac{c}{4}\sinh(x)-\frac{3c^2}{4}(\cosh^2(x)+\sinh^2(x))t$$

$$\begin{aligned}A_1=u_1h^{(1)}(u_0)&=\left(\frac{1}{4}u_0+A_0t\right)\frac{\left(-3\left(u_{0x}^2-u_0^2-\frac{1}{4}u_0u_{0x}\right)_x\right)_x}{u_{0x}}\\ &=\left(\frac{1}{4}u_0+A_0t\right)\frac{-3\left(2u_{0xx}u_{0x}-2u_{0x}u_0-\frac{1}{4}(u_{0x}u_{0x}+u_0u_{0xx})\right)_x}{u_{0x}}\\ &=3c\left(\frac{c}{4}\sinh(x)-\frac{3c^2}{4}\left(\cosh^2(x)+\sinh^2(x)\right)t\right)\sinh(x)\end{aligned}$$

$$\begin{aligned}u_2(x,t)&=\frac{1}{4}u_{1xx}+\int_0^t A_1\,ds=\frac{c}{16}\sinh(x)-3c^2\left(1+2\sinh^2(x)\right)t+3\left(\frac{c^2}{4}\sinh(x)t+\frac{3c^3}{8}(1+2\sinh^2(x))t^2\right)\sinh(x)\\ &=\frac{c}{16}\sinh(x)-3c^2\left(1+\frac{7}{4}\sinh^2(x)\right)t+\frac{9c^3}{8}\left(\sinh(x)^2+2\sinh^3(x)\right)t^2\end{aligned}$$

$$A_2=u_2h^{(1)}(u_0)+u_1^2h^{(2)}(u_0)=-3u_2u_0-3\left(\frac{c}{4}\sinh(x)+\frac{3c^2}{4}(\cosh^2(x)+\sinh^2(x))t\right)^2$$

$$
\begin{aligned}
u_3(x,t) &= \frac{1}{4}u_{2xx} + \int_0^t A_2\,\mathrm{d}s \\
&= \frac{c}{64}\sinh(x) - \frac{42c^2}{16}(\cosh^2(x)+\sinh^2(x))t + \frac{9c^3}{8}((\sinh(x)^2+\cosh^2(x)) + 2\sinh^3(x) + 12\sinh(x)\cosh^2(x))t^2 \\
&\quad - 3\sinh(x)\left(\frac{c}{16}\sinh(x)t - \frac{3c^2}{2}\left(1+\frac{7}{4}\sinh^2(x)\right)t^2 + \frac{9c^3}{24}(\sinh(x)^2 + 2\sinh^3(x))t^3\right) \\
&\quad - \frac{4}{c^2(1+2\sinh^2(x))}\left(\frac{c}{4}\sinh(x) + \frac{3c^2}{4}(1+2\sinh^2(x))t\right)^3
\end{aligned}
$$

So, the approximate solution, truncated in the second term, is

$$
\begin{aligned}
u(x,t) &\approx u_0 + u_1(x,t) + u_2(x,t) \\
&= \frac{c^3}{2}\left(\frac{585}{8}\cosh(x)\sinh(x)^2 - \frac{9}{2}\sinh(x)^3 - 27\cosh(x)^2\sinh(x)\right)t^2 + \frac{c^3}{2}\left(\frac{585}{8}\cosh(x)^3 - \frac{9}{2}\frac{\cosh(x)^4}{\sinh(x)}\right)t^2 \\
&\quad + \frac{c^2}{2}\left(-\frac{93}{8}\sinh(x)^2 - \frac{93}{8}(\cosh(x)^2 - 3\cosh(x)\sinh(x))\right)t + \frac{21c}{16}\sinh(x)
\end{aligned}
$$

The graph of $u(x,t)$ is plotted in Fig. 1. From the figure, we see that the approximate solution is similar to a single peakon solution of the CH equation.

Example 4.2

$$
\begin{cases}
u_t - \frac{1}{4}u_{xxt} + \frac{3}{2}(u^2)_x - \frac{1}{8}(u_x^2)_x - \frac{1}{4}(uu_{xx})_x = 0 \\
u_0 = u(x,0) - \frac{1}{4}u_{xx}(x,0) = c_1\cosh(x), \quad c_1 = \text{constant.}
\end{cases}
\tag{4.26}
$$

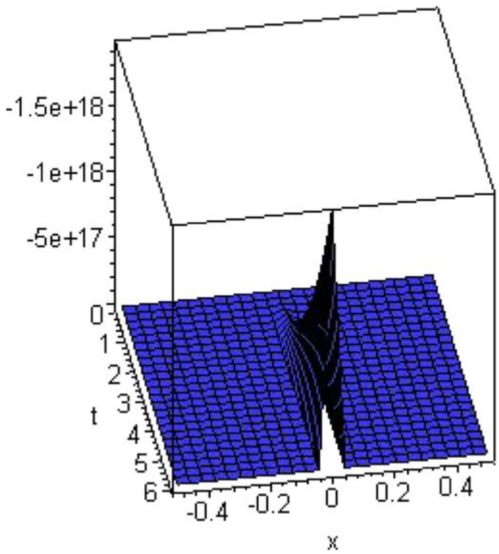

Fig. 1. Approximate solution for $c = 1$.

