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一类耦合非线性扩散方程及 Liouville 完全可积系统*

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摘 要 本文引入了一个保谱问题，并导出了相应的耦合非线性演化方程族，得到了两个新的 Liouville 意义下的有限维完全可积系统.

关键词 耦合非线性扩散方程；经典可积系统.

本文将引入一个谱问题，并导出相应的耦合非线性演化方程族. 该族方程的第一个非平凡系统为

$$u_t = -\left(\frac{1}{\sqrt{1+2v+u^2}}\right)_{xx},\ v_t = \left(\frac{u}{\sqrt{1+2v+u^2}}\right)_{xx} \tag{1}$$

当 $v=-\frac{1}{2}$ 时，(1)导致著名的非线性扩散方程[1,2] $u_t=(u^{-2}u_x)_x$. 依据文[3,4]的思想，本文作者得到了两个新的 Liouville 意义下的有限维完全可积系统. 进一步作者将证明，非线性化后的谱问题是一个完全可积系统，并揭示该系统与一个驻定非线性演化方程间的关系.

1 演化方程

考虑谱问题

$$y_x = Uy,\quad U = \lambda\begin{pmatrix} 1+v & u+v \\ u-v & -1-v \end{pmatrix} \tag{2}$$

及辅助问题

$$y_t = Vy,\quad V = \lambda\begin{pmatrix} A+C & A+2B-C \\ -A+2B+C & -A-C \end{pmatrix} \tag{3}$$

其中 A, B, C 是 u, v 和 u, v 关于 x 的导数以及谱参数 λ 的函数. 设 $\lambda_t=0$, 相容条件(2)和(3)产生零曲率方程 $U_t-V_x+UV-VU=0$, 它等价于方程

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$$\begin{cases} u_t = 2B_x + 2\lambda(1+2v)C - 2\lambda A, \\ v_t = A_x - 2\lambda(1+2v)B + 2\lambda uA, \\ C_x - 2\lambda uC + 2\lambda B = 0 \end{cases} \tag{4}$$

令 $A=\sum_{j=0}^{m}A_j\lambda^{m-j}, B=\sum_{j=0}^{m}B_j\lambda^{m-j}, C=\sum_{j=0}^{m-1}C_j\lambda^{m-j}$ 并代入(4)式，合并 λ 的同类项得

$$A_0 = (1+2v)C_0,\ uA_0 = (1+2v)B_0,\ B_0 = uC_0 \tag{5}$$

$$\begin{cases} B_{jx} + (1+2v)C_{j+1} - A_{j+1} = 0,\ j = 0,1,\cdots,m-2, \\ A_{jx} - (1+2v)B_{j+1} + 2uA_{j+1} = 0,\ j = 0,1,\cdots,m-1, \\ C_{jx} - 2uC_{j+1} + 2B_{j+1} = 0, \quad j = 0,1,\cdots,m-2 \end{cases} \tag{6}$$

$$B_{m-1x} = A_m,\ C_{m-1x} = -2B_m \tag{7}$$

$$u_t = 2B_{mx},\ v_t = A_{mx}. \tag{8}$$

由(5)和(6)式，有(α 为常数)

$$C_0 = \frac{\alpha}{\sqrt{1+2v+u^2}},\ B_0 = \frac{\alpha u}{\sqrt{1+2v+u^2}},\ A_0 = \frac{\alpha(1+2v)}{\sqrt{1+2v+u^2}} \tag{9}$$

$$A_{jx} = -2uB_{jx} - (1+2v)C_{jx},\ j = 1,2,\cdots,m-1 \tag{10}$$

从(10)，(6)，(7)和(8)式，得到递推关系及演化方程

$$KG_{j-1} = JG_j,\ j = 1,2,\cdots,m-1 \tag{11}$$

$$(u_t, v_t)^T = KG_{m-1} \tag{12}$$

其中 K, J 是两个斜称算子

$$K = \begin{pmatrix} 0 & -\partial^2 \\ \partial^2 & 0 \end{pmatrix},\ J = \begin{pmatrix} 2\partial & -2\partial u \\ -2u\partial & -(1+2v)\partial - \partial(1+2v) \end{pmatrix},$$

$$G_j = \begin{pmatrix} B_j \\ C_j \end{pmatrix}.$$

(9)，(11)和(12)等价于非线性演化方程族

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = K\mathscr{L}^{m-1}G_0,\ G_0 = \frac{\alpha}{\sqrt{1+2v+u^2}}\begin{pmatrix} u \\ 1 \end{pmatrix},\ m = 1,2,\cdots, \tag{13}$$

