In: Discrete and Computational Mathematics
Editor: G. M. N'Guerekata, F. Liu, pp. 343-353
ISBN 978-1-60021-810-1

Chapter 24

R-MATRIX STRUCTURE AND A NEW INTEGRABLE PEAKON EQUATION

Zhijun Qiao*
Department of Mathematics
The University of Texas – Pan American
1201 W University Dr, Edinburg, TX 78541, USA

Abstract

We use the r-matrix formulation to show the integrability of geodesic flow on an N-dimensional space with coordinates q_k, with $k = 1, \ldots, N$, equipped with the co-metric $g^{ij} = e^{-|q_i - q_j|}\left(2 - e^{-|q_i - q_j|}\right)$. This flow is canonically generated by a quadratic conserved quantity of the integrable partial differential equation $m_t + um_x + 3mu_x = 0$, $m = u - \alpha^2 u_{xx}$ (α is a constant). The isospectral eigenvalue problem associated with this equation is used to find a Lax matrix for the geodesic flow. By employing this Lax matrix we obtain its dynamical r-matrix.

2000 Mathematics Subject Classification. 35Q53, 58F07, 35Q35.
Key words and phrases. Peakon equation, Hamiltonian, Lax matrix, r-matrix structure.

1 Introduction

The $b = 3$ Peakon Equation and Its Isospectral Problem

We begin with the case $b = 3$ of the $b-$weighted peakon equation. This is the evolutionary equation defined on the real line as,

$$m_t + um_x + bmu_x = 0\,, \quad m = u - \alpha^2 u_{xx}\,, \quad \lim_{|x|\to\infty} m = 0\,, \tag{1.1}$$

in which the subscripts denote partial derivatives with respect to the independent variables x and t. For any values of the dimensionless constant $b > 1$ and constant lengthscale α, this

*E-mail address: qiao@utpa.edu

equation admits stable exact N-peakon solutions

$$u(x,t)=\sum_{i=1}^{N} p_i(t)\, e^{-|x-q_i(t)|/\alpha}\,, \tag{1.2}$$

in which the $2N$ time-dependent functions $p_i(t)$ and $q_i(t)$, $i=1,2,\dots,N$, satisfy a system of ordinary differential equations whose character depends on the value of the bifurcation parameter b. The case $b=2$ in (1.1) is the dispersionless limit of the integrable Camassa-Holm (CH) equation that was discovered for shallow water waves in [2]. Later, the CH equation is generalized to the CH integrable hierarchy in [6], which presents the relationship between the CH hierarchy of PDEs and finite-dimensional integrable systems and gives algebraic-geometric solutions on some symplectic sub-manifold. A general idea is provided to generate the integrable hierarchy in [8]. Very recently, we found new cusp and smooth soliton solutions for the CH equation [10].

An equation equivalent to the case $b=3$ of the peakon equation (1.1) was singled out for special attention among a family of related equations by Degasperis and Procesi in [3]. The peakon equation (1.1) was shown to be completely integrable for the case $b=3$ in [4]. The Lax pair consists of a third order eigenvalue problem and a second-order evolutionary equation for the eigenfunction,

$$\psi_{xxx} = \frac{1}{\alpha^2}\psi_x-\lambda m\psi, \tag{1.3}$$

$$\psi_t = -\frac{1}{\alpha^2\lambda}\psi_{xx}-u\psi_x+(u_x+\frac{2}{3\lambda})\psi. \tag{1.4}$$

Compatibility $\psi_{xxxt}=\psi_{txxx}$ implies Eq. (1.1) with $b=3$ provided $d\lambda/dt=0$. Thus, equation (1.1) with $b=3$, namely, the Degasperis-Procesi (DP) equation [3]

$$m_t+um_x+3\,mu_x=0\,,\quad m=u-\alpha^2u_{xx}\,, \tag{1.5}$$

was shown to be integrable by the inverse spectral transform for the isospectral eigenvalue problem (1.3) and to possess an infinite sequence of conservation laws. Recently, the DP equation is extended to two different integrable hierarchies, which are restricted to 3×3 constrained finite-dimensional integrable systems and have exact solutions in parametric form [7, 9]. Similar to the CH nonlocal solitons, we also found new peakons, cuspon and smooth soliton solutins for the DP equation [12].

Henceforth, we shall set $\alpha=1$ for the sake of simplicity in notation (The length scale α is easily restored by using dimensional considerations.)

