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A new integrable equation with no smooth solitons

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Abstract

In this paper, we propose a new completely integrable equation:

$$m_t = \frac{1}{2}\left(\frac{1}{m^2}\right)_{xxx} - \frac{1}{2}\left(\frac{1}{m^2}\right)_x,$$

which has no smooth solitons. This equation is shown to have bi-Hamiltonian structure and Lax pair, which imply integrability of the equation. Studying this new equation, we develop two new kinds of soliton solutions under the inhomogeneous boundary condition $\lim_{|x|\to\infty} m = B$ where B is nonzero constant. One is continuous and piecewise smooth "W/M"-shape-peaks solitary solution and the other one-single-peak soliton. The two new kinds of peaked solitons can not be written as the regular type peakon: $c\mathrm{e}^{-|x-ct|}$, where c is a constant. We will provide graphs to show those new kinds of peaked solitons.

1. Introduction

Recently, the study of peaked and cusped soliton equations has arisen lot of attractive attention. The typical representative of such equations is the well-known Harry–Dym (HD) equation [8]

$$u_t = \left(\frac{1}{\sqrt{u}}\right)_{xxx}.$$

Wadati et al. [18] generalized the HD equation to an integrable hierarchy. In their paper [19,20], Wadati et al. first time proposed the cusp soliton, which is a kind of peaked soliton whose left and right derivatives equal infinities, for the HD equation. Later, there are several authors studying the cusp and peaked soliton solutions for the integrable equations [2–5,9,11,13,15–17].

In this paper, we propose a new peaked soliton equation:

$$m_t = \frac{1}{2}\left(\frac{1}{m^2}\right)_{xxx} - \frac{1}{2}\left(\frac{1}{m^2}\right)_x, \tag{1}$$

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where m is a scalar function and subscripts denote the partial derivatives. This equation is shown to have bi-Hamiltonian structure, and Lax pair that implies its integrability. Through studying equation (1), we develop two new kinds of soliton solutions under the inhomogeneous boundary condition $\lim_{|x|\to\infty} m = B$, where B is nonzero constant. One is continuous and piecewise smooth "W/M"-shape-peaks solitary solution and the other one-single-peak soliton. The two new kinds of peaked solitons cannot be equivalent to the regular peakon: $ce^{-|x-ct|}$, where c is a constant. There is no smooth soliton found for the new Eq. (1). We will take some graphs to show how these three peaks soltions and one-single-peak solitons look like.

2. Hamiltonian structure and integrability

Eq. (1) can be cast in the following Hamiltonian structure:

$$m_t = \frac{1}{2}\left(\frac{1}{m^2}\right)_{xxx} - \frac{1}{2}\left(\frac{1}{m^2}\right)_x = J\frac{\delta H_1^+}{\delta m} = K\frac{\delta H_0^+}{\delta m}, \tag{2}$$

where

$$J = -\partial m \partial^{-1} m \partial, \tag{3}$$

$$K = \partial^3 - \partial,\ \partial = \frac{\partial}{\partial x}, \tag{4}$$

$$H_0^+ = -\frac{1}{2}\int_\Omega \frac{1}{m}\,\mathrm{d}x,$$

$$H_1^+ = -\frac{1}{2}\int_\Omega \left(\frac{1}{4m^3} + \left(\frac{4}{5m^5} + \frac{4}{7m^7}\right)m_x^2\right))\,\mathrm{d}x,$$

$\Omega = (x_0, x_0 + T)$ or $\Omega = (-\infty, +\infty)$ is the domain of m that needs to be periodic with T or to approach the same constant as x goes to $\pm\infty$, and H_0^+, H_1^+ are two Hamiltonian functions. Both operator K and operator J are Hamiltonian, and furthermore our Eq. (1) is bi-Hamiltonian (see Remark 1).