这里

$$\mathscr{L} = J^{-1}K = \begin{pmatrix} \frac{-u}{2\sqrt{1+2v+u^2}}\partial^{-1}\frac{1}{\sqrt{1+2v+u^2}}\partial^2 & -\frac{1}{2}\partial + \frac{u}{2\sqrt{1+2v+u^2}}\partial^{-1}\cdot\frac{1}{\sqrt{1+2v+u^2}}\partial^2 \\ \frac{-1}{2\sqrt{1+2v+u^2}}\partial^{-1}\frac{1}{\sqrt{1+2v+u^2}}\partial^2 & \frac{1}{2\sqrt{1+2v+u^2}}\partial^{-1}\cdot\frac{u}{\sqrt{1+2v+u2}}\partial^2 \end{pmatrix}$$

当 $m=1, \alpha=1$ 时，(13)导致系统(1).

2 两个有限维对合系

在辛空间($R^{2N}, dp\wedge dq$)中,定义两个函数的 Poisson 括号

$$(F,G)=\sum_{j=1}^{N}\left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j}-\frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j}\right)$$
$$=<F_q,G_p>-<F_p,G_q>,$$

其中$<\cdot,\cdot>$是 R^N 中的标准内积,$q=(q_1,\cdots,q_N)^T$, $p=(p_1,\cdots,p_N)^T$. 令 $\lambda_1<\cdots<\lambda_N$. 定义

$$\Gamma_k=\sum_{\substack{j\neq k\\ j=1}}^{N}\frac{\lambda_k\lambda_j B_{kj}^2}{\lambda_k-\lambda_j},\ B_{kj}=p_kq_j-q_kp_j \tag{14}$$

引理1 $(\Gamma_i,\Gamma_k)=0$ (15)

证明 注意到公式

$$(B_{il}^2,B_{kj}^2)=4B_{ik}B_{ij}B_{lk}\delta_{lj}+4B_{ik}B_{ij}\delta_{lk}B_{lj}$$
$$+4B_{ik}\delta_{ij}B_{lk}B_{lj}+4\delta_{ik}B_{ij}B_{lk}B_{lj}$$

其中 δ_{ij}是 Kronecker 记号,我们有

$$(\Gamma_i,\Gamma_k)=4\lambda_k\lambda_i\sum_{l\neq i}\sum_{j\neq k}\lambda_j\lambda_l(\lambda_i-\lambda_l)^{-1}(\lambda_k-\lambda_j)^{-1}(B_{il}^2,B_{kj}^2)$$
$$=4\lambda_k\lambda_iB_{ik}\Big[\sum_{j\neq k,i}\frac{\lambda_j^2B_{ij}B_{jk}}{(\lambda_i-\lambda_j)(\lambda_k-\lambda_j)}+\sum_{j\neq k}\frac{\lambda_j\lambda_kB_{ij}B_{kj}}{(\lambda_i-\lambda_k)(\lambda_k-\lambda_j)}$$
$$+\sum_{j\neq i}\frac{\lambda_i\lambda_jB_{jk}B_{ji}}{(\lambda_i-\lambda_k)(\lambda_j-\lambda_i)}\Big]$$
$$=4\lambda_k\lambda_i(\lambda_k-\lambda_i)^{-1}B_{ik}\Big(\sum_j\lambda_jB_{jk}B_{ji}-\sum_j\lambda_jB_{jk}B_{ji}\Big)$$
$$=0$$

引理2 $(P_k^2,\Gamma_l)=4(\lambda_l-\lambda_k)^{-1}\lambda_k\lambda_lp_kp_lB_{kl}$, $(q_kp_k,q_lp_l)=0$ (16)

$(q_k^2,\Gamma_l)=4\lambda_k\lambda_l(\lambda_l-\lambda_k)^{-1}q_kq_lB_{kl}$,

$(q_kp_k,\Gamma_l)=2\lambda_l\lambda_k(\lambda_l-\lambda_k)^{-1}B_{kl}(q_kp_l+q_lp_k)$,

$(q_k^2,p_l^2)=4q_kp_l\delta_{kl}$, $(q_kp_k,p_l^2)=2p_lp_k\delta_{kl}$, (17)

$(q_kp_k,q_l^2)=-2q_kq_l\delta_{kl}$

证明 直接计算.

