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A New Integrable Hierarchy, Parametric Solutions and Traveling Wave Solutions

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Abstract. In this paper we give a new integrable hierarchy. In the hierarchy there are the following representatives:

$$u_t = \partial_x^5 u^{-2/3},$$
$$u_t = \partial_x^5 \frac{(u^{-1/3})_{xx} - 2(u^{-1/6})_x^2}{u},$$
$$u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0.$$

The first two are the positive members of the hierarchy, and the first equation was a reduction of an integrable (2 + 1)-dimensional system (see B. G. Konopelchenko and V. G. Dubrovsky, *Phys. Lett. A* **102** (1984), 15–17). The third one is the first negative member. All nonlinear equations in the hierarchy are shown to have 3×3 Lax pairs through solving a key 3×3 matrix equation, and therefore they are integrable. Under a constraint between the potential function and eigenfunctions, the 3×3 Lax pair and its adjoint representation are nonlinearized to be two Liouville-integrable Hamiltonian systems. On the basis of the integrability of $6N$-dimensional systems we give the parametric solution of all positive members in the hierarchy. In particular, we obtain the parametric solution of the equation $u_t = \partial_x^5 u^{-2/3}$. Finally, we present the traveling wave solutions (TWSs) of the above three representative equations. The TWSs of the first two equations have singularities, but the TWS of the 3rd one is continuous. The parametric solution of the 5th-order equation $u_t = \partial_x^5 u^{-2/3}$ can not contain its singular TWS. We also analyse Gaussian initial solutions for the equations $u_t = \partial_x^5 u^{-2/3}$, and $u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0$. Both of them are stable.

Mathematics Subject Classifications (2000): 37K10, 58F07, 35Q35.

Key words: Hamiltonian system, matrix equation, zero curvature representation, parametric solution, traveling wave solution.

1. Introduction

The inverse scattering transformation (IST) method plays a very important role in solving integrable nonlinear evolution equations (NLEEs) [17]. These NLEEs include the well-known KdV equation [22] which is related to a 2nd order operator (i.e. Hill operator) problem [23, 25], the remarkable Ablowitz–Kaup–Newell–Segur (AKNS) equations [1, 2] which are associated with the Zakharov–Shabat (ZS) spectral problem [33], and other higher-dimensional integrable equations.

In the theory of integrable system, it is significant for us to find new integrable evolution equations. Kaup [19] studied the inverse scattering problem for cubic eigenvalue equations of the form $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi$, and showed a 5th-order partial differential equation (PDE) $Q_t + Q_{xxxxx} + 30(Q_{xxx}Q + (5/2)Q_{xx}Q_x) + 180Q_xQ^2 = 0$ (called the KK equation) integrable. Afterwards, Kupershmidt [21] constructed a super-KdV equation and presented the integrability of the equation through giving bi-Hamiltonian property and Lax form. Recently, Degasperis and Procesi [12] proposed a new integrable equation: $m_t + um_x + 3mu_x = 0$, $m = u - u_{xx}$, called the DP equation, which has the peaked soliton solution.

The DP equation is actually a member with $b = 3$ in the family $m_t + um_x + bmu_x = 0$, $m = u - u_{xx}$, $b = constant$. It has been already proven that only $b = 2, 3$ are integrable cases [26]. With $b = 2$, it works out the equation $m_t + um_x + 2mu_x = 0$, which was first derived in Camassa and Holm [8] (1993) by using asymptotic expansions for Euler's equations governing inviscid incompressible flow in the shallow water regime. It was thereby shown to be bi-Hamiltonian and integrable and to have the peaked soliton solution. Its billiard solutions, piecewise smooth solutions and algebro-geoemtric solutions were successively treated in Alber et al. [3–6] (1994, 1995, 1999, 2001), Constantin and McKean [10] (1999) and in Qiao [29] (2003). Before Camassa and Holm [8] (1993), families of integrable equations similar to shallow water equation were known to be derivable in the general context of hereditary symmetries in Fokas and Fuchssteiner [16] (1981). However, this equation was not written explicitly, nor was it derived physically as a water wave equation and its solution properties were not studied before Camassa and Holm [8] (1993). See Fuchssteiner [15] (1996) for an insightful history of how the shallow water equation is associated with the hereditary symmetries and symplectic structures.

The DP equation (i.e. the equation with $b = 3$) was proven integrable, associated with a 3rd-order spectral problem [11]: $\psi_{xxx} = \psi_x - \lambda m\psi$, and related to the canonical Hamiltonian system under a new nonlinear Poisson bracket (called Peakon bracket) [18]. In 2002, we extended the DP equation to an integrable hierarchy and dealt with its parametric solution and stationary solutions [28].

In this paper, we propose a new integrable hierarchy. In particular, the following three representatives in the hierarchy

$$u_t = \partial_x^5 u^{-2/3}, \tag{1}$$

$$u_t = \partial_x^5 \frac{(u^{-1/3})_{xx} - 2(u^{-1/6})_x^2}{u}, \tag{2}$$

$$u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0, \tag{3}$$

are shown to have bi-Hamiltonian operator structure and to be integrable. The first two are the positive members of the hierarchy. But the third one is the first negative member of the hierarchy. Konopelchenko and Dubrovsky [20] pointed out that Equation (1) is a reduction of a $(2 + 1)$-dimensional equation. Here we will deal with its spectral problem and parametric representation of solution from the point of constraint view. All nonlinear equations in the hierarchy are shown to have 3×3 Lax pairs through solving a key 3×3 matrix equation, and therefore they are integrable. After being imposed on a constraint between the potential function and eigenfunctions, the 3×3 Lax pair and its adjoint representation are nonlinearized to be two Liouville-integrable Hamiltonian systems. On the basis of the integrability of $6N$-dimensional systems we give the parametric solution of all positive members in the hierarchy. In particular, we obtain the parametric solution of the equation $u_t = \partial_x^5 u^{-2/3}$. Furthermore, we obtain the traveling wave solutions (TWSs) for Equations (1), (2), and (3). The first two look like a class of cusp soliton solutions (but not cusp soliton [32]). The TWSs of Equations (1) and (2) have singularities, but the TWS of Equation (3) is continuous. Additionally, the parametric solution of the 5th-order Equation (1) can not include its singular TWS. Equation (3) has a compacton-like and a parabolic cylinder solution. We also analyse the initial Gaussian solutions for equations $u_t = \partial_x^5 u^{-2/3}$ and $u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0$. Both of them are stable (see Figures 1 and 2).