Remark 1. Apparently, the operator $K = \partial^3 - \partial$ is Hamiltonian (see [10], chapter 7) because of constant coefficients and skew-symmetric property. From Ref. [10], we also know that the operator J is Hamiltonian if and only if $\mathrm{Pr}V_{J\theta}(\mathscr{A}_J) = 0$, where

$$\mathscr{A}_J = \frac{1}{2}\int (\theta \wedge J\theta)\,\mathrm{d}x$$

is the associated bi-vector of J, and θ is a basic uni-vector corresponding to m. Let $P = \partial^{-1} m\theta_x$, then $P_x = m\theta_x$, $J\theta = -(mP)_x$, $\mathscr{A} = -\frac{1}{2}\int \theta \wedge (mP)_x\,\mathrm{d}x$, and

$$\begin{aligned}\mathrm{Pr}V_{J\theta}(\mathscr{A}_J) &= -\frac{1}{2}\int \left(\theta \wedge J\theta \wedge (mP)_x - \theta_x \wedge J\theta \wedge P\right)\mathrm{d}x = \frac{1}{2}\int \left(\theta \wedge (mP)_x \wedge (mP)_x + \theta_x \wedge (m_x P + mP_x) \wedge P\right)\mathrm{d}x \\ &= \frac{1}{2}\int \left(\theta_x \wedge (m_x P + m^2\theta_x) \wedge P\right)\mathrm{d}x = 0.\end{aligned}$$

So, J is Hamiltonian. In a similar way, we can prove that $K + J$ is also Hamiltonian. Therefore, K and J form a Hamiltonian pair.

In order to show the integrability of this equation, let us consider the following spectral problem

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\lambda m \\ -\frac{1}{2}\lambda m & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \equiv U(m,\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{5}$$

where λ is a spectral parameter, m is a scalar potential function periodic or approaching the same constant at both infinities, and $\psi = (\psi_1, \psi_2)^{\mathrm{T}}$ is the spectral function corresponding to the spectral parameter λ. Then, we have

$$K\nabla\lambda = \lambda^2 J\nabla\lambda, \tag{6}$$

where $\nabla\lambda = \frac{\lambda}{2}(\psi_1^2 + \psi_2^2)$.

Remark 2. Eq. (6) plays a very important role in the discussions of the periodic solutions of the new wave equation (1), which we will deal with in a subsequent paper [16]. Actually, on the basis of those two operators, following our earlier method [12,14] we are able to generate a new integrable hierarchy.

A direct calculation leads to the following statement.

Eq. (1) has the following Lax pair:

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x = U(m,\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{7}$$

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_t = V(m,\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{8}$$

where

$$U(m,\lambda) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\lambda m \\ -\frac{1}{2}\lambda m & \frac{1}{2}\end{pmatrix},$$

$$V(m,\lambda) = \frac{\lambda}{2}\begin{pmatrix} -\frac{\lambda}{m} & \lambda^2 + \frac{m(m_x - m_{xx}) + 3m_x^2}{m^4} \\ -\lambda^2 + \frac{m(m_x + m_{xx}) - 3m_x^2}{m^4} & \frac{\lambda}{m}\end{pmatrix}.$$

In fact, one can use mathematical software Maple to check that the compatibility condition $\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_{xt} = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_{tx}$, namely

$$U_t - V_x + [U,V] = 0$$

generates equation (1).

So the wave equation (1) is accordingly completely integrable by the Inverse Scattering Transformation [1].

3. W/M-shape-peaks solitons and new one-single-peak solitons

3.1. Traveling wave setting

Let $m(x,t) = \frac{1}{\sqrt{v(x,t)}}$, then Eq. (1) becomes

$$-\frac{\frac{\partial}{\partial t}v(x,t)}{v(x,t)^{(3/2)}} = \frac{\partial^3}{\partial x^3}v(x,t) - \frac{\partial}{\partial x}v(x,t). \tag{9}$$

Let us consider the traveling wave solutions of the Eq. (9) through a generic setting $v(x,t) = U(\xi)$, where $\xi = x - ct$, and c is the wave speed. Substituting it into Eq. (9) yields the following ODE:

$$U_{\xi\xi\xi} - U_\xi = cU^{-3/2}U_\xi. \tag{10}$$

Apparently $U =$ constant is a solution, which is not interesting for us. Let us find non-trivial solutions. Taking indefinite integral twice on both sides of the ODE (10), we obtain

$$\frac{2c}{\sqrt{U}} + U_{\xi\xi} - U + C_1 = 0, \tag{11}$$

$$4c\sqrt{U} + C_1 U - \frac{U^2}{2} + \frac{1}{2}U_\xi^2 + C_2 = 0, \tag{12}$$

where C_1 and C_2 are two constants to be determined.

