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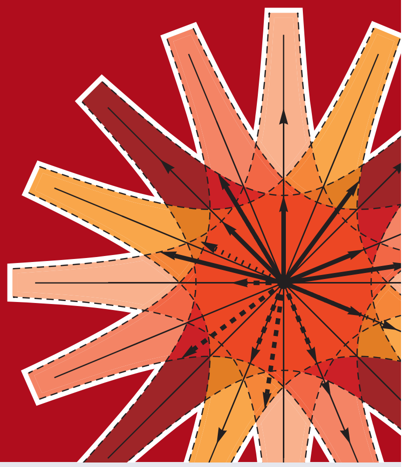
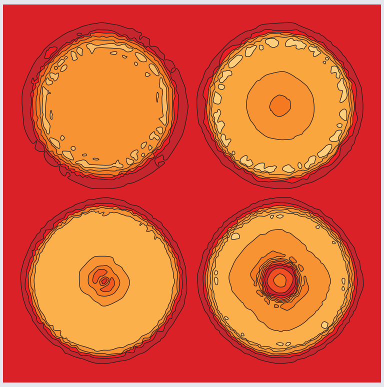
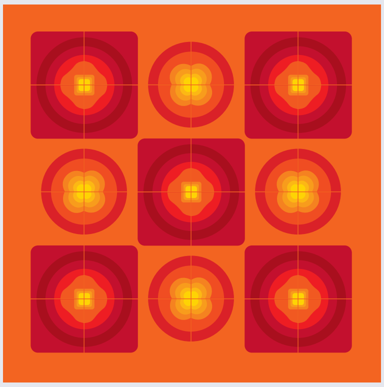


Image: Ornamental multiplication of space-time figures of temperature transformation rules (adapted from T. S. Bíró and P. Ván 2010 *EPL* **89** 30001; artistic impression by Frédérique Swist).



Biaxial strain on lens-shaped quantum rings of different inner radii, adapted from **Zhang *et al*** 2008 EPL **83** 67004.



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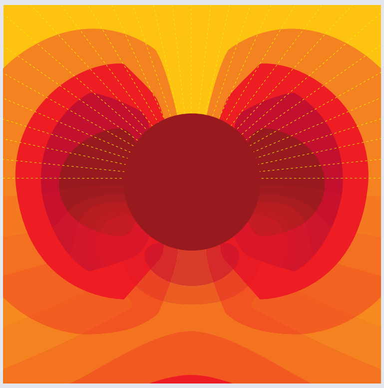

Artistic impression of velocity and normal stress profiles around a sphere that moves through a polymer solution, adapted from **R Tuinier, J K G Dhont and T-H Fan** 2006 *EPL* **75** 929.

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Negative-order KdV equation with both solitons and kink wave solutions

ZHIJUN QIAO[1(a)] and JIBIN LI[2]

[1] *Department of Mathematics, The University of Texas-Pan American 1201 W University Drive Edinburg, TX 78539, US*
[2] *Department of Mathematics, Zhejiang Normal University - Jinhua, Zhejiang, 321004, China*

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Abstract – In this paper, we report an interesting integrable equation that has both solitons and kink solutions. The integrable equation we study is $(\frac{-u_{xx}}{u})_t = 2uu_x$, which actually comes from the negative KdV hierarchy and could be transformed to the Camassa-Holm equation through a gauge transform. The Lax pair of the equation is derived to guarantee its integrability, and furthermore the equation is shown to have classical solitons, periodic soliton and kink solutions.

Introduction. – Soliton theory and integrable systems play an important role in the study of nonlinear water wave equations. They have many significant applications in fluid mechanics, nonlinear optics, classical and quantum fields theories etc. Particularly in recent years, more focuses have been pulled to integrable systems with non-smooth solitons, such as peakons, cuspons, since the study of the remarkable Camassa-Holm (CH) equation with peakon solutions [2]. Henceforth, much progress have been made in the study of non-smooth solitons for integrable equations [3–30].

In this paper, we consider the following integrable equation:

$$\left(\frac{-u_{xx}}{u}\right)_t = 2uu_x, \tag{1}$$

which is actually the first member in the negative KdV hierarchy [26]. Equation (1) is proven equivalent to the Camassa-Holm (CH) equation: $m_t + m_x u + 2mu_x = 0, m = u - u_{xx}$ through a gauge transform (see Remark 1 in the paper). Therefore, we find a simpler reduced form of the CH equation. The Lax pair of the eq. (1) is derived to guarantee its integrability, and furthermore the equation is shown to have classical solitons, periodic solitons and kink solutions.