The first few conservation laws are given in the notation of [4] with $\alpha=1$ by

$$\begin{array}{rclrcl} H_{-1} &=& \frac{1}{6}\int u^3\,dx\,, & H_0 &=& \int m\,dx\,,\\ H_1 &=& \frac{1}{2}\int(v_{xx}^2+5v_x^2+4v^2)\,dx\,, & H_5 &=& \int m^{1/3}\,dx\,. \end{array} \tag{1.6}$$

We shall pay special attention to the quadratic conservation law H_1, in which the quantity v is defined as

$$v:=(4-\partial_x^2)^{-1}u\equiv(4-\partial_x^2)^{-1}(1-\partial_x^2)^{-1}m\,. \tag{1.7}$$

Lax Matrix for N-Peakon Dynamics

Substituting the N-peakon solution,

$$u(x,t) = \sum_{j=1}^{N} p_j(t)e^{-|x-q_j(t)|}, \quad m(x,t) = 2\sum_{j=1}^{N} p_j(t)\delta(x-q_j(t)), \tag{1.8}$$

into the isospectral eigenvalue problem (1.3) yields

$$\frac{1}{\lambda}\psi(x,t) = \frac{1}{2}\sum_{j=1}^{N}\left[1+sgn(x-q_j(t))\left(1-e^{-|x-q_j(t)|}\right)\right]p_j\psi(q_j(t))\,. \tag{1.9}$$

Setting $\psi(q_i(t),t) = \psi_i(t)$ then gives the following matrix eigenvalue problem,

$$\frac{2}{\lambda}\psi_i = \sum_{j=1}^{N}\tilde{L}_{ij}\psi_j, \tag{1.10}$$

where

$$\tilde{L}_{ij} = \left[1+sgn(q_i-q_j)\left(1-e^{-|q_i-q_j|}\right)\right]p_j\,. \tag{1.11}$$

Let $\tilde{L}$ denote the $N\times N$ matrix $\tilde{L}_{ij}$. One can use the two conserved quantities $\mathrm{tr}\tilde{L}$ and $\mathrm{tr}\tilde{L}^2$ to solve the 2-peakon subdynamics of the the N-peakon dynamics q_k, p_k, with $k=1,\ldots,N$, $\alpha=1$, satisfying

$$\begin{array}{rcl} \dot{p}_j & = & 2\sum_{k=1}^{N}p_j p_k\, sgn(q_j-q_k)e^{-|q_j-q_k|}, \\ \dot{q}_j & = & \sum_{k=1}^{N}p_k e^{-|q_j-q_k|}. \end{array} \tag{1.12}$$

Two-peakon interaction has explicit formulas for its phase shifts as functions of the asymptotic speeds [4].

2 A Geodesic Pulson Flow Related to $B=3$ Peakons

The quantity used for determining the two-peakon $N=2$ collision laws,

$$H_1 = \frac{1}{2}\mathrm{tr}\tilde{L}^2 = \frac{1}{2}\sum_{i,j=1}^{N}p_i p_j e^{-|q_i-q_j|}\left(2-e^{-|q_i-q_j|}\right), \tag{2.1}$$

is also the quadratic conservation law H_1 in (1.6) for the $b=3$ peakon equation (1.5), when H_1 is evaluated on the $N-$peakon solution (1.8).

The dynamics (1.12) is a non-canonical Hamiltonian system. However, the canonical Hamiltonian dynamics generated by H_1 is geodesic motion on an N-dimensional space with co-metric $g^{ij} = e^{-|q_i-q_j|}\left(2-e^{-|q_i-q_j|}\right)$. As we shall show by finding its $r-$matrix structure in the remainder of the present paper, the geodesic motion canonically generated by the conservation law $H_1 = Tr\tilde{L}^2$ in (2.1) provides a new $2N$-dimensional integrable system,

$$\dot{q}_k = \frac{\partial H_1}{\partial p_k} = \sum_{j=1}^{N}p_j e^{-|q_k-q_j|}\left(2-e^{-|q_k-q_j|}\right), \tag{2.2}$$

$$\dot{p}_k = -\frac{\partial H_1}{\partial q_k} = -2p_k\sum_{j=1}^{N}p_j sgn(q_j-q_k)e^{-|q_k-q_j|}\left(1-e^{-|q_k-q_j|}\right). \tag{2.3}$$

We emphasize that these canonical geodesic H_1-dynamics for p_k, q_k, are not the same as the non-canonical $N-$peakon dynamics in (1.12). Rather, this is the geodesic flow for the dynamics of "pulson solutions" associated with the Hamiltonian H_1 in (1.6). In fact, the canonical geodesic flow (2.2-2.3) is obtained by substituting the $N-$pulson solution

$$v(x,t) = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|}\left(2 - e^{-|x-q_j(t)|}\right) \tag{2.4}$$

into the 5th-order evolutionary equation defined on the real line by,

$$m_t = -vm_x - 2mv_x = -(m\partial_x + \partial_x m)\frac{\delta H_1}{\delta m}, \quad m = (4-\partial_x^2)(1-\partial_x^2)v. \tag{2.5}$$

Being in the pulson family of integral PDEs, the initial value problem for (2.5) is dominated by the pulson solutions (2.4), as shown in Figure 1. Moreover, these pulsons scatter elastically amongst themselves, as shown in Figure 2. The remainder of the paper is devoted to showing integrability of the canonical dynamics in (2.2) and (2.3) produced by Hamiltonian (2.1).