The whole paper is organized as follows. In the next section we describe how to connect the above three equations to a spectral problem and how to cast them into a new hierarchy of NLEEs, and also give the bi-Hamiltonian operators for the whole hierarchy. In Section 3, we construct the zero curvature representations for the new hierarchy through solving a key 3×3 matrix equation. In particular, we obtain the Lax pair of Equations (1), (2), (3), and therefore they are integrable. In Section 4, we show that the 3rd order spectral problem and its adjoint representation related to the above three equations are nonlinearized as a completely integrable Hamiltonian system under a constraint in $\mathbb{R}^{6N}$. In Section 5 we present the parametric solution

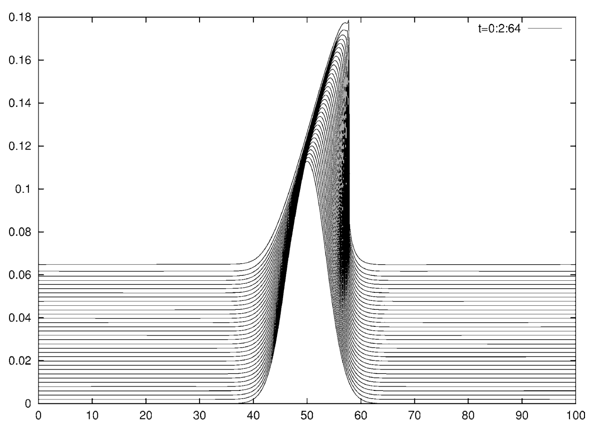


Figure 1. Stable solution for the equation $u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0$ under the Gaussian initial condition. A shock is developed during the time integration. This figure is very like the Burgers case $u_{xxt} + 3u_{xx}u_x + u_{xxx}u + \epsilon u_{xxxx} = 0$ through adding small viscosity term ϵu_{xxxx} to the equation. For instance, when $\epsilon = -0.01$, the equation $u_{xxt} + 3u_{xx}u_x + u_{xxx}u + \epsilon u_{xxxx} = 0$ has Figure 2.

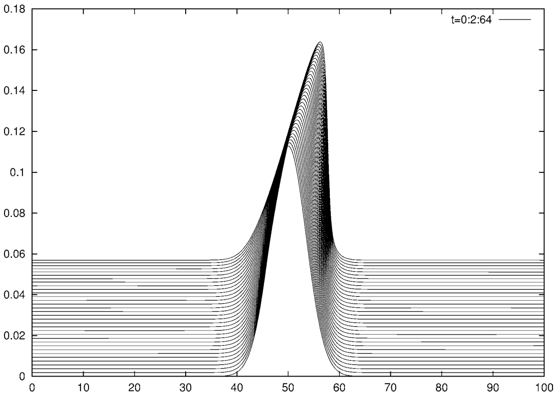


Figure 2. Stable solution for the equation $u_{xxt} + 3u_{xx}u_x + u_{xxx}u + \epsilon u_{xxxx} = 0, \epsilon = -0.01$ under the Gaussian initial condition. This figure is almost same as Figure 1.

for the positive hierarchy of NLEEs. We particularly get the parametric solution of Equation (1). Moreover, in section 6 we obtain the traveling wave solutions for Equations (1), (2), and (3), and also analyse the initial Gaussian solutions for the equations $u_t = \partial_x^5 u^{-2/3}$, and $u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0$. Finally, in Section 7 we give some conclusions.

2. Spectral Problem, Hamiltonian Operators, and a New Hierarchy

Let us consider the following spectral problem

$$\psi_{xxx} = -\lambda u \psi \tag{4}$$

and its adjoint representation

$$\psi^*_{xxx} = \lambda u \psi^*. \tag{5}$$

Then, we have their functional gradient $\delta\lambda/\delta u$ with respect to the potential u

$$\frac{\delta\lambda}{\delta u} = \frac{\lambda\psi\psi^*}{E} \equiv \frac{\nabla\lambda}{E}, \tag{6}$$

where

$$\begin{aligned} \nabla\lambda &= \lambda\psi\psi^*, \\ E &= \int_\Omega u\psi\psi^* \,\mathrm{d}x = constant, \end{aligned} \tag{7}$$

and $\Omega = (-\infty, \infty)$ or $\Omega = (0, T)$. In this procedure, we need the boundary condition of u decaying at infinities or of u being periodic with period T. Usually, we compute the functional gradient $\delta\lambda/\delta u$ of the eigenvalue λ with respect to the potential u by using the method in [13, 9, 31]. Fokas and Anderson constructed hereditary symmetries and Hamiltonian systems by using the isospectral eigenvalue problems [13]. Later, Cao developed the functional gradient procedure to the nonlinearization method [9], which closely connects finite-dimensional integrable systems to nonlinear integrable partial differential equations (also see details in [27]).

Taking derivatives five times on both sides of Equation (7), we find

$$(\nabla\lambda)_{xxxxx} = -3\lambda^2(2u\partial + \partial u)(\psi\psi_x^* - \psi^*\psi_x),$$
$$(\psi\psi_x^* - \psi^*\psi_x)_{xxx} = (u\partial + 2\partial u)\nabla\lambda,$$

which directly lead to

$$K\nabla\lambda = \lambda^2 J\nabla\lambda, \tag{8}$$

where

$$K = \partial^5, \tag{9}$$
$$J = -3(2u\partial + \partial u)\partial^{-3}(u\partial + 2\partial u). \tag{10}$$

Obviously, K, J are antisymmetric, and both of them are Hamiltonian operators because *they satisfy the Jacobi identity.*

Now, according to this pair of Hamiltonian operators, we define the hierarchy of nonlinear evolution equations associated with the spectral problems (4) and (5). Let $G_0 \in \operatorname{Ker} J = \{G \in C^\infty(\mathbb{R}) \mid JG = 0\}$ and $G_{-1} \in \operatorname{Ker} K = \{G \in C^\infty(\mathbb{R}) \mid KG = 0\}$. We define the Lenard sequence

$$G_j = \begin{cases} \mathcal{L}^j G_0, & j \geqslant 0, \\ \mathcal{L}^{j+1} G_{-1}, & j < 0, \end{cases} \quad j \in \mathbb{Z}, \tag{11}$$

where $\mathcal{L} = J^{-1}K$ is called the recursion operator. Therefore we obtain a new hierarchy of nonlinear evolution equations:

$$u_{t_k} = JG_k, \quad \forall k \in \mathbb{Z}. \tag{12}$$

Apparently, this hierarchy includes the positive members ($k \geqslant 0$) and the negative members ($k < 0$), and possesses the bi-Hamiltonian structure because of the Hamiltonian properties of K, J.

