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Involutive Solutions and Commutator Representations of the Dirac Hierarchy

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Abstract The Dirac hierarchy of isospectral evolution equations associated with the Dirac spectral problem are studied in this paper. The commutator representations of Dirac hierarchy are first presented, and then through the nonlinearization of Lax pair the involutive solutions of Dirac hierarchy are obtained.

Keywords Dirac hierarchy; commutator representation; involutive system; involutive solution

Consider the Dirac spectral problem

$$L\varphi \equiv L(p,q)\varphi \equiv \begin{pmatrix} -q & p-\partial \\ p+\partial & q \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \lambda\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \partial = \partial/\partial x, \tag{1}$$

where p,q are two potentials, λ is a spectral parameter.

$$L_*(\xi) \triangleq \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} L(u+\varepsilon\xi) = \begin{pmatrix} -\xi_2 & \xi_1 \\ \xi_1 & \xi_2 \end{pmatrix}, \quad u = \begin{pmatrix} p \\ q \end{pmatrix}, \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \tag{2}$$

L_* is an injective homomorphism.

The functional gradient $\nabla\lambda = (2\varphi_1\varphi_2, \varphi_2^2 - \varphi_1^2)^T$ of eigenvalue λ with respect to p,q satisfies

$$K\nabla\lambda = \lambda J\nabla\lambda, \tag{3}$$

where

$$K = \frac{1}{2}\begin{pmatrix} \partial - 4q\partial^{-1}q & 4q\partial^{-1}p \\ 4p\partial^{-1}q & \partial - 4p\partial^{-1}p \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \partial\partial^{-1} = \partial^{-1}\partial = 1 \tag{4}$$

are called the Lenard's operator pair of (1).

Theorem 1 Let $G^{(1)}(x), G^{(2)}(x)$ be two arbitary smooth functions, $G = (G^{(1)}, G^{(2)})^T$. Then the following operator equation with respect to $V = V(G)$,

$$[V, L] = L_*(KG) - L_*(JG)L \tag{5}$$

possesses the operator solution

$$V = V(G) = \begin{pmatrix} -\frac{1}{2}G^{(1)} & -\frac{1}{2}G^{(2)} + \partial^{-1}(pG^{(2)} - qG^{(1)}) \\ -\frac{1}{2}G^{(2)} - \partial^{-1}(pG^{(2)} - qG^{(1)}) & \frac{1}{2}G^{(1)} \end{pmatrix}, \tag{6}$$

where $[\cdot,\cdot]$ is the commutator; $L = L(p,q)$, K, J are defined by (1), (4) respectively.

Proof Substitute (6) into (5), directly calculate.

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Define the Lenard's recursive sequence G_j: $G_0=(p,q)^T$, $KG_j=JG_{j+1}$ $(j=0,1,2,\cdots)$. $X_m(p,q)=JG_m$ $(m=0,1,2,\cdots)$ are called the Dirac vector fields, and $X_m(p,q)$ produce the Dirac hierarchy $(p,q)_t^T=X_m(p,q)$, $m=0,1,2,\cdots$, with the representative equation

$$\begin{pmatrix} p \\ q \end{pmatrix}_t = X_3(p,q) = \frac{1}{8}\begin{pmatrix} -p_{xxx}+6(p^2+q^2)p_x \\ -q_{xxx}+6(p^2+q^2)q_x \end{pmatrix}, \tag{8}$$

which can be reduced to the remarkable MKDV equation $p_t=-\frac{1}{8}p_{xxx}-\frac{3}{4}p^2p_x$ as $q=0$.