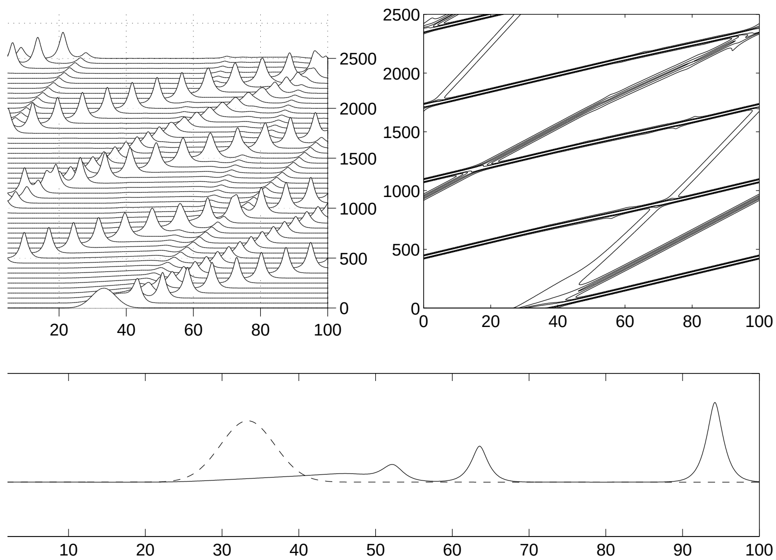

Figure 1: Pulson solutions (2.4) of equation (2.5) emerge from a Gaussian of unit area and width $\sigma = 5$ centered about $x = 33$ on a periodic domain of length $L = 100$. The fastest pulson crosses the domain four times and collides elastically with the slower ones

R-Matrix Results for the Geodesic H_1-Dynamics

To show integrability of the canonical dynamics in (2.2) and (2.3) produced by Hamiltonian (2.1), we shall use the r-matrix method, which reveals that trL^n for a certain Lax matrix L are constants in involution [1]. Here L is the Lax matrix for the geodesic pulson dynamics in (2.2-2.3). We emphasize that the two dynamical systems are completely different and the geodesic pulson flow (2.2-2.3) does not commute with the peakon dynamics (1.12).

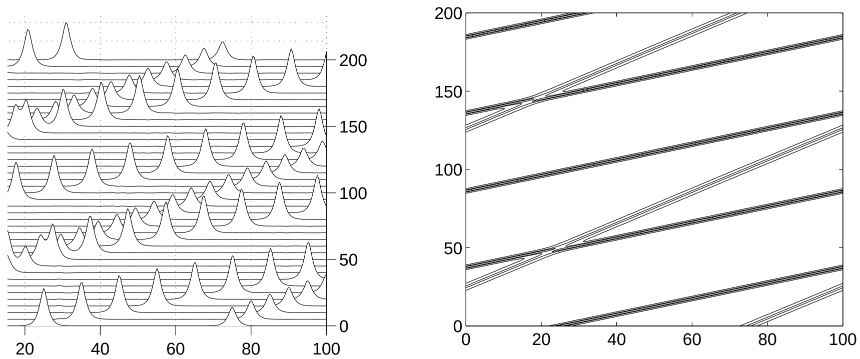

Figure 2: Two rear-end collisions of pulson solutions (2.4) of equation (2.5). The initial positions are $x = 25$ and $x = 75$. The faster pulson moves at twice the speed of the slower one. For this ratio of speeds, both collisions result in a phase shift to the right for the faster space-time trajectory, but no phase shift for the slower one

To find the r-matrix structure for these canonical H_1-dynamics for p_k, q_k, we shall start with a Lax matrix in [8] for the peakon dynamics of Eqn. (1.5),

$$L = \sum_{i,j=1}^{N} L_{ij} E_{ij}, \tag{2.6}$$

where

$$L_{ij} = \sqrt{p_i p_j} A_{ij}, \tag{2.7}$$

$$A_{ij} = A(q_i - q_j) = \sqrt{(2 - e^{-|q_i - q_j|}) e^{-|q_i - q_j|}}. \tag{2.8}$$