Let us follow the procedure in Example 4.1 and notice $u_0^2 - u_{0x}^2 = c_1^2$, we obtain the following formulas.

$$
\begin{aligned}
u_1(x,t) &= \frac{1}{4}u_{0xx} + \int_0^t h(u_0)\,\mathrm{d}s \\
&= \frac{c}{4}\cosh(x) - \frac{3c^2}{4}(4(\cosh(x)^2 + \sinh(x)^2) - \cosh(x)\sinh(x))t. \\
u_2(x,t) &= \frac{1}{4}u_{1xx} + \int_0^t A_1\,\mathrm{d}s \\
&= \frac{c^3}{2}\left(\frac{585}{8}\cosh(x)\sinh(x)^2\right)t^2 \\
&\quad + \frac{c^3}{2}\left(-\frac{9}{2}\sinh(x)^3 - 27\cosh(x)^2\sinh(x) + \frac{585}{8}\cosh(x)^3 - \frac{9}{4}\sinh(x)\cosh(x)^4\right)t^2 \\
&\quad + 3c^2\left(\sinh(x)^2 - 2\cosh(x)^2 + \frac{5}{16}\cosh(x)\sinh(x) + \frac{1}{16}\sinh(x)\cosh(x)^3\right)t + \frac{c}{16}\cosh(x)
\end{aligned}
$$

So, the approximate solution corresponding to Eq. (4.26) is

$$
\begin{aligned}
u(x,t) &\approx u_0 + u_1(x,t) + u_2(x,t) \\
&= \frac{c^3}{2}\left(\frac{585}{8}\cosh(x)\sinh(x)^2 - \frac{9}{4}\sinh(x)^3 - 27\cosh(x)^2\sinh(x)\right)t^2 \\
&\quad + \frac{c^3}{2}\left(\frac{585}{8}\cosh(x)^3 - \frac{9}{2}\sinh(x)\cosh(x)^4\right)t^2 \\
&\quad + c^2\left(-6\sinh(x)^2 - 9\cosh(x)^2 + \frac{27}{16}\cosh(x)\sinh(x) + \frac{3}{16\sinh(x)\cosh(x)^3}\right)t + \frac{21c}{16}\cosh(x)
\end{aligned}
$$

The graph of $u(x,t)$ is plotted in Fig. 2, which shows that the approximate solution is similar to a single anti-peakon solution of the CH equation.

Example 4.3

$$
\begin{cases}
u_t - \frac{1}{4}u_{xxt} + \frac{3}{2}(u^2)_x - \frac{1}{8}(u_x^2)_x - \frac{1}{4}(uu_{xx})_x = 0 \\
u_0 = u_{xx}(x,0) - \frac{1}{4}u_{xx}(x,0) = ae^x + be^{-x}
\end{cases}
\tag{4.27}
$$

In this case, by $u_{0xx} = u_0$, $e^x = \frac{u_0+u_{0x}}{2a}$, $e^{-x} = \frac{u_0-u_{0x}}{2b}$ and $u_0^2 - u_{0x}^2 = 4ab$, we obtain those u_j's below

$$
A_0 = -\frac{1}{4}\left(21a^2e^{2x} - 27b^2e^{-2x}\right)
$$

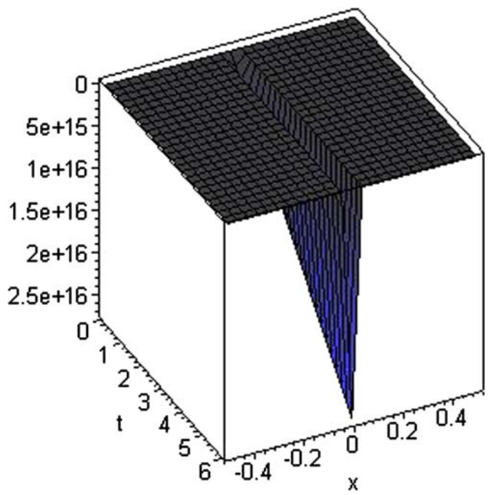

Fig. 2. Approximate solution for $c = 1$.