Let us now give specific equations in the hierarchy (12).

- Choosing $G_{-1} = 1/6 \in \operatorname{Ker} K$ yields the first negative member of the hierarchy:

  $$u_t + vu_x + 3v_x u = 0, \quad u = v_{xx}. \tag{13}$$

  This equation is actually: $v_{xxt} + 3v_{xx}v_x + v_{xxx}v = 0$ which is equivalent to $\partial^2(v_t + vv_x) = 0$. It has the compacton-like solution [30]. Obviously, $v = c_1x + c_0$ (c_1, c_0 are two constants) is a special solution of this equation.

- Choosing $G_0 = u^{-2/3} \in \operatorname{Ker} J$ leads to the second positive member of the hierarchy:

  $$u_t = \partial_x^5 u^{-2/3}. \tag{14}$$

  Konopelchenko and Dubrovsky pointed out that this equation is integrable and is a reduction of a $(2+1)$-dimensional equation [20]. But they did not study solutions of the equation. In the following, we study the relation between the equation and finite-dimensional integrable system and will find that it has parametric solution as well as the traveling wave solution which looks like a cusp.
- Choosing another element $G_0 = ((u^{-1/3})_{xx} - 2(u^{-1/6})_x^2)/u$ in the kernel Ker J gives the following positive member of the hierarchy:

  $$u_t = \partial_x^5 \frac{(u^{-1/3})_{xx} - 2(u^{-1/6})_x^2}{u}. \tag{15}$$

  This equation also has a cusp-like trveling wave solution. See this in Section 6.

Of course, we may generate further nonlinear equations by selecting other elements from the kernel elements of J, K. In the following, we will see that all equations in the hierarchy (12) are integrable. Particularly, *the above three Equations* (13), (14), (15) *are integrable*.

3. Zero Curvature Representations

Letting $\psi = \psi_1$, we change Equation (4) to the following 3×3 matrix spectral problem

$$\Psi_x = U(u, \lambda)\Psi, \tag{16}$$

$$U(u,\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda u & 0 & 0 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}. \tag{17}$$

Apparently, the Gateaux derivative matrix $U_*(\xi)$ of the spectral matrix U in the direction $\xi \in C^\infty(\mathbb{R})$ at point u is

$$U_*(\xi) \triangleq \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} U(u+\epsilon\xi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\lambda\xi & 0 & 0 \end{pmatrix} \tag{18}$$

which is obviously an injective homomorphism, i.e. $U_*(\xi) = 0 \Leftrightarrow \xi = 0$.

For any given C^∞-function G, we construct the following 3×3 matrix equation with respect to $V = V(G)$

$$V_x - [U, V] = U_*(KG - \lambda^2 JG). \tag{19}$$

THEOREM 1. *For spectral problem* (16) *and an arbitrary* C^{∞}*-function* G*, the matrix equation* (19) *has the following solution*

$$V = \lambda \begin{pmatrix} -G'' - 3\lambda\partial^{-2}\Upsilon G & 3(G' + \lambda\partial^{-3}\Upsilon G) & -6G \\ -G''' - 3\lambda\partial^{-1}uG' & 2G'' & 3(-G' + \lambda\partial^{-3}\Upsilon G) \\ -G'''' - 3\lambda^2 u\partial^{-3}\Upsilon G & G''' - 3\lambda\partial^{-1}uG' & -G'' + 3\lambda\partial^{-2}\Upsilon G \end{pmatrix}, \tag{20}$$

where $\partial = \partial_x = \partial/\partial x$*,* $\Upsilon = u\partial + 2\partial u$*, and the superscript "′" means the derivative in* x*. Therefore,* $J = -3\Upsilon^*\partial^{-3}\Upsilon$ *(*Υ^* *is the conjugate of* Υ*).*

Proof. Let us set

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix},$$

and subsitute it into Equation (19), which is an overdetermined equation. Using calculation techniques in [27], we obtain the following results:

$$\begin{aligned} V_{11} &= -\lambda G'' - 3\lambda^2\partial^{-2}\Upsilon G, \\ V_{12} &= 3(\lambda G' + \lambda^2\partial^{-3}\Upsilon G), \\ V_{13} &= -6\lambda G, \\ V_{21} &= -\lambda G''' - 3\lambda^2\partial^{-1}uG', \\ V_{22} &= 2\lambda G'', \\ V_{23} &= 3\lambda(-G' + \lambda\partial^{-3}\Upsilon G), \\ V_{31} &= -\lambda G'''' - 3\lambda^3 u\partial^{-3}\Upsilon G, \\ V_{32} &= \lambda G''' - 3\lambda^2\partial^{-1}uG', \\ V_{33} &= -\lambda G'' + 3\lambda^2\partial^{-2}\Upsilon G, \end{aligned}$$

which completes the proof. □

THEOREM 2. *Let* $G_0 \in \operatorname{Ker} J$*,* $G_{-1} \in \operatorname{Ker} K$*, and let each* G_j *be given through Equation* (11)*. Then,*

1. *each new vector field* $X_k = JG_k$*,* $k \in \mathbb{Z}$ *satisfies the following commutator representation*

$$V_{k,x} - [U, V_k] = U_*(X_k), \quad \forall k \in \mathbb{Z}; \tag{21}$$

2. *the new hierarchy* (12)*, i.e.*

$$u_{t_k} = X_k = JG_k, \quad \forall k \in \mathbb{Z}, \tag{22}$$

possesses the zero curvature representation

$$U_{t_k} - V_{k,x} + [U, V_k] = 0, \quad \forall k \in \mathbb{Z}, \tag{23}$$

where

$$V_k = \sum V(G_j)\lambda^{2(k-j-1)}, \quad \sum = \begin{cases} \sum_{j=0}^{k-1}, & k > 0, \\ 0, & k = 0, \\ -\sum_{j=k}^{-1}, & k < 0, \end{cases} \tag{24}$$

and $V(G_j)$ *is given by Equation* (20) *with* $G = G_j$.