The Lax matrix (2.6) also satisfies,

$$H_1 = \frac{1}{2} \mathrm{tr} L^2 = \frac{1}{2} \sum_{i,j=1}^{N} p_i p_j e^{-|q_i - q_j|} \left(2 - e^{-|q_i - q_j|}\right), \tag{2.9}$$

which is the Hamiltonian for the canonical dynamics in Eqs. (2.2) and (2.3). In Eq. (2.8), we have

$$A(x) = \sqrt{(2 - e^{-|x|}) e^{-|x|}}, \tag{2.10}$$

and the function $A(x)$ satisfies the following relations,

$$A'(x) = -sgn(x) \frac{1 - e^{-|x|}}{2 - e^{-|x|}} A(x), \tag{2.11}$$

$$A_{ij} = A_{ji}, \ \ A_{ii} = 1, \tag{2.12}$$

$$A'_{ij} = A'(q_i - q_j) = -A'(q_j - q_i) = -A'_{ji}, \ \ A'_{ii} = 0, \tag{2.13}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) A(x) A(y) = A'(x) A(y) + A(x) A'(y)$$

$$= -A(x)A(y)\left[sgn(x)\frac{1-e^{-|x|}}{2-e^{-|x|}} + sgn(y)\frac{1-e^{-|y|}}{2-e^{-|y|}}\right], \quad (2.14)$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)A(x)A(-x) = 0. \quad (2.15)$$

We shall work in the canonical matrix basis E_{ij},

$$(E_{ij})_{kl} = \delta_{ik}\delta_{jl}, \quad i,j,k,l = 1,\dots,N.$$

To find the r-matrix structure for the H_1-dynamics in Eqs. (2.2) and (2.3), we consider the so-called fundamental Poisson bracket [5]:

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2], \quad (2.16)$$

where

$$\begin{aligned}
L_1 &= L\otimes \mathbf{1} = \sum_{i,j=1}^{N} L_{ij}E_{ij}\otimes\mathbf{1},\\
L_2 &= \mathbf{1}\otimes L = \sum_{k,l=1}^{N} L_{kl}\mathbf{1}\otimes E_{kl},\\
r_{12} &= \sum_{i,j,s,t}^{N} r_{ij;st}E_{ij}\otimes E_{st},\\
r_{21} &= \sum_{i,j,s,t}^{N} r_{ij;st}E_{st}\otimes E_{ij},\\
\{L_1,L_2\} &= \sum_{i,j,k,l=1}^{N}\{L_{ij},L_{kl}\}E_{ij}\otimes E_{kl}.
\end{aligned}$$

Here $\{L_{ij}, L_{kl}\}$ is the standard Poisson bracket of two functions, $\mathbf{1}$ is the $N\times N$ unit matrix, and the quantities $r_{ij;st}$ are to be determined. In Eq. (2.16), $[\cdot,\cdot]$ denotes the usual commutator of matrices.

After a lengthy calculation for both sides of Eq. (2.16), we obtain the following key equalities (whose detailed verification is given in the Appendix):

$$\begin{aligned}
[r_{ll}, L]_{ll} &= 0,\\
[r_{jj}, L]_{ll} &= [r_{ll}, L]_{jj},\ j\neq l,\\
\left[r_{jl}, L\right]_{lj} &= \left[r_{lj}, L\right]_{lj} = 0,\\
[r_{ll}, L]_{lj} &= [r_{ll}, L]_{jl} = -\sqrt{p_j p_l}A'_{jl},\\
\left[r_{lj}, L\right]_{ll} &= \left[r_{jl}, L\right]_{ll} = -\sqrt{p_j p_l}A'_{jl},\\
[r_{ll}, L]_{kj} &= \left[r_{jl}, L\right]_{kk} = 0,\ j\neq l,k; k\neq l,\\
\left[r_{jl}, L\right]_{lk} &= \left[r_{lj}, L\right]_{kl} = \frac{1}{2}\sqrt{p_k p_j}(A_{jl}A_{lk})',\ j\neq l,k; k\neq l,\\
\left[r_{jl}, L\right]_{kl} &= \left[r_{lj}, L\right]_{lk} = \frac{1}{2}\sqrt{p_k p_j}(A_{jl}A_{lk})',\ j\neq l,k; k\neq l,
\end{aligned}$$

$$[r_{st}, L]_{jl} \quad = \quad 0, \quad \text{for different} \quad s, t, j, l.$$

where $r_{jl} = \sum_{k,m} r_{km;jl} E_{km}$, $r_{ll} = \sum_{k,m} r_{km;ll} E_{km}$, are two $N \times N$ matrices whose entries are to be determined, L is the Lax matrix, and $[\cdot, L]_{kl}$ stands for the k-th row and the l-th colum element of $[\cdot, L]$.