命题1 由

$E_k^{(1)}=aq_k^2+bp_k^2+cq_kp_k+\Gamma_k$, $\forall a,b,c\in R$ (18)

定义的 $E_1^{(1)},\cdots,E_N^{(1)}$ 构成N-对合系.

证明 $k=l$ 时,$(E_k^{(1)},E_l^{(1)})=0$ 显然成立.设 $k\neq l$,由引理2及 Poisson 括号的性质容易算得 $(E_k^{(1)},E_l^{(1)})=0$.

引理3

$$\begin{cases}(\Gamma_k, <\Lambda q, q>) = 4\lambda_k q_k^2 <\Lambda p, q> - 4\lambda_k p_k q_k <\Lambda q, q>, \\ (\Gamma_k, <\Lambda p, p>) = 4\lambda_k q_k p_k <\Lambda p, p> - 4\lambda_k p_k^2 <\Lambda q, p>, \\ (\Gamma_k, <\Lambda q, p>) = 2\lambda_k q_k^2 <\Lambda p, p> - 2\lambda_k p_k^2 <\Lambda q, q>\end{cases} \tag{19}$$

其中 $\Lambda = diag(\lambda_1, \cdots, \lambda_N)$.

证明 由引理2直接计算.

命题2 由

$$E_k^{(2)} = -2\Gamma_k + \frac{1}{2}(<\Lambda q, q> + <\Lambda p, p> - 2<\Lambda q, p>)(p_k^2 + q_k^2 + 2q_k p_k) + \frac{1-(<\Lambda p, p> - <\Lambda q, q>)^2}{2(<\Lambda p, p> + <\Lambda q, q> + 2<\Lambda q, p>)}(p_k^2 + q_k^2 + 2q_k p_k) \tag{20}$$

定义的 $E_1^{(2)}, \cdots, E_N^{(2)}$ 构成 N 对合系.

证明 令

$\theta_k = p_k^2 + q_k^2 + 2q_k p_k$, $\xi = <\Lambda q, q> + <\Lambda p, p> + 2<\Lambda q, p>$,

$\eta = <\Lambda p, p> + <\Lambda q, q> - 2<\Lambda q, p>$, $\zeta = <\Lambda p, p> - <\Lambda q, q>$.

由引理2及引理3知

$$\begin{cases}(\Gamma_k \xi = 4\lambda_k[(q_k^2 + p_k q_k) <\Lambda p, p> - <\Lambda q, q>(p_k^2 + p_k q_k) \\ \qquad + (q_k^2 - p_k^2) <\Lambda q, p>] \\ (\Gamma_k, \eta) = 4\lambda_k[(q_k p_k - q_k^2) <\Lambda p, p> + (p_k^2 - p_k p_k) <\Lambda q, q> \\ \qquad + (q_k^2 - p_k^2) <\Lambda q, p>] \\ (\Gamma_k, \zeta) = 4\lambda_k[p_k q_k <\Lambda q, q> + q_k p_k <\Lambda p, p> - (p_k^2 + q_k^2) <\Lambda q, p>] \\ (\Gamma_k, (1-\zeta^2)\xi^{-1}) = -2\zeta\xi^{-1}(\Gamma_k, \zeta) - (1-\zeta^2)\xi^{-1}(\Gamma_k, \xi)\end{cases} \tag{21}$$