Proof. 1. For $k = 0$, it is obvious. For $k < 0$, we have

$$\begin{aligned} V_{k,x} - [U, V_k] &= -\sum_{j=k}^{-1}(V_x(G_j) - [U, V(G_j)])\lambda^{2(k-j-1)} \\ &= -\sum_{j=k}^{-1} U_*(KG_j - \lambda^2 KG_{j-1})\lambda^{2(k-j-1)} \\ &= U_*\left(\sum_{j=k}^{-1} KG_{j-1}\lambda^{2(k-j)} - KG_j\lambda^{2(k-j-1)}\right) \\ &= U_*(KG_{k-1} - KG_{-1}\lambda^{2k}) \\ &= U_*(KG_{k-1}) \\ &= U_*(X_k). \end{aligned}$$

For the case of $k > 0$, it is similar to prove.
2. Noticing $U_{t_k} = U_*(u_{t_k})$, we obtain

$$U_{t_k} - V_{k,x} + [U, V_k] = U_*(u_{t_k} - X_k).$$

The injectiveness of U_* implies the second result holds. □

From Theorem 2, we immediately obtain the following corollary.

COROLLARY 1. *The new hierarchy* (12) *has Lax pair:*

$$\psi_{xxx} = -\lambda u\psi, \tag{25}$$

$$\psi_{t_k} = \sum \lambda^{2(k-j)-1}[-6G_j\psi_{xx} + 3(G_j' + \lambda\partial^{-3}\Upsilon G_j)\psi_x - (G_j'' + 3\lambda\partial^{-2}\Upsilon G_j)\psi], \tag{26}$$

where all symbols are the same as in Thereom 2 *and Thereom* 1.

So, all equations in the hierarchy (12) have Lax pairs and are therefore integrable. In particular, we have the following specific examples.

- When we choose $G_{-1} = 1/6$, Equation (13) has the following Lax pair:

$$\Psi_x = U(u, \lambda)\Psi, \tag{27}$$

$$\Psi_t = V(u, \lambda)\Psi, \tag{28}$$

where $u = u_{xx}$, $U(u, \lambda)$ is defined by Equation (17), and $V(u, \lambda)$ is given by

$$V(u, \lambda) = \begin{pmatrix} v_x & -v & \lambda^{-1} \\ 0 & 0 & -v \\ \lambda v u & 0 & -v_x \end{pmatrix}. \tag{29}$$

Apparently, Lax pair (27) and (28) is equivalent to

$$\psi_{xxx} = -\lambda u \psi, \tag{30}$$
$$\psi_t = \lambda^{-1}\psi_{xx} - v\psi_x + v_x\psi. \tag{31}$$

- In a similar way, choosing $G_0 = u^{-2/3}$ gives the Lax pair of Equation (14), i.e. $u_t = (u^{-2/3})_{xxxxx}$,

$$\psi_{xxx} = -\lambda u \psi, \tag{32}$$
$$\psi_t = -6\lambda u^{-2/3}\psi_{xx} + 3\lambda(u^{-2/3})_x\psi_x - \lambda(u^{-2/3})_{xx}\psi. \tag{33}$$

  This Lax pair is different from/inequivalent to the result in [20].
- Furthermore, through choosing $G_0 = ((u^{-1/3})_{xx} - 2(u^{-1/6})_x^2)/u$, we find that the new equation (2) has the Lax pair:

$$\psi_{xxx} = -\lambda u \psi, \tag{34}$$
$$\begin{aligned}\psi_t &= -6\lambda G_0\psi_{xx} + 3\lambda(G_0' + 3\lambda u^{-1/3})\psi_x \\ &\quad - \lambda(G_0'' + 9\lambda(u^{-1/3})_{xx})\psi.\end{aligned} \tag{35}$$

4. 6*N*-dimensional Integrable System

To discuss solutions of the hierarchy (12), we want to use the constrained method [9, 27] which leads finite-dimensional integrable systems to nonlinear integrable partial differential equations. Because Equation (4)/(16) is a 3rd order eigenvalue problem, we have to investigate itself together with its adjoint problem when we adopt the nonlinearized procedure. Ma and Strampp [24] already studied the AKNS and its adjoint problem, a 2×2 case, by using the so-called symmetry constraint method. Now, we are dealing with 3×3 spectral problem (16) related to the hierarchy (12).

Let us go back to spectral problem (16) and consider its adjoint problem

$$\Psi_x^* = \begin{pmatrix} 0 & 0 & u\lambda \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}\Psi^*, \quad \Psi^* = \begin{pmatrix} \psi_1^* \\ \psi_2^* \\ \psi_3^* \end{pmatrix}, \tag{36}$$

where $\psi^* = \psi_3^*$.

Let λ_j $(j = l, \ldots, N)$ be N distinct spectral values of (16) and (36), and q_{1j}, q_{2j}, q_{3j} and p_{1j}, p_{2j}, p_{3j} be the corresponding spectral functions, respectively. Then we have

$$\begin{aligned} q_{1x} &= q_2, \\ q_{2x} &= q_3, \\ q_{3x} &= -u\Lambda q_1, \end{aligned} \tag{37}$$

and

$$\begin{aligned} p_{1x} &= u\Lambda p_3, \\ p_{2x} &= -p_1, \\ p_{3x} &= -p_2, \end{aligned} \tag{38}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, $q_k = (q_{k1}, q_{k2}, \ldots, q_{kN})^{\mathrm{T}}$, $p_k = (p_{k1}, p_{k2}, \ldots, p_{kN})^{\mathrm{T}}$, $k = 1, 2, 3$.