In matrix notation, all the above equalities can be rewritten as

$$\begin{aligned} [r_{jl}, L] &= B^{jl}, \quad j \neq l, && (2.17) \\ [r_{ll}, L] &= B^{ll}, && (2.18) \end{aligned}$$

where B^{jl}, B^{ll} are the following two $N \times N$ matrices:

$$B^{jl} = \begin{pmatrix} 0 & \cdots & \frac{1}{2}\sqrt{p_1 p_l}(A_{lj}A_{j1})' & \cdots & \frac{1}{2}\sqrt{p_1 p_j}(A_{jl}A_{l1})' & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2}\sqrt{p_1 p_l}(A_{lj}A_{j1})' & \cdots & -\sqrt{p_j p_l}A'_{lj} & \cdots & 0 & \cdots & \frac{1}{2}\sqrt{p_N p_l}(A_{lj}A_{jN})' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{2}\sqrt{p_1 p_j}(A_{jl}A_{l1})' & \cdots & 0 & \cdots & -\sqrt{p_j p_l}A'_{jl} & \cdots & \frac{1}{2}\sqrt{p_N p_j}(A_{jl}A_{lN})' \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{2}\sqrt{p_N p_l}(A_{lj}A_{jN})' & \cdots & \frac{1}{2}\sqrt{p_N p_j}(A_{jl}A_{lN})' & \cdots & 0 \end{pmatrix},$$

and

$$B^{ll} \quad = \quad \begin{pmatrix} 0 & \cdots & -\sqrt{p_1 p_l}A'_{1l} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ -\sqrt{p_1 p_l}A'_{1l} & \cdots & 0 & \cdots & -\sqrt{p_N p_l}A'_{Nl} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\sqrt{p_N p_l}A'_{Nl} & \cdots & 0 \end{pmatrix}.$$

By solving Eqs. (2.17) and (2.18), we have the following r-matrix structure:

$$\begin{aligned} r_{12} \quad = \quad & \sum_{j,l=1}^{N} \left(\frac{A'_{lj}}{A_{lj}} E_{jl} \otimes (E_{jl} + E_{lj}) + \frac{A'_{lj}}{A_{lj}} E_{ll} \otimes E_{jj} \right) \\ & + \frac{1}{2} \sum_{j,k,l=1}^{N} \sqrt{\frac{p_k}{p_j}} \left(\frac{A'_{kj}A_{kl}}{A_{kj}A_{lj}} + \frac{(A_{jk}A_{lj})'}{A_{lj}} \right) E_{ll} \otimes E_{jk}. \end{aligned} \qquad (2.19)$$

Perhaps not unexpectedly, this non-contant r-matrix for the geodesic H_1-dynamics differs from the constant r-matrix associated with the CH equation ($b = 2$) discovered by Ragnisco and Bruschi in [11]. According to the r-matrix method [1], the auxiliary matrix M corresponding to L may be also constructed. But, in our case the r-matrix is dynamical, and therefore the construction of M is a rather complicated process, which we defer to another time.

Concluding Remarks

In this paper, we found the r-matrix formulation for the integrable geodesic motion generated canonically by the quadratic quantity H_1 in (2.1). This quantity arises by restriction to the peakon sector of a quadratic conservation law in the hierarchy of integrable equations associated with the isospectral problem for the 1+1 integrable partial differential equation (1.5). This equation was singled out in [3] and was proven to be completely integrable by the isospectral transform in [4]. Remarkably, the quantity H_1 appears as a conservation law in two different finite-dimensional integrable dynamical systems, namely, equations (1.12) and (2.2-2.3). We emphasize that these two systems are quite different: one is non-canonical peakon dynamics; the other is canonical geodesic flow for pulsons. However, the two systems have the integrals $\sum p_j$ and $\sum p_j p_k A_{jk}^2$ in common, and they are both integrable but for different reasons: the first because it is a finite-dimensional reduction of an integrable PDE, the second because it has an $r-$matrix.

Acknowledgments

The author thanks for the support of the UTPA-FRC award. The partial work of this paper was done when the author visited DSU and co-organized the annual 2006 DoD-Applied Math workshop at DSU. The author is grateful for Dr. Fengshan Liu and Dr. Xiquan Shi for their invitation and hospitality.

References

[1] O. Babelon and C.M. Viallet, *Phys. Lett. B* **237**(1990), 411–416.

[2] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett*. **71**(1993), 1661–1664.

[3] A. Degasperis and M. Procesi, Asymptotic integrability. In: A. Degasperis and G. Gaeta (Eds.), *Symmetry and Perturbation Theory*, World Scientific, 1999 23–37.

[4] A. Degasperis, D. D. Holm and A. N. W. Hone, A new integrable equation with peakon solutions. *Theor. Math. Phys.* **133**(2002), 1463–1474.

[5] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*. Springer-Verlag, Berlin, 1987.