容易计算

$$\begin{cases}(\theta_k, \xi) = 0, (\theta_k, \eta) = 8\lambda_k(p_k^2 - q_k^2) \\ (\theta_k, \theta_l) = 0, (k \neq l), (\theta_k, \zeta) = 4\lambda_k\theta_k \\ (\xi, \zeta) = 4(<\Lambda^2 p, p> + <\Lambda^2 q, q> + 2<\Lambda^2 q, p>) \\ (\eta, \xi) = 8<\Lambda^2 q, q> - 8<\Lambda^2 p, p> \\ (\eta, \zeta) = 4(2<\Lambda^2 q, p> - <\Lambda^2 p, p> - <\Lambda^2 q, q>)\end{cases} \tag{22}$$

令

$I_1 = -(\Gamma_k, \eta\theta_l) + (\Gamma_l, \eta\theta_k)$

$I_2 = -(\Gamma_k, (1-\zeta^2)\xi^{-1}\theta_l) + (\Gamma_l, (1-\zeta^2)\xi^{-1}\theta_k)$

$I_3 = \frac{1}{4}(\eta\theta_k, \eta\theta_l)$

$I_4 = \frac{1}{4}(\eta\theta_k, (1-\zeta^2)\xi^{-1}\theta_l) - \frac{1}{4}(\eta\theta_l, (1-\zeta^2)\theta_k\xi^{-1})$

$I_5 = \frac{1}{4}((1-\zeta^2)\theta_k\xi^{-1}, (1-\zeta^2)\theta_l\xi^{-1})$

由(21),(22)及 Poisson 括号的性质, 可得

$I_1 = -4\lambda_k\theta_l[(q_k p_k - q_k^2) <\Lambda p, p> + (p_k^2 - p_k q_k) <\Lambda q, q> + (q_k^2 - p_k^2) <\Lambda q, p$

$>]$

$$+4\lambda_l\theta_k[(q_lp_l-q_l^2)<\Lambda p,p>+(p_l^2-p_lq_l)<\Lambda q,q>+(q_l^2-p_l^2)<\Lambda q,p>]$$

$$I_2=4\lambda_k\theta_l(1-\zeta^2)\xi^{-2}[(q_k^2+p_kq_k)<\Lambda p,p>-(p_k^2+q_kp_k)<\Lambda q,q> +(q_k^2-p_k^2)<\Lambda q,p>]$$

$$-4\lambda_l\theta_k(1-\zeta^2)\xi^{-2}[(q_l^2+q_lp_l)<\Lambda p,p>-(p_l^2+p_lq_l)<\Lambda q,q> +(q_l^2-p_l^2)<\Lambda q,p>]$$

$$+8\lambda_k\theta_l\zeta\xi^{-1}[q_kp_k<\Lambda q,q>+q_kp_k<\Lambda p,p>-(p_k^2+q_k^2)<\Lambda q,p>]$$

$$+8\lambda_l\theta_k\zeta\xi^{-1}[(p_l^2+q_l^2)<\Lambda q,p>-q_lp_l<\Lambda q,q>-<\Lambda p,p>q_lp_l]$$

$$I_3=2\lambda_k\theta_l(p_k^2+q_k^2-2q_kp_k)\zeta-2\lambda_l\theta_k(p_l^2+q_l^2-2q_lp_l)\zeta$$

$$I_4=2(\lambda_l-\lambda_k)\eta\zeta\theta_k\theta_l\xi^{-1}+2(1-\zeta^2)\xi^{-1}[\lambda_k\theta_l(p_k^2-q_k^2)-\lambda_l\theta_k(p_l^2-q_l^2)]$$

$$I_5=2(\lambda_l-\lambda_k)\theta_k\theta_l(1-\zeta^2)\zeta\xi^{-2}$$

比较上面各式中的 λ_k 与 λ_l 的系数，可得 $(E_k^{(2)},E_l^{(2)})=\sum_{j=1}^{5}I_j=0$.