Let us consider the above two systems in the symplectic space $(\mathbb{R}^{6N}, \mathrm{d}p \wedge \mathrm{d}q)$, and introduce the following constraint:

$$u^{-2/3} = \sum_{j=1}^{N} \nabla\lambda_j, \tag{39}$$

where $\nabla\lambda_j = \lambda_j q_{1j} p_{3j}$ is the functional gradient of λ_j for spectral problems (16) and (36). Then Equation (39) reads

$$u = \langle \Lambda q_1, p_3 \rangle^{-3/2}. \tag{40}$$

Under this constraint, Equation (37) and its adjoint problem (38) are cast in a canonical Hamiltonian form in $\mathbb{R}^{6N}$:

$$\begin{aligned} q_x &= \{q, H^+\}, \\ p_x &= \{p, H^+\}, \end{aligned} \tag{41}$$

with the Hamiltonian

$$H^+ = \langle q_2, p_1 \rangle + \langle q_3, p_2 \rangle + \frac{2}{\sqrt{\langle \Lambda q_1, p_3 \rangle}}, \tag{42}$$

where $p = (p_1, p_2, p_3)^{\mathrm{T}}$, $q = (q_1, q_2, q_3)^{\mathrm{T}} \in \mathbb{R}^{6N}$, $\langle \cdot\,, \cdot \rangle$ stands for the standard inner product in $\mathbb{R}^N$, and $\{\cdot\,, \cdot\}$ represents the Poisson bracket of two functions F_1, F_2 defined by:

$$\{F_1, F_2\} = \sum_{i=1}^{3} \left(\left\langle \frac{\partial F_1}{\partial q_i}, \frac{\partial F_2}{\partial p_i} \right\rangle - \left\langle \frac{\partial F_1}{\partial p_i}, \frac{\partial F_2}{\partial q_i} \right\rangle \right) \tag{43}$$

which is antisymmetric and bilinear and satifies the Jacobi identity.

To see the integrability of system (41), we take into account the time part $\Psi_t = V_k \Psi$ and its adjoint problem $\Psi_t^* = -V_k^{\mathrm{T}} \Psi^*$, where V_k is defined by $V_k = \sum_{j=0}^{k-1} V(G_j) \lambda^{2(k-j-1)}$, and $V(G_j)$ is given by Equation (20) with $G = G_j$.

Let us first look at V_1 case. Then the corresponding time part is:

$$\Psi_t = \lambda \begin{pmatrix} -(u^{-2/3})_{xx} & 3(u^{-2/3})_x & -6u^{-2/3} \\ -(u^{-2/3})_{xxx} + 6\lambda u^{1/3} & 2(u^{-2/3})_{xx} & -3(u^{-2/3})_x \\ -(u^{-2/3})_{xxxx} & (u^{-2/3})_{xxx} + 6\lambda u^{1/3} & -(u^{-2/3})_{xx} \end{pmatrix} \Psi, \quad (44)$$

and its adjoint problem is:

$$\Psi_t^* = \lambda \begin{pmatrix} (u^{-2/3})_{xx} & (u^{-2/3})_{xxx} + 6\lambda u^{1/3} & -(u^{-2/3})_{xxxx} \\ -3(u^{-2/3})_x & -2(u^{-2/3})_{xx} & -(u^{-2/3})_{xxx} - 6\lambda u^{-1/3} \\ 6u^{-2/3} & 3(u^{-2/3})_x & (u^{-2/3})_{xx} \end{pmatrix} \Psi^*. \quad (45)$$

Noticing the following relations

$$\begin{aligned} &u^{1/3} = \langle \Lambda q_1, p_3 \rangle^{-1/2}, \\ &(u^{-2/3})_x = \langle \Lambda q_2, p_3 \rangle - \langle \Lambda q_1, p_2 \rangle, \\ &(u^{-2/3})_{xx} = \langle \Lambda q_3, p_3 \rangle + \langle \Lambda q_1, p_1 \rangle - 2\langle \lambda q_2, p_2 \rangle, \\ &(u^{-2/3})_{xxx} = 3(\langle \Lambda q_2, p_1 \rangle - \langle \Lambda q_3, p_2 \rangle), \\ &(u^{-2/3})_{xxxx} = 6\langle \Lambda q_3, p_1 \rangle + 3\langle \Lambda q_1, p_3 \rangle^{-3/2}(\langle \Lambda^2 q_1, p_2 \rangle + \langle \Lambda^2 q_2, p_3 \rangle), \end{aligned}$$

we obtain nonlinearized systems of the time parts (44) and (45), and furthermore cast them into the following canonical Hamiltonian system in $\mathbb{R}^{6N}$:

$$\begin{aligned} q_{t_1} &= \{q, F_1^+\}, \\ p_{t_1} &= \{p, F_1^+\}, \end{aligned} \quad (46)$$

with the Hamiltonian

$$\begin{aligned} F_1^+ = &-\frac{1}{2}(\langle \Lambda q_1, p_1 \rangle + \langle \Lambda q_3, p_3 \rangle)^2 \\ &+ 2\langle \Lambda q_2, p_2 \rangle(\langle \Lambda q_1, p_1 \rangle + \langle \Lambda q_3, p_3 \rangle - \langle \Lambda q_2, p_2 \rangle) \\ &+ 3(\langle \Lambda q_2, p_3 \rangle - \langle \Lambda q_1, p_2 \rangle)(\langle \Lambda q_2, p_1 \rangle - \langle \Lambda q_3, p_2 \rangle) \\ &- 6\langle \Lambda q_1, p_3 \rangle \langle \Lambda q_3, p_1 \rangle \\ &+ \frac{6}{\sqrt{\langle \Lambda q_1, p_3 \rangle}}(\langle \Lambda^2 q_1, p_2 \rangle + \langle \Lambda^2 q_2, p_3 \rangle). \end{aligned} \quad (47)$$

A direct computation leads to the following theorem.

THEOREM 3.

$$\{H^+, F_1^+\} = 0, \quad (48)$$

that is, two Hamiltonian flows commute in $\mathbb{R}^{6N}$.