Theorem 2 The Dirac hierarchy (7) have the commutator representations

$$L_t=[W_m,L],\quad m=0,1,2,\cdots, \tag{9}$$

where the operator

$$\begin{aligned} W_m &= \sum_{j=1}^{m} V(G_{j-1})L^{m-j} \\ &= \sum_{j=0}^{m}\begin{pmatrix} -\frac{1}{2}G_{j-1}^{(1)} & -\frac{1}{2}G_{j-1}^{(2)}+\partial^{-1}(pG_{j-1}^{(2)}-qG_{j-1}^{(1)}) \\ -\frac{1}{2}G_{j-1}^{(2)}-\partial^{-1}(pG_{j-1}^{(2)}-qG_{j-1}^{(1)}) & \frac{1}{2}G_{j-1}^{(1)} \end{pmatrix}L^{m-j} \end{aligned} \tag{10}$$

is the Lax operator of Dirac vector fields $X_m(p,q)$, i.e. W_m satisfies $[W_m,L]=L_*(X_m)$.

Corollary 1 The potentials p,q satisfy a stationary Dirac system $X_N+\alpha_1X_{N-1}+\cdots+\alpha_NX_0=0$ if and only if $[W_N+\alpha_1W_{N-1}+\cdots+\alpha_NW_0,L]=0$. Here $\alpha_k(k=1,2,\cdots,N)$ are some constants.

Let $\lambda_j(j=1,2,\cdots,N)$ be N different eigenvalues of (1) and $(\varphi_{1j},\varphi_{2j})^T$ the associated eigenfunctions, then

$$\begin{cases} \varphi_{1x}=\Lambda\varphi_2-p\varphi_1-q\varphi_2 \\ \varphi_{2x}=-\Lambda\varphi_1+p\varphi_2-q\varphi_1, \end{cases} \tag{11}$$

where $\varphi_k=(\varphi_{k1},\cdots,\varphi_{kN})^T$, $k=1,2$; $\Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_N)$.

Under the Bargmann constraint[1]

$$\begin{cases} p=-\langle\varphi_1,\varphi_2\rangle \\ q=-\frac{1}{2}(\langle\varphi_2,\varphi_2\rangle-\langle\varphi_1,\varphi_1\rangle), \end{cases} \tag{12}$$

(11) is nonlinearized as a completely integrable Hamiltonian system[2]

$$(H):\quad \begin{cases} \varphi_{1x}=\Lambda\varphi_2+\langle\varphi_1,\varphi_2\rangle\varphi_1+\frac{1}{2}(\langle\varphi_2,\varphi_2\rangle-\langle\varphi_1,\varphi_1\rangle)\varphi_2=\partial H/\partial\varphi_2 \\ \varphi_{2x}=-\Lambda\varphi_1-\langle\varphi_1,\varphi_2\rangle\varphi_2+\frac{1}{2}(\langle\varphi_2,\varphi_2\rangle-\langle\varphi_1,\varphi_1\rangle)\varphi_1=-\partial H/\partial\varphi_1 \end{cases} \tag{13}$$

with the Hamiltonial function

$$\begin{aligned} H=&\frac{1}{2}(\langle\Lambda\varphi_2,\varphi_2\rangle+\langle\Lambda\varphi_1,\varphi_1\rangle)+\frac{1}{2}(\langle\varphi_1,\varphi_2\rangle^2-\langle\varphi_1,\varphi_1\rangle\langle\varphi_2,\varphi_2\rangle) \\ &+\frac{1}{8}(\langle\varphi_1,\varphi_1\rangle+\langle\varphi_2,\varphi_2\rangle)^2 \end{aligned} \tag{14}$$

whose involutive systems are

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$$F_m = \frac{1}{4}(\langle\Lambda^m\varphi_2,\varphi_2\rangle + \langle\Lambda^m\varphi_1,\varphi_1\rangle) + \frac{1}{4}\sum_{j=1}^{m}\begin{vmatrix}\langle\varphi_1,\Lambda^{j-1}\varphi_2\rangle & \langle\varphi_1,\Lambda^{j-1}\varphi_1\rangle\\ \langle\varphi_2,\Lambda^{m-j}\varphi_2\rangle & \langle\varphi_1,\Lambda^{m-j}\varphi_2\rangle\end{vmatrix}, m = 1,2,\cdots, \tag{15}$$

which are generated by the nonlinearlization of the time part $y_{t_m}=W_m y$ of Lax form (9).