[6] Z. J. Qiao, The Camassa-Holm hierarchy, N-dimensional integrable systems, and algebro-geometric solution on a symplectic submanifold. *Communications in Mathematical Physics*, **239**(2003), 309–341.

[7] Z. J. Qiao, Integrable hierarchy, 3×3 constrained systems, and parametric and stationary solutions. *Acta Applicandae Mathematicae*, **83**(2004), 199–220.

[8] Z. J. Qiao, Generalized r-matrix structure and algebro-geometric solutions for integrable systems. *Rev. Math. Phys*. **13**(2001), 545–586.

[9] Z. J. Qiao and S. Li, A new integrable hierarchy, parametric solution, and traveling wave solution. *Mathematical Physics, Analysis and Geometry*, **7**(2004), 289–308.

[10] Z. J. Qiao and G. P. Zhang, On peaked and smooth solitons for the Camassa-Holm equation. *Europhysics Letters*, **73**(2006), 657–663.

[11] O. Ragnisco and M. Bruschi, Peakons, *r*-matrix and Toda lattice. *Physica A*, **228**(1996), 150–159.

[12] G. P. Zhang and Z. J. Qiao, New Soliton Solutions for the Degasperis-Procesi Equation. *Preprint*, 2006.

Appendix

The following computations are needed in verifying Eq. (2.16).

First, we calculate the left hand side of Eq. (2.16).

$$\begin{aligned}
\frac{\partial L_{ij}}{\partial q_m} &= \sqrt{p_i p_j} A'_{ij}(\delta_{im} - \delta_{jm}), \\
\frac{\partial L_{kl}}{\partial p_m} &= \frac{A_{kl}}{2\sqrt{p_k p_l}}(p_l \delta_{km} + p_k \delta_{lm}),
\end{aligned}$$

$$\begin{aligned}
\{L_{ij}, L_{kl}\} &= \sum_{m=1}^{N} \left(\frac{\partial L_{ij}}{\partial q_m}\frac{\partial L_{kl}}{\partial p_m} - \frac{\partial L_{kl}}{\partial q_m}\frac{\partial L_{ij}}{\partial p_m} \right) \\
&= \frac{1}{2}\sum_{m=1}^{N} \Bigg[\sqrt{p_i p_j} A'_{ij} \frac{A_{kl}}{\sqrt{p_k p_l}} (\delta_{im} - \delta_{jm})(p_l \delta_{km} + p_k \delta_{lm}) \\
&\quad - \sqrt{p_k p_l} A'_{kl} \frac{A_{ij}}{\sqrt{p_i p_j}} (\delta_{km} - \delta_{lm})(p_j \delta_{im} + p_i \delta_{jm}) \Bigg] \\
&= \frac{1}{2}\left[\sqrt{p_i p_k}(-A'_{ij}A_{kj} + A'_{kj}A_{ij})\delta_{jl} + \sqrt{p_l p_j}(A'_{ij}A_{il} - A'_{il}A_{ij})\delta_{ik} \right. \\
&\quad \left. - \sqrt{p_i p_l}(A_{ij}A_{jl})'\delta_{kj} + \sqrt{p_k p_j}(A_{ij}A_{ki})'\delta_{il} \right] \\
&= \frac{1}{2}\left[\sqrt{p_k p_i}(A_{kj}A_{ji})'\delta_{jl} + \sqrt{p_l p_j}(A_{li}A_{ij})'\delta_{ik} \right. \\
&\quad \left. + \sqrt{p_i p_l}(A_{lj}A_{ji})'\delta_{kj} + \sqrt{p_k p_j}(A_{ki}A_{ij})'\delta_{il} \right],
\end{aligned}$$

where the superscript $'$ means Eq. (2.14) with the argument.

Thus, we obtain the following formula,

$$\begin{aligned}
\{L_1, L_2\} &= \sum_{i,j,k,l=1}^{N} \{L_{ij}, L_{kl}\} E_{ij} \otimes E_{kl} \\
&= \frac{1}{2}\sum_{j,k,l=1}^{N} \left[\sqrt{p_k p_j}(A_{kl}A_{lj})' E_{jl} \otimes E_{kl} + \sqrt{p_k p_j}(A_{kl}A_{lj})' E_{lj} \otimes E_{lk} \right. \\
&\quad \left. + \sqrt{p_k p_j}(A_{kl}A_{lj})' E_{jl} \otimes E_{lk} + \sqrt{p_k p_j}(A_{kl}A_{lj})' E_{lj} \otimes E_{kl} \right] \\
&= \frac{1}{2}\sum_{j,k,l=1}^{N} \sqrt{p_k p_j}(A_{kl}A_{lj})' \left[E_{jl} \otimes E_{kl} + E_{lj} \otimes E_{lk} + E_{jl} \otimes E_{lk} + E_{lj} \otimes E_{kl} \right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{j,k,l=1,j\neq k,l;k\neq l}^{N}\sqrt{p_kp_j}(A_{kl}A_{lj})'\left(E_{jl}+E_{lj}\right)\otimes(E_{kl}+E_{lk})\\
&\quad+\sum_{k,l=1}^{N}\sqrt{p_kp_l}A'_{kl}\left(E_{ll}\otimes(E_{kl}+E_{lk})-(E_{kl}+E_{lk})\otimes E_{ll}\right). \qquad (2.20)
\end{aligned}
$$