For general case V_k, $k > 0$, $k \in \mathbb{Z}$, we consider the following Hamiltonian functions

$$
\begin{aligned}
F_k^+ = &-\frac{1}{2}\sum_{j=0}^{k-1}(\langle\Lambda^{2j+1}q_1, p_1\rangle + \langle\Lambda^{2j+1}q_3, p_3\rangle)(\langle\Lambda^{2(k-j)-1}q_1, p_1\rangle \\
&\qquad + \langle\Lambda^{2(k-j)-1}q_3, p_3\rangle) \\
&+ 2\sum_{j=0}^{k-1}\langle\Lambda^{2j+1}q_2, p_2\rangle(\langle\Lambda^{2(k-j)-1}q_1, p_1\rangle + \langle\Lambda^{2(k-j)-1}q_3, p_3\rangle \\
&\qquad - \langle\Lambda^{2(k-j)-1}q_2, p_2\rangle) \\
&+ 3\sum_{j=0}^{k-1}(\langle\Lambda^{2j+1}q_2, p_3\rangle - \langle\Lambda^{2j+1}q_1, p_2\rangle)(\langle\Lambda^{2(k-j)-1}q_2, p_1\rangle \\
&\qquad - \langle\Lambda^{2(k-j)-1}q_3, p_2\rangle) \\
&- 6\sum_{j=0}^{k-1}\langle\Lambda^{2j+1}q_1, p_3\rangle\langle\Lambda^{2(k-j)-1}q_3, p_1\rangle \\
&- \frac{3}{2}\sum_{j=0}^{k}(\langle\Lambda^{2j}q_1, p_1\rangle - \langle\Lambda^{2j}q_3, p_3\rangle)(\langle\Lambda^{2(k-j)}q_1, p_1\rangle \\
&\qquad - \langle\Lambda^{2(k-j)}q_3, p_3\rangle) \\
&- 3\sum_{j=0}^{k}(\langle\Lambda^{2j}q_2, p_3\rangle - \langle\Lambda^{2j}q_1, p_2\rangle)(\langle\Lambda^{2(k-j)}q_2, p_1\rangle \\
&\qquad + \langle\Lambda^{2(k-j)}q_3, p_2\rangle) \\
&+ 3H^+(\langle\Lambda^{2k}q_1, p_2\rangle + \langle\Lambda^{2k}q_2, p_3\rangle). \qquad (49)
\end{aligned}
$$

Through a lengthy calculation, we find

$$\{H^+, F_k^+\} = 0, \qquad \{F_l^+, F_k^+\} = 0, \quad k, l = 1, 2, \ldots. \qquad (50)$$

That is,

THEOREM 4. *All canonical Hamiltonian flows* (F_k^+) *commute with the Hamiltonian system* (41). *In particular, the Hamiltonian systems* (41) *and* (46) *are compatible and therefore integrable in the Liouville sense.*

Remark 1. In the proof procedure of this theorem, we have used the following two facts: $\langle q_1, p_2\rangle + \langle q_2, p_3\rangle = c_1$, and $\langle q_1, p_1\rangle - \langle q_3, p_3\rangle = c_2$. They always hold along x-flow in $\mathbb{R}^{6N}$. Here c_1, c_2 are two constants.

Remark 2. In fact, the involutive functions F_k^+ are generated from nonlinearization of the time part $\Psi_t = V_k\Psi$ and its adjoint problem $\Psi_t^* = -V_k^{\mathrm{T}}\Psi^*$ under the constraint (39), where V_k is defined by $V_k = \sum_{j=0}^{k-1} V(G_j)\lambda^{2(k-j-1)}$, and $V(G_j)$ is given by Equation (20) with $G = G_j$. In this calculation, we have used the following equalities:

$$
\begin{aligned}
G_j &= -\langle \Lambda^{2j+1} q_1, p_3\rangle, \quad j = 0, 1, 2 \ldots, \\
G'_j &= \langle \Lambda^{2j+1} q_2, p_3\rangle - \langle \Lambda^{2j+1} q_1, p_2\rangle, \\
G''_J &= \langle \Lambda^{2j+1} q_3, p_3\rangle + \langle \Lambda^{2j+1} q_1, p_1\rangle - 2\langle \Lambda^{2j+1} q_2, p_2\rangle, \\
G'''_j &= 3(\langle \Lambda^{2j+1} q_2, p_1\rangle - \langle \Lambda^{2j+1} q_3, p_2), \\
G''''_j &= 6\langle \Lambda^{2j+1} q_3, p_1\rangle + 3\langle \Lambda q_1, p_3\rangle^{-3/2}(\langle \Lambda^{2j+2} q_1, p_2\rangle + \langle \Lambda^{2j+2} q_2, p_3\rangle), \\
\partial^{-1} m G'_j &= \langle \Lambda^{2j} q_3, p_2\rangle + \langle \Lambda^{2j} q_2, p_1\rangle, \\
\partial^{-2} \Upsilon G_j &= \langle \Lambda^{2j} q_1, p_1\rangle - \langle \Lambda^{2j} q_3, p_3\rangle, \\
\partial^{-3} \Upsilon G_j &= -(\langle \Lambda^{2j} q_1, p_2\rangle + \langle \Lambda^{2j} q_2, p_3\rangle).
\end{aligned}
$$

5. Parametric Solution

Since Hamiltonian flows (H^+) and (F_k^+) are completely integrable in $\mathbb{R}^{6N}$ and their Poisson brackets $\{H^+, F_k^+\} = 0$ $(k = 1, 2, \ldots)$, their phase flows $g^x_{H^+}$, $g^{t_k}_{F_k^+}$ commute [7]. Thus, we can define their compatible solution as follows:

$$
\begin{pmatrix} q(x, t_k) \\ p(x, t_k) \end{pmatrix} = g^x_{H^+} g^{t_k}_{F_k^+} \begin{pmatrix} q(x^0, t_k^0) \\ p(x^0, t_k^0) \end{pmatrix}, \quad k = 1, 2, \ldots, \tag{51}
$$

where x^0, t_k^0 are the initial values of phase flows $g^x_{H^+}$, $g^{t_k}_{F_k^+}$.

THEOREM 5. *Let $q(x, t_k) = (q_1, q_2, q_3)^{\mathrm{T}}$, $p(x, t_k) = (p_1, p_2, p_3)^{\mathrm{T}}$ be a solution of the compatible Hamiltonian systems (H_+) and (F_k^+) in $\mathbb{R}^{6N}$. Then*

$$
u = \frac{1}{\sqrt{\langle \Lambda q_1(x, t_k), p_3(x, t_k)\rangle^3}} \tag{52}
$$

satisfies the positive equation of the hierarchy

$$
u_{t_k} = J \mathcal{L}^k \cdot u^{-2/3}, \quad k = 1, 2, \ldots, \tag{53}
$$

where the operators $\mathcal{L} = J^{-1}K$, J, K are given by Equations (10) *and* (9)*, respectively.*

Proof. Direct computation completes the proof. □

In particular, we have the following result.