在 R^N 上定义一个双线性函数

$$Q_z(\beta,\mu)\equiv<(z-\Lambda)^{-1}\beta,\mu>=\sum_{k=1}^{N}\beta_k\mu_k(z-\lambda_k)^{-1}.$$

命题3
$$\begin{vmatrix}Q_z(\Lambda q,q) & Q_z(\Lambda q,p)\\ Q_z(\Lambda p,q) & Q_z(\Lambda p,p)\end{vmatrix}=\sum_{k=1}^{N}\frac{\Gamma_k}{z-\lambda_k} \tag{23}$$

证明 由于 $(z-\lambda_i)^{-1}\quad(z-\lambda_j)^{-1}=(z-\lambda_i)^{-1}(\lambda_i-\lambda_j)^{-1}+(z-\lambda_j)^{-1}(\lambda_j-\lambda_i)^{-1}$,

故 $Q_z(\Lambda q,q)Q_z(\Lambda p,p)-Q_z^2(\Lambda q,p)=\sum_{i,j}\lambda_i\lambda_j(q_i^2p_j^2-q_iq_jp_ip_j)(z-\lambda_i)^{-1}(z-\lambda_j)^{-1}$

$$=\frac{1}{2}\sum_{i,j}\lambda_i\lambda_jB_{ij}^2(z-\lambda_i)^{-1}(z-\lambda_j)^{-1}=\frac{1}{2}(\sum_i\Gamma_i(z-\lambda_i)^{-1}+\sum_j\Gamma_j(z-\lambda_j)^{-1})$$

$$=\sum_k\Gamma_k(z-\lambda_k)^{-1}.$$

由(23)式，作者分别得 $E_k^{(1)}$ 和 $E_k^{(2)}$ 的母函数

$$aQ(q,q)+bQ(p,p)+cQ(q,p)+\begin{vmatrix}Q_z(\Lambda q,p) & Q_z(\Lambda q,p)\\ Q_z(\Lambda p,q) & Q_z(\Lambda p,q)\end{vmatrix}=\sum_{k=1}^{N}\frac{E_k^{(1)}}{z-\lambda_k} \tag{24}$$

$$\frac{1}{2}(<\Lambda q,q>+<\Lambda p,p>-2<\Lambda q,p>)(Q_z(q,q)+Q_z(p,p)+2Q_z(q,p))$$

$$+\frac{1-(<\Lambda p,p>-<\Lambda q,q>)^2}{2(<\Lambda p,p>+<\Lambda q,q>+2<\Lambda q,p>}(Q_z(q,q)+Q_z(p,p)+2Q_z(q,p))$$

$$-2\begin{vmatrix}Q_z(\Lambda q,q) & Q_z(\Lambda p,q)\\ Q_z(\Lambda q,p) & Q_z(\Lambda p,p)\end{vmatrix}=\sum_{k=1}^{N}\frac{E_k^{(2)}}{z-\lambda_k} \tag{25}$$

命题4 令 $F_s^{(1)}=\sum_{k=1}^{N}\lambda_k^sE_k^{(1)},\ s=0,\pm1,\cdots$, 则

$$F_0^{(1)}=a<q,q>+b<p,p>+c<q,p>$$

$$F_m^{(1)}=a<\Lambda^mq,q>+b<\Lambda^mp,p>+c<\Lambda^mq,p> +\sum_{i+j=m-1}\begin{vmatrix}<\Lambda^{j+1}q,q> & <\Lambda^{j+1}q,p>\\ <\Lambda^{i+1}p,q> & <\Lambda^{i+1}p,p>\end{vmatrix} \tag{26}$$

$$F_{-m-1}^{(1)} = a < \Lambda^{-m-1}q, q > + b < \Lambda^{-m-1}p, p > + c < \Lambda^{-m-1}q, p > - \sum_{i+j=m} \begin{vmatrix} < \Lambda^{-j}q, q > & < \Lambda^{-j}q, p > \\ < \Lambda^{-i}q, p > & < \Lambda^{-i}p, p > \end{vmatrix}$$

$$m = 0, 1, 2, \cdots, 及 (F_s^{(1)}, F_l^{(1)}) = 0, \ \forall s, l \in Z. \tag{27}$$