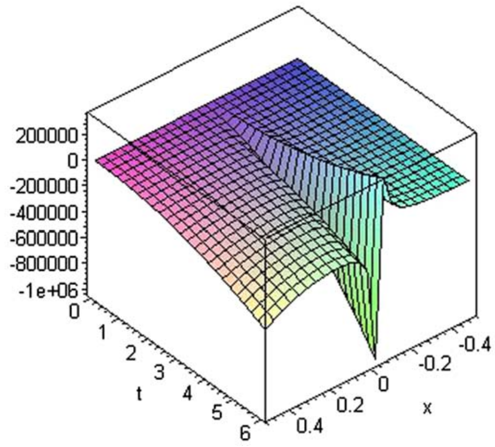

Fig. 3. Approximate solution for $a = -5$, $b = 2$.

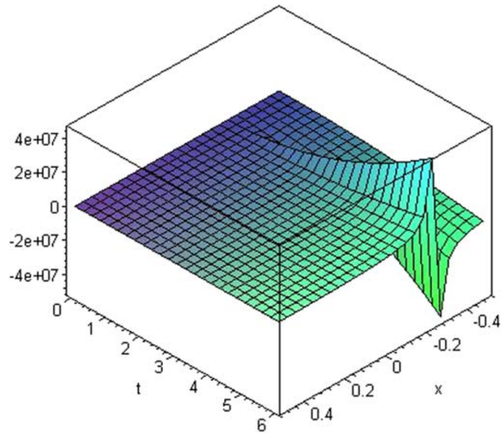

Fig. 4. Approximate solution for $a = -5$, $b = 10$.

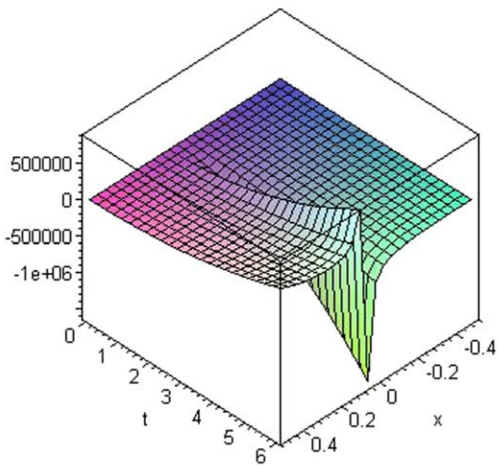

Fig. 5. Approximate solution for $a = 5$, $b = 2$.

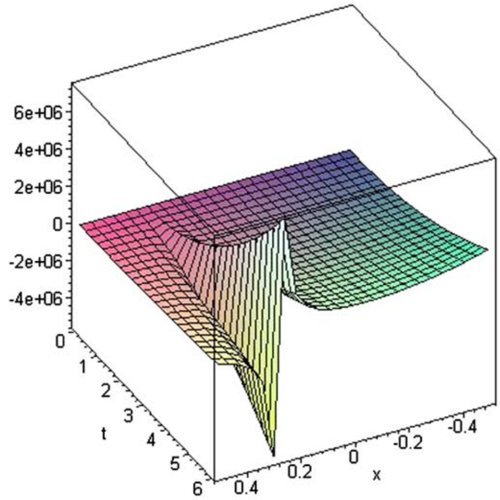

Fig. 6. Approximate solution for $a = 5$, $b = 10$.

$$u_1 = \frac{1}{4}u_{0xx} + \int_0^t A_0\,\mathrm{d}s = \frac{ae^x + be^{-x}}{4} - \frac{1}{4}\left(21a^2e^{2x} + 27b^2e^{-2x}\right)t$$

$$u_2(x,t) = \frac{ae^x + be^{-x}}{16} - \left(\frac{21a^2}{4}e^{2x} + \frac{27b^2}{4}e^{-2x} - \frac{1}{ae^2x - b}\left(\frac{27ab^2}{8} - \frac{21a^2b}{8}e^{2x} + \frac{27b^3}{8}e^{-2x}\right)\right)t + \frac{1}{ae^2x - b}\left(\frac{441}{16}a^4e^{5x} - \frac{729}{16}b^4e^{-3x}\right)t^2$$

So, the approximate solution corresponding to Eq. (4.27) is

$$\begin{aligned} u(x,t) &\approx u_0 + u_1(x,t) + u_2(x,t) \\ &= \frac{21}{16}(ae^x + be^{-x}) - \left(\frac{21a^2}{2}e^{2x} + \frac{27b^2}{2}e^{-2x} - \frac{1}{ae^2x - b}\left(\frac{27ab^2}{8} - \frac{21a^2b}{8}e^{2x} + \frac{27b^3}{8}e^{-2x}\right)\right)t \\ &\quad + \frac{1}{ae^2x - b}\left(\frac{441}{16}a^4e^{5x} - \frac{729}{16}b^4e^{-3x}\right)t^2 \end{aligned}$$

The graphs of $u(x,t)$ for different a's and b's are plotted in Figs. 3–6. Those figures reveal that the approximate solutions are describing the interactions of two anti-peakons for the CH equation.