THEOREM 6. *Let $p(x, t)$, $q(x, t)$ $(p(x, t) = (p_1, p_2, p_3)^{\mathrm{T}}, q(x, t) = (q_1, q_2, q_3)^{\mathrm{T}})$ be a common solution of the two integrable compatible flows* (41) *and* (46)*, then*

$$
u = \frac{1}{\sqrt{\langle \Lambda q_1(x, t), p_3(x, t)\rangle^3}} \tag{54}
$$

satisfies the equation:

$$u_t = \partial_x^5 u^{-2/3}. \tag{55}$$

Proof. Taking derivatives in x five times on both sides of Equation (54), we obtain

$$\begin{aligned}\partial_x^5 u^{-2/3} &= 9u(\langle \Lambda^2 q_3, p_3\rangle - \langle \Lambda^2 q_1, p_1\rangle)\\ &\quad + 3u_x(\langle \Lambda^2 q_1, p_2\rangle + \langle \Lambda^2 q_2, p_3\rangle),\end{aligned} \tag{56}$$

where

$$u_x = -\frac{3}{2}u\frac{(\langle \Lambda^2 q_1, p_2\rangle + \langle \Lambda^2 q_2, p_3\rangle)(\langle \Lambda q_2, p_3\rangle - \langle \Lambda q_1, p_2\rangle)}{\langle \Lambda q_1, p_3\rangle}.$$

On the other hand, taking derivative in t on both sides of Equation (54) yields

$$\begin{aligned}u_t &= -\frac{3}{2}u\frac{\langle \Lambda p_3, \dot{q}_1\rangle + \langle \Lambda q_1, \dot{p}_3\rangle}{\langle \Lambda q_1, p_3\rangle}\\ &= -\frac{3}{2}u\frac{\langle \Lambda p_3, \frac{\partial F_1^+}{\partial p_1}\rangle - \langle \Lambda q_1, \frac{\partial F_1^+}{\partial q_3}\rangle}{\langle \Lambda q_1, p_3\rangle}.\end{aligned}$$

Substituting expression of F_1^+ into the above equality and calculating, we find that final result is the same as the right-hand side of Equation (56), which completes the proof. □

6. Traveling Wave Solutions

First, let us compute the traveling wave solution of Equation (3). Set $u = f(\xi)$, $\xi = x - ct$ (c is some constant speed), then after substituting this setting into Equation (3) we obtain

$$-cf''' + 3f''f' + f'''f = 0,$$

i.e.

$$(f^2 - 2cf)''' = 0.$$

Therefore,

$$(f - c)^2 = A\xi^2 + B\xi + C, \quad \forall A, B, C \in \mathbb{R}. \tag{57}$$

So, the equation $u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0$ has the following traveling wave solution

$$u(x,t) = c \pm \sqrt{A(x-ct)^2 + B(x-ct) + C}. \tag{58}$$

Let us discuss specific cases as follows:

- When $c = 0$, we get stationary solution

$$u(x) = \pm\sqrt{Ax^2 + Bx + C}, \quad \forall A, B, C \in \mathbb{R}, \tag{59}$$

which may be a straight line, circle, ellipse, parabola, and hyperbola according to different choices of constants A, B, C.

- When $c \neq 0$ and $A \neq 0$, then we have

$$u(x,t) = c \pm \sqrt{A\left(x - ct + \frac{B}{2A}\right)^2 + \frac{4AC - B^2}{4A}}, \quad \forall A, B, C \in \mathbb{R}. \tag{60}$$

Therefore with $4AC - B^2 = 0$ this solution becomes

$$u(x,t) = c \pm \sqrt{A}\left|\left(x - ct + \frac{B}{2A}\right)\right|, \quad \forall A > 0, B \in \mathbb{R}. \tag{61}$$

Setting $c = 1$, $A = 1$, $B = 0$ yields

$$u(x,t) = 1 - |x - t|, \tag{62}$$

and

$$u(x,t) = 1 + |x - t|. \tag{63}$$

The former looks like a compacton solution [30, 14]. The latter is a "V"-type solution.

- When $c \neq 0$ and $A = 0$, then we have

$$u(x,t) = c \pm \sqrt{B(x - ct) + C}, \quad \forall B, C \in \mathbb{R}, \tag{64}$$

which is a parabolic traveling wave solution if $B \neq 0$ and becomes a constant solution if $B = 0$. In particular,

$$u(x,t) = 1 + \sqrt{x - t}, \quad x - t \geqslant 0, \tag{65}$$

and

$$u(x,t) = 1 - \sqrt{x - t}, \quad x - t \geqslant 0 \tag{66}$$

are two specific solutions.

So, the 3rd-order equation $u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0$ has the continuous traveling wave solution (58). In addition, we also have the Gaussian initial solution of this 3rd-order equation, which is stable (see Figure 1).

Second, we give the traveling wave solution of the 5th-order equation (1). Set $u = \xi^{-\gamma}$, $\xi = x - ct$ (c is a constant speed to be determined), then after substituting this setting into Equation (1) we obtain

$$\gamma = \tfrac{12}{5}, \qquad c = -\tfrac{336}{625}. \tag{67}$$

So, the 5th-order equation (1) has the following traveling wave solution

$$u = (x + \tfrac{336}{625}t)^{-12/5}. \tag{68}$$

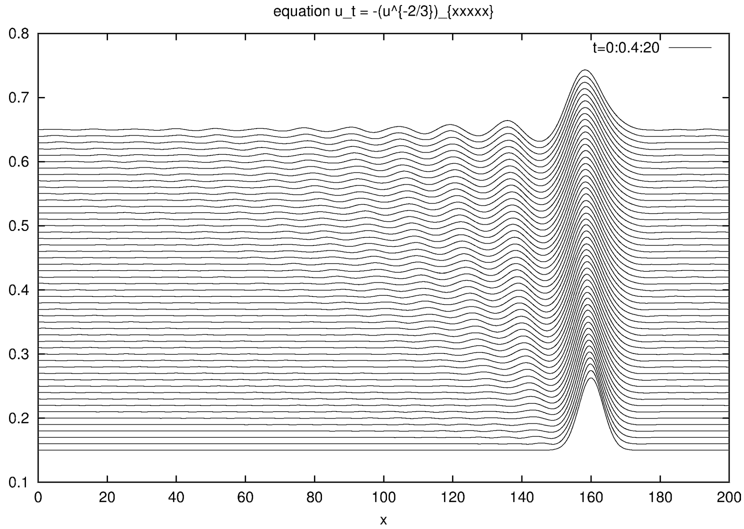


Figure 3. This is the stable solution for the 5th-order equation $u_t = \partial_x^5 u^{-2/3}$ under the Gaussian initial condition.