To have solitary traveling wave solutions, we set $U = V^2$ and impose the boundary condition

$$\lim_{\xi\to\pm\infty} V = A, \quad A > 0, \tag{13}$$

which implies $m \to \frac{1}{A}$ as x approaches $\pm\infty$ (see paper [15,17] for more details). Substituting the boundary condition (13) into the ODEs (11) and (12) generates the following two constants

$$C_1 = A^2 - \frac{2c}{A}, \tag{14}$$

$$C_2 = -\frac{1}{2}A^4 - 2cA. \tag{15}$$

So the ODE (12) becomes

$$U'^2 = U^2 + \frac{2(2c - A^3)}{A}U - 8c\sqrt{U} + A^4 + 4cA. \tag{16}$$

3.2. W/M-shape-peaks solitons

Setting $U = V^2$ and taking integral on both sides of the ODE (16), we arrive at

$$2\ln(A + V + \sqrt{(A+V)^2 + \frac{4c}{A}}) - \frac{\sqrt{A^3}}{\sqrt{A^3 + c}}\left(2\ln 2 + \ln\frac{A^3 + 2c + A^2V + \sqrt{(A^3+c)(AV^2 + 2A^2V + A^3 + 4c)}}{A(V-A)}\right)$$
$$= -|\xi|.$$

In general, we can not get an explicit form of V. But, if $\frac{\sqrt{A^3}}{\sqrt{A^3+c}} = 2$, namely, $c = -\frac{3}{4A^3}$, then we have

$$2\ln(A + V + \sqrt{V^2 + 2AV - 2A^2}) - 2\ln 2 - 2\ln\frac{A(-A + 2V + \sqrt{V^2 + 2AV - 2A^2})}{V - A} = -|\xi|,$$

which implies

$$V = A\frac{3 + 2X + 3X^2 - \sqrt{3(3 + 2X + 3X^2)(X-1)^2}}{4X},$$
$$X = e^{-\frac{1}{2}|\xi| + \ln 2},$$
$$\xi = x + \frac{3}{4}A^3 t.$$

Since $m = \frac{1}{V}$, we denote $B = \frac{1}{A} \neq 0$, then $m \to B$ as $\xi \to \pm\infty$, therefore we obtain the following explicit solution of Eq. (1):

$$m(x,t) = \frac{B}{2}\left(1 + \sqrt{6}\frac{\sinh|\frac{s}{2}|}{\sqrt{3\cosh s + 1}}\right),$$
$$s = \frac{1}{2}\left|x + \frac{3}{4B^3}t\right| - \ln 2, \tag{17}$$

whose 3D and 2D graphs are plotted in Fig. 1 for $B = 1$. This solution is of W-shape-peaks soliton [15,16] and has three peaks, and its profile looks like a "W" type wave. So, we called it "W-shape-peaks" soliton. Three peaks occur at $x = -\frac{3}{4}t_0$, $x = -\frac{3}{4}t_0 - 2\ln 2$, $x = -\frac{3}{4}t_0 + 2\ln 2$, for each time t_0. See graph 1 for more details.

We can also set $m = -\frac{1}{V}$ and take negative $B \neq 0$ as its infinities limit. The graph, corresponding to $B = -1$ and the solution form (17), is a "M-shape-peaks" soliton solution of Eq. (1), see Fig. 2.