5. Conclusions

In this paper, we successfully apply the Adomian polynomial decomposition method to solve the CH equation in an explicitly approximate form. The initial values we adopted are smooth, but the most interesting is: the approximate solutions are weak solutions with some peaks (see graphs in Figs. 1–6). The approximate solutions in Figs. 1, 2 show the single peakons of the CH equation, while the approximate solutions in Figs. 3–6 provide the interactions of the two anti-peakons. In comparison with the existing method to obtain two exact anti-peakons, our procedure just works on the polynomial and algebraic computations. In the future, we plan to generalize our method to multi-soliton solutions for the CH equation and other higher order equations. In the recent literatures, there are also other methods to deal with nonlinear partial differential equations [3,17], where smooth solutions were obtained. Our paper presents some peaked (i.e. continuous but non-smooth) explicit solutions for the CH equation (1.1).

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References

[1] Abbaoui K, Cherruault Y. Convergence of Adomian's method applied to nonlinear equations. Math Comput Model 1994;20(9): 60–73.
[2] Abbaoui K, Cherruault Y. New ideas for proving the convergence of decomposition method. Comput Math Appl 1995;29(7): 103–8.
[3] Abdou MA, Soliman AA. Variational iteration method for solving Burger's and coupled Burger's equations. J Comput Appl Math 2005;181:245–51.
[4] Ablowitz MJ, Segur H. Soliton and the inverse scattering transform. Philadelphia: SIAM; 1981.
[5] Adomian G. Solving frontier problems of physics: the decomposition method. Dordrecht/Norwell, MA: Kluwer Academic; 1994.
[6] Alber MS, Camassa R, Fedorov YN, Holm DD, Marsden JE. The complex geometry of weak piecewise smooth solutions of integrable nonlinear PDE's of shallow water and Dym type. Commun Math Phys 2001;221:197–227.
[7] Alber MS, Camassa R, Holm DD, Marsden JE. The geometry of peaked solitons and billiard solutions of a class of integrable PDEs. Lett Math Phys 1994;32:137–51.
[8] Alber MS, Fedorov YN. Wave solution of evolution equations and Hamiltonian flows on nonlinear subvarieties of generalized Jacobians. J Phys A: Math Gen 2000;33:8409–25.
[9] Calogero F. An integrable Hamiltonian system. Phys Lett A 1995;201:306–10.
[10] Camassa R, Holm DD. An integrable shallow water equation with peaked solitons. Phys Rev Lett 1993;71:1661–4.
[11] Constantin A, McKean HP. A shallow water equation on the circle. Comm Pure Appl Math 1999;52:949–82.
[12] Cherruault Y. Convergence of Adomian's method. Kybernetes 1989;18(2):31–8.
[13] Degasperis A, Procesi M. Asymptotic integrability. In: Degasperis A, Gaeta G, editors. Symmetry and perturbation theory. World Scientific; 1999. p. 23–37.
[14] Dubrovin B. Theta-functions and nonlinear equations. Russ Math Surv 1981;36:11–92.
[15] Fuchssteiner B, Fokas AS. Symplectic structures, their Baecklund transformations and hereditaries. Physica D 1981;4:47–66.
[16] Gesztesy F, Holden H. Algebraic–geometric solutions of the Camassa–Holm hierarchy, (private communication). Rev Mat Iberoamer 2003;19:73–142.
[17] He JH. Variational iteration method for autonomous ordinary differential systems. Appl Math Comput 2000;114:115–23.
[18] Newell AC. Soliton in mathematical physics. Philadelphia: SIAM; 1985.
[19] Novikov SP, Manakov SV, Pitaevskii LP, Zakharov VE. Theory of solitons. The inverse scattering method. New York: Plenum; 1984.
[20] Qiao ZJ. The Camassa–Holm hierarchy, N-dimensional integrable systems, and algebro-geometric solution on a symplectic submanifold. Commun Math Phys 2003;239:309–41.
[21] Qiao ZJ. Integrable hierarchy, 3×3 constrained systems, and parametric and stationary solutions. Acta Appl Math 2004;83: 199–220.
[22] Qiao ZJ. Generalized r-matrix structure and algebro-geometric solutions for integrable systems. Rev Math Phys 2001;13:545–86.
[23] Ragnisco O, Bruschi M. Peakons, r-matrix and Toda lattice. Physica A 1996;228:150–9.
[24] Suris YB. A discrete time peakons lattice. Phys Lett A 1996;217:321–9.