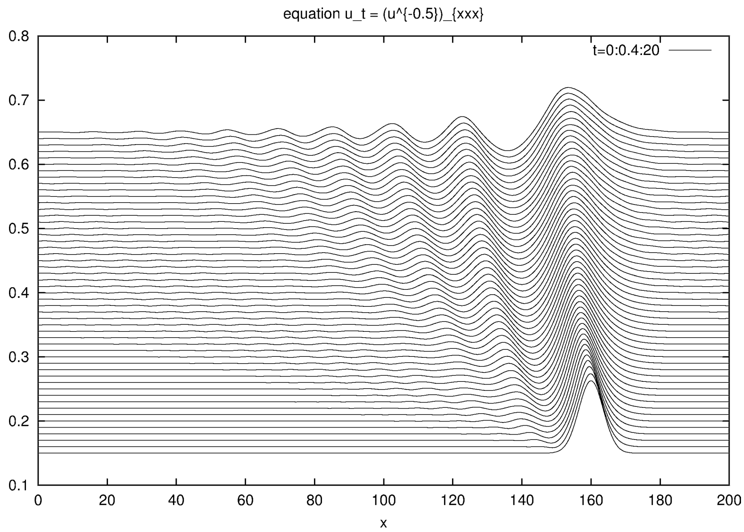
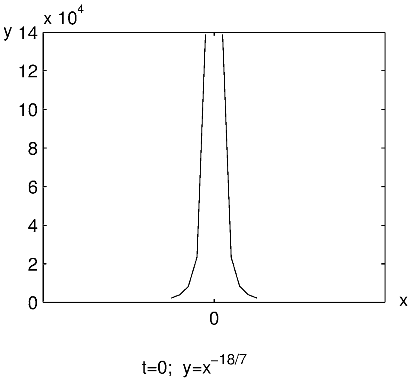


Figure 4. This is the stable solution for the Harry–Dym equation $u_t = \partial_x^3 u^{-1/2}$ under the Gaussian initial condition.

Although at each time solution (68) has singular point at $x = -(336/625)t$, this 5th-order equation has the smooth and stable traveling wave solution under the Gaussian initial condition (see Figure 3).

So, Figure 3 of the equation $u_t = \partial_x^5 u^{-2/3}$ has a slight difference from Figure 4 of the Harry–Dym equation $u_t = \partial_x^3 u^{-1/2}$.



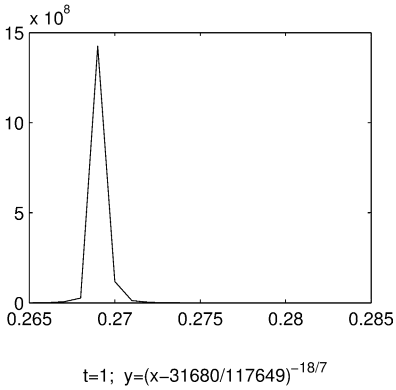


Figure 5. Solution near singular point.

Third, we give the traveling wave solution for the new integrable 7th-order equation (2). Set $u = \xi^{-\gamma}$, $\xi = x - ct$ (c is a constant speed to be determined), then we have

$$\gamma = \tfrac{18}{7}, \qquad c = \tfrac{31680}{117649}. \tag{69}$$

So, the 7th-order equation (1) has the following traveling wave solution (see Figure 5)

$$u = (x - \tfrac{31680}{117649}t)^{-18/7}. \tag{70}$$

Furthermore, we propose the following new equations:

$$u_t = \partial_x^l u^{-m/n}, \quad l \geqslant 1, n \neq 0, m, n \in \mathbb{Z}. \tag{71}$$

This equation has the following traveling wave solution

$$u(x,t) = (x - ct)^{-n(l-1)/(m+n)}, \quad c = \frac{m}{n}\prod_{k=1}^{l-1}\left(\frac{m(l-1)}{m+n} - k\right). \tag{72}$$

Apparently, if $mn + n^2 > 0$ this solution has singularity at point $x = ct$ at each time, and if $mn + n^2 < 0$ this solution is a polynomial traveling wave solution which is smooth.

Remark 3. Here are the cusp-like traveling wave solutions with singularities

$$u(x,t) = (x - \tfrac{2}{9}t)^{-4/3} \tag{73}$$

and

$$u(x,t) = (x + \tfrac{336}{626}t)^{-12/5} \tag{74}$$

for the Harry–Dym equation $u_t = \partial^3(u^{-1/2})$ and the 5th-order equation $u_t = \partial^5(u^{-2/3})$.

7. Conclusions

In Section 5, we obtain the parametric solution (54) of the 5th-order equation (1). This parametric solution does not include its traveling wave solution $u = (x + (336/625)t)^{-12/5}$ because the parametric solution is smooth everywhere, but the traveling wave solution has singularity.

The traveling wave solutions $u = (x + (336/625)t)^{-12/5}$ for the equation $u_t = \partial^5 u^{-2/3}$ and $u = (x-(31680/117649)t)^{-18/7}$ for the equation $u_t = \partial_x^5(((u^{-1/3})_{xx} - 2(u^{-1/6})_x^2)/u)$ are singular at each time. That is, the singularity property travels with the time t (see Figure 5). Actually, when $n(m+n) > 0$ the traveling wave solution (72) for general Equation (71) is also matching this property. A natural question arises here: is Equation (71) integrable for all $l \geqslant 1, m, n \in \mathbb{Z}$ or for what kind of $l \geqslant 1, m, n \in \mathbb{Z}$ is it integrable? We will discuss this elsewhere.

The Harry–Dym equation has the cusp-like traveling wave solution $u(x,t) = (x - (2/9)t)^{-4/3}$, but this is not cusp soliton which Wadati described in [32], because the current traveling wave solution is singular, but the cusp is continuous.

If we consider other constraints between the potential and eigenfunctions, then we can still get parametric solutions for the other two equations

$$u_t = \partial_x^5 \frac{(u^{-1/3})_{xx} - 2(u^{-1/6})_x^2}{u},$$
$$u_{xxt} + 3u_{xx}u_x + u_{xxx}u = 0.$$

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