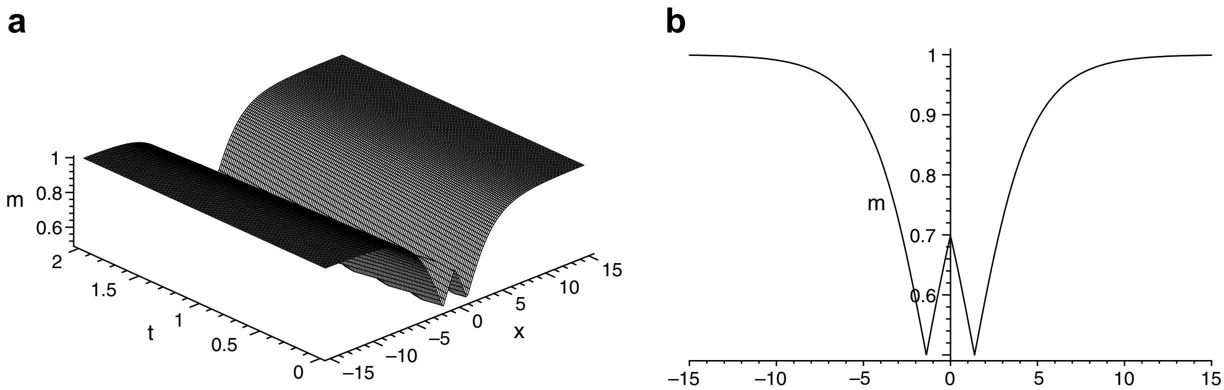

Fig. 1. (a) 3D graph of the explicit solution $m(x,t)$ defined by (17) when $B = 1$, wave speed $c = -3/4$, and intervals of x, t, m: $-15 \leqslant x \leqslant 15$, $0 \leqslant t \leqslant 2$, $0 \leqslant m \leqslant 1$. (b) 2D graph of the explicit solution $m(x,t)$ defined by (17) at $t = 0$. This is a W-shape-peaks soliton solution.

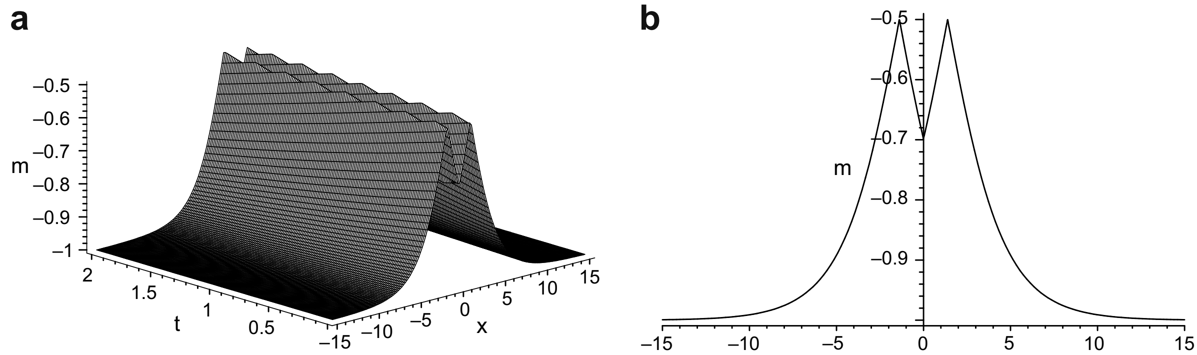

Fig. 2. (a) 3D graph of solution (17) when wave speed $c = 3/4$, and intervals of x, t, m: $-15 \leqslant x \leqslant 15$, $0 \leqslant t \leqslant 2$, $0 \leqslant m \leqslant 1$. (b) 2D graph of solution (17) at $t = 0$. This is a M-shape-peaks soliton solution.

3.3. One-single-peak solitons

We already know that Eq. (1) has three peaks (either W-shape-peaks or M-shape-peaks) soliton solutions. Let us consider the solution $m(x, t)$, defined by (17), without the absolute value of $x + \frac{3}{4B^3}t$. So, we create

$$M(x,t) = B\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{\frac{X^2 - 2X + 1}{3X^2 + 2X + 3}}\right),$$
$$X = \mathrm{e}^{-\frac{1}{2}\left(x + \frac{3}{4B^3}t\right)}, \quad B > 0. \tag{18}$$

Note no absolute value in X's expression. A direct verification reveals that $M(x, t)$ still satisfies Eq. (1).