证明　当 $|z| > max\{|\lambda_1|, \cdots, |\lambda_N|\}$ 时, 于是有 $(z-\lambda_k)^{-1} = \sum_{m=0}^{\infty} z^{-m-1}\lambda_k^m$ 及 $Q_z(\beta, \mu) = \sum_{m=0}^{\infty} < \Lambda^m\beta, \mu > z^{-m-1}$. 将 Q_z 的 Laurent 展式及 $(z-\lambda_k)^{-1}$ 的幂级数展式分别代入(24)的两端, 比较 z^{-2-m} 的系数便得(26)式. 当 $|z| < min\{|\lambda_1|, \cdots, |\lambda_N|\}$ 时, 将 $(z-\lambda_k)^{-1} = -\sum_{m=0}^{\infty} z^m \lambda_k^{-m-1}$ 及 $Q_z(\beta, \mu) = -\sum_{m=0}^{\infty} z^m < \Lambda^{-m-1}\beta, \mu >$ 代入(24)式两端, 便得(27)式.

显然, 由 $E_k^{(1)}$ 的对合性可推出 $\{F_s^{(1)}\}$ 的对合性.

同样, 借助(25)式, 则得下述命题.

命题 5　令 $F_s^{(2)} = \sum_{k=1}^{N} \lambda_k^s E_k^{(2)}, s = 0, \pm 1, \cdots$, 则

$$F_0^{(2)} = \frac{1}{2}(< \Lambda q, q > + < \Lambda p, p > - 2 < \Lambda q, p >)(< p, p > + < q, q > + 2 < q, p >) + \frac{1 - (< \Lambda p, p > - < \Lambda q, q >)^2}{2(< \Lambda p, p > + < \Lambda q, q > + 2 < \Lambda q, p >)}(< p, p > + < q, q > + 2 < p, q >) \tag{28}$$

$$F_m^{(2)} = \frac{1}{2}(< \Lambda q, q > + < \Lambda p, p > - 2 < \Lambda q, p >)(< \Lambda^m q, q > + < \Lambda^m p, p > + 2 < \Lambda^m q, p >) + \frac{1 - (< \Lambda p, p > - < \Lambda q, q >)^2}{2(< \Lambda p, p > + < \Lambda q, q > + 2 < \Lambda q, p >)}(< \Lambda^m q, q > + < \Lambda^m p, p > + 2 < \Lambda^m q, p >) - 2\sum_{i+j=m-1} \begin{vmatrix} < \Lambda^{j+1}q, q > & < \Lambda^{j+1}p, q > \\ < \Lambda^{i+1}q, p > & < \Lambda^{i+1}p, p > \end{vmatrix}$$

$$F_{-m-1}^{(2)} = \frac{1}{2}(< \Lambda q, q > + < \Lambda p, p > - 2 < \Lambda q, p >)(< \Lambda^{-m-1}q, q > + < \Lambda^{-m-1}p, p > + 2 < \Lambda^{-m-1}q, p >) + \frac{1 - (< \Lambda p, p > - < \Lambda q, q >)^2}{2(< \Lambda p, p > + < \Lambda q, q > + 2 < \Lambda q, p >}(< \Lambda^{-m-1}q, q > + < \Lambda^{-m-1}p, p > + 2 < \Lambda^{-m-1}q, p >) + 2\sum_{i+j=m} \begin{vmatrix} < \Lambda^{-j}q, q > & < \Lambda^{-j}q, p > \\ < \Lambda^{-i}p, q > & < \Lambda^{-i}p, p > \end{vmatrix} \tag{29}$$

$m = 0, 1, 2, \cdots$ 及 $(F_s^{(2)}, F_l^{(2)}) = 0, \ \forall s, l \in Z$.

定理 1 Hamilton 系统

$$q_{t_s}=\frac{\partial F_s^{(j)}}{\partial p},\ p_{t_s}=-\frac{\partial F_s^{(j)}}{\partial q},\ j=1,2;s=0,\pm 1,\pm 2,\cdots \tag{30}$$

在 Liouviue 意义下都是完全可积的.