Theorem 3 Let $(\varphi_1,\varphi_2)^T$ be a solution of (13), then $p=-\langle\varphi_1,\varphi_2\rangle$, $q=-\frac{1}{2}(\langle\varphi_2,\varphi_2\rangle-\langle\varphi_1,\varphi_1\rangle)$ is the solution of a stationary Dirac equation $X_N+a_1X_{N-1}+\cdots+a_NX_0=0$, where constants $a_k(k=1,2,\cdots,N)$ are suitably chosen.

Lemma 1 $\langle\frac{\partial F_m}{\partial\varphi_1},\frac{\partial F_n}{\partial\varphi_2}\rangle=\langle\frac{\partial F_n}{\partial\varphi_1},\frac{\partial F_m}{\partial\varphi_2}\rangle$, $\forall\ m,n.$ (16)

Theorem 4

1) F_m defined by (15) are involutive in pairs, i. e. $(F_k,F_l)=0, \forall\ k,l.$

2) $(H,F_m)=0, \forall\ m.$ Here $(\cdot,\cdot)$ stands for the standard Poisson bracket[3] in the symplectic space $(R^{2N},\mathrm{d}\varphi_2\wedge\mathrm{d}\varphi_1)$, *i. e.*

$$(F,G) = \sum_{j=1}^{N}\left(\frac{\partial F}{\partial\varphi_{1j}}\frac{\partial G}{\partial\varphi_{2j}} - \frac{\partial F}{\partial\varphi_{2j}}\frac{\partial G}{\partial\varphi_{1j}}\right) = \langle\frac{\partial F}{\partial\varphi_1},\frac{\partial G}{\partial\varphi_2}\rangle - \langle\frac{\partial F}{\partial\varphi_2},\frac{\partial G}{\partial\varphi_1}\rangle\ . \tag{17}$$

Theorem 5 The Hamiltonian systems $(R^{2N},\mathrm{d}\varphi_2\wedge\mathrm{d}\varphi_1,F_m)$ defined by (15), and $(R^{2N},\mathrm{d}\varphi_2\wedge\mathrm{d}\varphi_1,H)$ are completely integrable in the Liouville sense.

Consider the canonical equation of F_m-flow

$$(F_m):\quad \frac{\partial}{\partial t_m}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix}\partial F_m/\partial\varphi_2\\ -\partial F_m/\partial\varphi_1\end{pmatrix}\ . \tag{18}$$

Denote the solution operator of its initial-value problem by $g_m^{t_m}$. Since $(F_k,F_l)=0$, the Hamiltonain systems (F_k) and (F_l) are compatible, and their phase-flows $g_k^{t_k}, g_l^{t_l}$ commute. Denote the flow variables of $(H),(F_m)$ by x,t_m respcetively. Define

$$\begin{pmatrix}\varphi_1(x,t_m)\\ \varphi_2(x,t_m)\end{pmatrix} = g_0^x g_m^{t_m}\begin{pmatrix}\varphi_1(0,0)\\ \varphi_2(0,0)\end{pmatrix}\ , \tag{19}$$

which is called the involutive solution of consistent equations (H) and (F_m).

Theorem 6 Let $(\varphi_1(x,t_m),\varphi_2(x,t_m))^T$ be an involutive solution of consistent equations (H) and (F_m). Then $p(x,t_m)=-\langle\varphi_1,\varphi_2\rangle$, $q(x,t_m)=-\frac{1}{2}(\langle\varphi_2,\varphi_2\rangle-\langle\varphi_1,\varphi_2\rangle)$ satisfy a m-th Dirac equation

$$(p,q)^T_{t_m} = X_m + c_1X_{m-1} + \cdots + c_mX_0, m = 1,2,\cdots, \tag{20}$$

where constants $c_j(j=1,2,\cdots,m)$ are independent of x.

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