We view solution (18) as a function of $\xi = x + \frac{3}{4B^3}t$. Then apparently, $M(\xi)$ has the following properties:

$$M(0) = \frac{1}{2}B, \quad M'(0+) = \frac{\sqrt{6}}{8}B, \quad M'(0-) = -\frac{\sqrt{6}}{8}B.$$

So, we found a continuous and piecewise-smooth (but not smooth) soliton solution for our new Eq. (1). See the graphs of $M(x, t)$ in Fig. 3.

Regarding negative $B < 0$, let us take $B = -1$ as a representative. In this case, we have $M(0) = -\frac{1}{2}$, $M'(0+) = -\frac{\sqrt{6}}{8}$, $M'(0-) = \frac{\sqrt{6}}{8}$ which imply that $M(\xi)$ is an anti-peaked continuous and piecewise-smooth soliton. See Fig. 4 for more details.

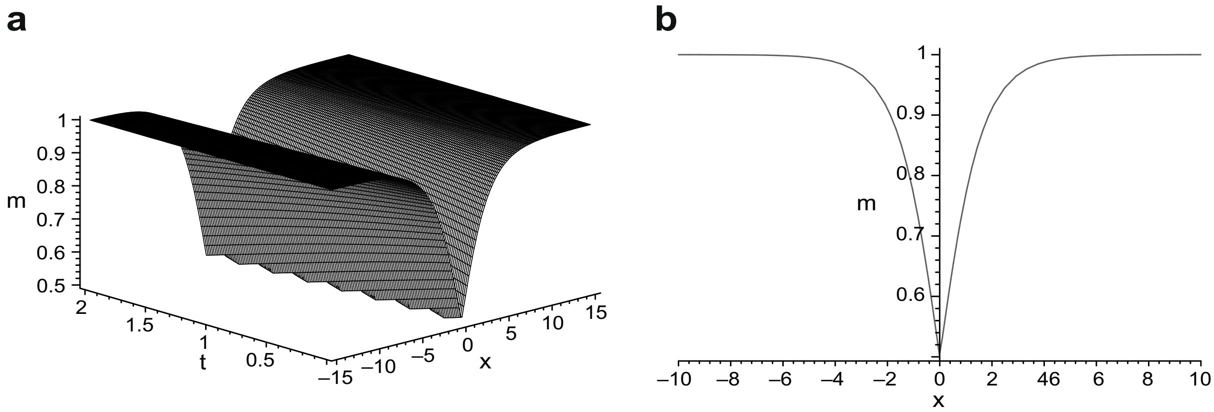

Fig. 3. (a) 3D graph of the explicit solution $M(x, t)$ defined by (18) when $B = 1$, wave speed $c = -3/4$, and intervals of x, t, M: $-15 \leqslant x \leqslant 15$, $0 \leqslant t \leqslant 2$, $0 \leqslant M \leqslant 1$. (b) 2D graph of the explicit solution $M(x, t)$ defined by (18) at $t = 0$. This is a single peak soliton solution.