3 谱问题的非线性化

设 $\lambda_j(j=1,2,\cdots,N)$ 是 N 个不同的特征参数, $y_j(x)=(q_j,p_j)^T$ 是(2)的与之相应的解.容易证明

命题 6 $K\nabla\lambda_j=\lambda_j J\nabla\lambda_j,\ \nabla_{(u,v)}\lambda_j=(\lambda_j(p_j^2-q_j^2),\lambda_j(p_j^2+q_j^2+2q_jp_j))^T$ (31)

考虑约束: $G_0|_{\alpha=1}=\sum\limits_{j=1}^{N}\nabla\lambda_j$, 即

$$u=\frac{<\Lambda p,p>-<\Lambda q,q>}{<\Lambda p,p>+<\Lambda q,q>+2<\Lambda q,p>},$$

$$v=-\frac{1}{2}+\frac{1-(<\Lambda p,p>-<\Lambda q,q>)^2}{2(<\Lambda p,p>+<\Lambda q,q>+2<\Lambda q,p>)^2} \tag{32}$$

在(32)下,(2)的非线性化给出 Hamilton 系统

$$\begin{cases} q'=(1+v)\Lambda q+(u+v)\Lambda p=\dfrac{\partial H}{\partial p}, \\ p'=(u-v)\Lambda q-(1+v)\Lambda p=-\dfrac{\partial H}{\partial q} \end{cases} \tag{33}$$

其中 $'=\partial/\partial x$, u,v 由(32)定义, Hamilton 函数为

$$H=-\frac{1}{4}(<\Lambda p,p>+<\Lambda q,q>-2<\Lambda q,p>)$$
$$-\frac{1-(<\Lambda p,p>-<\Lambda q,q>)^2}{4(<\Lambda q,q>+<\Lambda p,p>+2<\Lambda q,p>)}$$

定理 2 设 (q,p) 是 Hamilton 系统(33)的一个解,则由(32)定义的 u,v 满足驻定的非线性演化方程

$$K\mathscr{L}^N G_0|_{\alpha=1}+\sum_{l=0}^{N-1}\beta_{N-l}K\mathscr{L}^l G_0|_{\alpha=1}=0 \tag{34}$$

证明 将算子 $(J^{-1}K)^l$ 作用到 $G_0|_{\alpha=1}=\sum\limits_{j=1}^{N}\nabla\lambda_j$ 上,利用(31)和(11),则有

$$\mathscr{L}^l G_0|_{\alpha=1}=\sum_{j=1}^{N}\lambda_j^l\nabla\lambda_j \tag{35}$$

考虑多项式 $(\beta_0=1)$

$$P(\lambda)=(\lambda-\lambda_1)\cdots(\lambda-\lambda_N)=\beta_0\lambda^N+\beta_1\lambda^{N-1}+\cdots+\beta_N.$$

将算子 $K\sum\limits_{l=0}^{N}\beta_{N-l}\cdot$ 作用于(35),便得(34).

定理 3 Hamilton 系统(33)是 Liouville 意义下的完全可积系统.

证明 经直接计算,可以验证 $(H,F_m^{(2)})=0,\ m=1,2,\cdots$.

定理 4 设 (q,p) 是 Hamilton 系统 $\{q'=H_p,\ p'=-H_q\}$ 和 $\{q_t=F_{2,p}^{(2)},\ p_t=-F_{2,q}^{(2)}\}$ 的一个解,则由(32)定义的 (u,v) 是方程(1)的一个解.

证明 显然，$\{q'=H_p, p'=-H_q\}$，亦即(33)是(1)的 Lax 对的空间部分. 借助 $G_0|_{a=1}=\sum_{j=1}^{N}\nabla\lambda_j$，(5)，(7)($m=1$)以及(33)，$\{q_t=F^{(2)}_{2,p}, p_t=-F^{(2)}_{2,q}\}$导致(1)的 Lax 对的时间部分. 因此，由(32)定义的(u,v)是(1)的解.

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A Hierarchy of Coupled Nonlinear Diffusion Equations and Liouville Completely Integrable Systems

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ABSTRACT An isospectral problem is proposed. The corresponding hierarchy of coupled nonlinear evolution equations is derived, they are extensions of the hierarchy of nonlinear diffusion equations. Two new hierarchies of finite－dimensional completely integrable systems in the liouviue sense are obtained, one of which is the spatial and time parts of nonlinearied Lax pairs of the hierarchy of coupled evolution equations.

KEY WORDS Coupled nonlinear diffusion equations, Classical integrable systems.