Next, we compute the right hand side of Eq. (2.16),

$$
\begin{aligned}
&[r_{12},L_1]-[r_{21},L_2]\\
=&\sum_{i,j,s,t,k,l=1}^{N} r_{ij;st}L_{kl}\left[(E_{ij}\otimes E_{st})(E_{kl}\otimes\mathbf{1})-(E_{kl}\otimes\mathbf{1})(E_{ij}\otimes E_{st})\right.\\
&\left.-(E_{st}\otimes E_{ij})(\mathbf{1}\otimes E_{kl})+(\mathbf{1}\otimes E_{kl})(E_{st}\otimes E_{ij})\right]\\
=&\sum_{i,j,s,k,l=1}^{N}\left[r_{ij;sk}\left(L_{jl}(E_{il}\otimes E_{sk})-L_{li}(E_{lj}\otimes E_{sk})\right)\right.\\
&\left.-r_{ij;ss}L_{kl}\left((E_{ss}\otimes E_{ij}E_{kl})-(E_{ss}\otimes E_{kl}E_{ij})\right)\right]\\
=&\sum_{i,j,s,k,l=1}^{N}\left[r_{ij;sk}\left(L_{jl}(E_{il}\otimes E_{sk})-L_{li}(E_{lj}\otimes E_{sk})\right)\right]\\
&+\sum_{i,j,s,l=1}^{N}\left[-r_{ji;ss}L_{il}(E_{ss}\otimes E_{jl})+r_{ij;ss}L_{li}(E_{ss}\otimes E_{lj})\right]\\
=&\sum_{i,j,s,t,l,s\neq t,j\neq l}\left[r_{ji;st}L_{il}(E_{jl}\otimes E_{st})-r_{ij;st}L_{li}(E_{lj}\otimes E_{st})\right]\\
&+\sum_{i,j,s,l,s\neq j,l,j\neq l}\left[r_{ji;ss}L_{il}(E_{jl}\otimes E_{ss}-E_{ss}\otimes E_{jl})-r_{ij;ss}L_{li}(E_{lj}\otimes E_{ss}-E_{ss}\otimes E_{lj})\right.\\
&\left.+(r_{si;lj}L_{is}-r_{is;lj}L_{si})E_{ss}\otimes E_{lj})\right]\\
&+\sum_{i,j,l,j\neq l}\left[(r_{li;lj}L_{il}-r_{il;lj}L_{li}+r_{ij;ll}L_{li}-r_{li;ll}L_{ij})E_{ll}\otimes E_{lj}\right.\\
&+(r_{li;jl}L_{il}-r_{il;jl}L_{li}-r_{ji;ll}L_{il}+r_{il;ll}L_{ji})E_{ll}\otimes E_{jl}\\
&+(r_{ji;ll}L_{il}-r_{il;ll}L_{ji})(E_{jl}\otimes E_{ll}-E_{ll}\otimes E_{jl})\\
&\left.-(r_{ij;ll}L_{li}-r_{li;ll}L_{ij})(E_{lj}\otimes E_{ll}-E_{ll}\otimes E_{lj})\right]\\
&+\sum_{i,j,l,j\neq l}\left[(r_{li;jj}L_{il}-r_{il;jj}L_{li}+r_{ij;ll}L_{ji}-r_{ji;ll}L_{ij})E_{ll}\otimes E_{jj}\right.\\
&\left.+(r_{li;jj}L_{il}-r_{il;jj}L_{li})(E_{ll}\otimes E_{jj}-E_{jj}\otimes E_{ll})\right]\\
&+\sum_{i,l}4(r_{li;ll}L_{il}-r_{il;ll}L_{li})E_{ll}\otimes E_{ll} \qquad (2.21)
\end{aligned}
$$