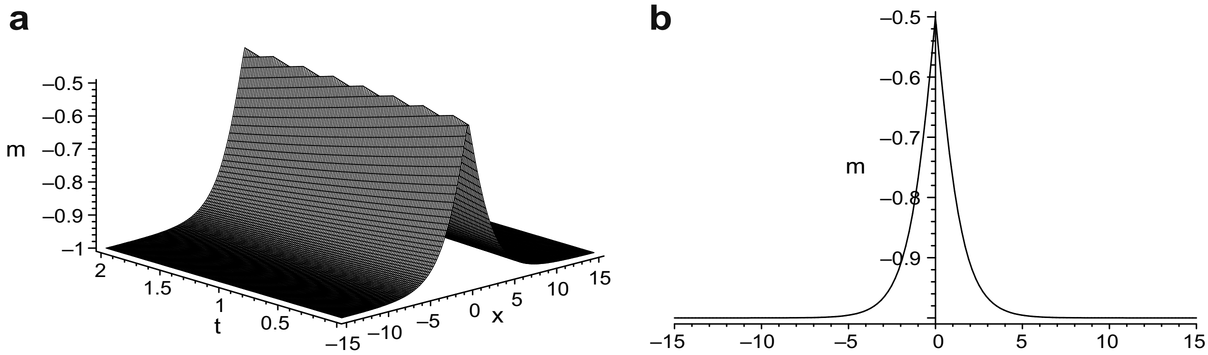

Fig. 4. 3D and 2D graphs of a continuous and piecewise-smooth soliton solution for Eq. (1) with negative amplitude $B = -1$. This is a single peak soliton solution.

4. Conclusions and open problems

In the paper, we present a new integrable equation (1). Through the regular traveling wave setting for our Eq. (1), we develop two new types of soliton solutions: one is "W-shape-peaks"/"M-shape-peaks" soliton (three peaks, continuous and piecewise smooth, but not smooth, see Figs. 1 and 2), and the other is one-single-peak soliton solution (also continuous and piecewise smooth, but not smooth, see Figs. 3 and 4). Those solutions are apparently different from regular peakons. No smooth solitons are found for our equation, but our equations are completely integrable. Namely, in this paper we provide an integrable system with no smooth solitons.

We try to construct the interaction (both collision and chase) of two single peaked solitons, two W-shape-peaks solitons (WW), two M-shape-peaks solitons (MM), WM, MW, and one single peaked and the other M/W solitons. But, that is a really hard procedure because of the following two major reasons:

1. So far we have an effective numerical scheme to solve our PDE (10). The solutions are not smooth, e.g. our solitons (18) and (19) have three peaks and one peak, respectively. We tried using the superposition of two single solitons (19) with same A and different wave speed c as an initial condition of the PDE (10). However, the usual Finite Difference schemes could neither capture the collision nor the chase. The authors have the impression that in order to capture the wave interactions numerically some special techniques need to be developed for this specific Eq. (10). This is the task of our future study.
2. We do not have a theoretical ansatz to deal with the peaked 2-soliton or N-soliton solutions of our equation like the CH equation with $\sum_{j=1}^{N} p_j(t)\mathrm{e}^{-|x-q_j(t)|}$, although we are seeking for. We tried extending our peaked soliton solutions (18) and (19) to the form of $\sum_{j=1}^{N} p_j(t)m(\mathrm{e}^{-|x-q_j(t)|})$ for the purpose of multi-soliton solutions. However, that is not the case for our equation. There is no smooth soliton for our equation though it is completely integrable. This causes difficulty to discuss multi-solitons. Finding what ansatz is appropriate for our equation will be a crucial work for discussing interaction of two peaked solitons.

Furthermore, we suggest a more general partial differential equation:

$$m_t = \frac{1}{2}\left(\frac{1}{m^k}\right)_{xxx} - \frac{1}{2}\left(\frac{1}{m^k}\right)_x \tag{19}$$

with a constant $k \in \mathbb{R}$. When $k = 2, 1/2, 0, -1$, the equation is integrable. For $k = 0$, that is trivial case; for $k = -1$, linear case; for $k = 1/2$, Harry-Dym type case; for $k = 2$, we already discussed in this paper. Any other integrable cases? We will study in the near future.

The ODE (16) has a physical meaning and can be cast into the Newton equation $U'^2 = S(U) - S(A^2)$ of a particle with a new potential $S(U) = U^2 + \frac{2(2c-A^3)}{A}U - 8c\sqrt{U}$, or can be converted to $V'^2 = T(V) - T(A)$ with $U = V^2$, $T(V) = \frac{V^2}{4} - \frac{2c}{V} + \frac{A(A^3+4c)}{4V^2}$. In the paper, we successfully solved this new Newton system with new one-single-peak solitons and M/W-shape-peaks solitons. The new Newton system might have potential applications in the study of engineering of loop solitons on a vortex filament with axial flow [6,7].

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