The first term of Eq. (2.21) is:

$$
\begin{aligned}
&\sum_{i,j,s,t,l,s\neq t,j\neq l}(r_{ji;st}L_{il}-r_{il;st}L_{ji})E_{jl}\otimes E_{st}\\
=&\sum_{i,j,s,t,l;s\neq t,j,l;t\neq j,l;j\neq l}(r_{ji;st}L_{il}-r_{il;st}L_{ji})E_{jl}\otimes E_{st}\\
&+\sum_{i,j,k,l,j\neq k,l;k\neq l}\left[\left(r_{li;kl}L_{ij}-r_{ij;kl}L_{li}\right)E_{lj}\otimes E_{kl}+\left(r_{ki;lj}L_{il}-r_{il;lj}L_{ki}\right)E_{kl}\otimes E_{lj}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left(r_{ji;kl}L_{il}-r_{il;kl}L_{ji}\right)E_{jl}\otimes E_{kl}+\left(r_{li;lk}L_{ij}-r_{ij;lk}L_{li}\right)E_{lj}\otimes E_{lk}\big]\\
&+2\sum_{i,j,l,j\neq l}\left[\left(r_{li;jl}L_{ij}-r_{ij;jl}L_{li}\right)E_{lj}\otimes E_{jl}+\left(r_{li;lj}L_{ij}-r_{ij;lj}L_{li}\right)E_{lj}\otimes E_{lj}\right].
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
& [r_{12},L_1]-[r_{21},L_2]\\
= & \sum_{i,j,s,t,l;s\neq t,j,l;t\neq j,l;j\neq l}(r_{ji;st}L_{il}-r_{il;st}L_{ji})E_{jl}\otimes E_{st}\\
&+\sum_{i,j,k,l,j\neq k,l;k\neq l}\left[\left(r_{li;kl}L_{ij}-r_{ij;kl}L_{li}\right)E_{lj}\otimes E_{kl}+\left(r_{ji;lk}L_{il}-r_{il;lk}L_{ji}\right)E_{jl}\otimes E_{lk}\right.\\
&+\left(r_{ji;kl}L_{il}-r_{il;kl}L_{ji}\right)E_{jl}\otimes E_{kl}+\left(r_{li;lk}L_{ij}-r_{ij;lk}L_{li}\right)E_{lj}\otimes E_{lk}\big]\\
&+\sum_{i,j,k,l;k\neq j,l,j\neq l}\big[(-r_{ji;kk}L_{il}+r_{il;kk}L_{ji}+r_{ki;jl}L_{ik}-r_{ik;jl}L_{ki})E_{kk}\otimes E_{jl}\\
&+(r_{ji;kk}L_{il}-r_{il;kk}L_{ji})E_{jl}\otimes E_{kk}\big]\\
&+\sum_{i,j,l;j\neq l}\big[(r_{li;lj}L_{il}-r_{il;lj}L_{li}+2\left(r_{ij;ll}L_{li}-r_{li;ll}L_{ij}\right))E_{ll}\otimes E_{lj}\\
&+\left(r_{li;jl}L_{il}-r_{il;jl}L_{li}+2(-r_{ji;ll}L_{il}+r_{il;ll}L_{ji})\right)E_{ll}\otimes E_{jl}\\
&+\left(r_{li;ll}L_{ij}-r_{ij;ll}L_{li}\right)E_{lj}\otimes E_{ll}+2\left(r_{li;jl}L_{ij}-r_{ij;jl}L_{li}\right)E_{lj}\otimes E_{jl}\\
&+\left(r_{ji;ll}L_{il}-r_{il;ll}L_{ji}\right)E_{jl}\otimes E_{ll}+2\left(r_{li;lj}L_{ij}-r_{ij;lj}L_{li}\right)E_{lj}\otimes E_{lj}\big]\\
&+2\sum_{i,j,l,j\neq l}\left(r_{li;jj}L_{il}-r_{il;jj}L_{li}-r_{ji;ll}L_{ij}+r_{ij;ll}L_{ji}\right)E_{ll}\otimes E_{jj}\\
&+4\sum_{i,l}(r_{li;ll}L_{il}-r_{il;ll}L_{li})E_{ll}\otimes E_{ll}\\
= & \{L_1,L_2\} \quad \text{by Eqs. (2.20) and (2.21). This finishes the proof of Eq. (2.16).}
\end{aligned}
$$

References

[1] H. Dullin, G. Gottwald and D. D. Holm, An integrable shallow water equation with linear and nonlinear dispersion. *Phys. Rev. Lett.*, **87**(2001), 1945–1940.

[2] A. Degasperis, D. D. Holm and A. N. W. Hone, A Class of Equations with Peakon and Pulson Solutions. In preparation, 2002.

[3] O. Fringer and D. D. Holm, Integrable vs. nonintegrable geodesic soliton behavior. *Physica D*, **150**(2001), 237–263.

[4] D. D. Holm and Z. Qiao, On the relationship between the $b=3$ peakon equation and finite-dimensional integrable systems. In preparation, 2001.