(a)E-mail: qiao@utpa.edu

Derivation of eq. (1) and Lax representation. – Let us consider the Schrödinger-KdV spectral problem:

$$L\psi \equiv \psi_{xx} + v\psi = \lambda\psi, \tag{2}$$

where λ is an eigenvalue, ψ is the eigenfunction corresponding to the eigenvalue λ, and v is a potential function. One can easily get the following Lenard operator relation:

$$K\nabla\lambda = \lambda J\nabla\lambda, \tag{3}$$

where $\nabla\lambda \equiv \frac{\delta\lambda}{\delta v} = \psi^2$ is the functional gradient of the spectral problem (2) with respect to v, $K = \frac{1}{4}\partial^3 + \frac{1}{2}(v\partial + \partial v)$ and $J = \partial$ are two Hamiltonian operators as known in the literature [1].

By setting $v = -\frac{u_{xx}}{u}$, we have the product form of operators K, $\mathcal{L}$, L and their inverses

$$K = \frac{1}{4}u^{-2}\partial u^2\partial u^2\partial u^{-2},\ K^{-1} = 4u^2\partial^{-1}u^{-2}\partial^{-1}u^{-2}\partial^{-1}u^2,$$
$$\mathcal{L} = \frac{1}{4}\partial^{-1}u^{-2}\partial u^2\partial u^2\partial u^{-2},$$
$$\mathcal{L}^{-1} = 4u^2\partial^{-1}u^{-2}\partial^{-1}u^{-2}\partial^{-1}u^2\partial,$$
$$L = \partial^2 + v = u^{-1}\partial u^2\partial u^{-1},\ L^{-1} = u\partial^{-1}u^{-2}\partial^{-1}u,$$

where $\mathcal{L} = J^{-1}K$ and its inverse $\mathcal{L}^{-1} = K^{-1}J$ are the recursion operators for the positive-order and negative-order KdV hierarchy that we study below.

Now, according to Lenard's operators K and J, we construct the entire KdV hierarchy, and then we show the integrability of the hierarchy through solving a key operator equation.

Let $G_0 \in \text{Ker}\ \ J = \{G \in C^\infty(\mathbb{R}) | JG = 0\}$ and $G_{-1} \in \text{Ker}\ \ K = \{G \in C^\infty(\mathbb{R}) | KG = 0\}$. We define Lenard's sequence

$$G_j = \begin{cases} \mathcal{L}^j G_0, & j \in \mathbb{Z}, \\ \mathcal{L}^{j+1} G_{-1}, & j \in \mathbb{Z}, \end{cases} \tag{4}$$

where $\mathcal{L}, \mathcal{L}^{-1}$ are defined by eq. (2). Therefore we generate a hierarchy of nonlinear evolution equations (NLEEs):

$$v_{t_k} = JG_k = KG_{k-1}, \quad \forall k \in \mathbb{Z}, \tag{5}$$

which is called the entire KdV hierarchy. We will see below that the positive order ($k \geqslant 0$) gives the regular KdV hierarchy usually mentioned in the literature [1], while the negative order ($k < 0$) produces some interesting equations gauge-equivalent to the Camassa-Holm equation [2]. Apparently, this hierarchy possesses the bi-Hamiltonian structure because of the Hamiltonian properties of K, J. Let us now give special equations in the entire KdV hierarchy (5).

- Choosing $G_0 = 2 \in \text{Ker}\ J$ (therefore $G_1 = u$) leads to the second positive member of the hierarchy (5)

  $$v_{t_2} = \frac{1}{2} v_{xxx} + \frac{3}{2} v v_x, \tag{6}$$

  which is exactly the well-known KdV equation. Here there is nothing new. Therefore, the positive order ($k \geqslant 0$) in the hierarchy (5) yields the regular KdV hierarchy usually studied in the literature [1].

- Now, let us find kernel elements $G_{-1} \in \text{Ker}\ K$ in order to get the negative member of the hierarchy (5). Due to the product form of K and K^{-1}, $G_{-1} = K^{-1}0$ has the following three seed solutions:

  $$G_{-1}^1 = f(t_n) u^2, \quad G_{-1}^2 = g(t_n) u^2 \partial^{-1} u^{-2},$$
  $$G_{-1}^3 = h(t_n) u^2 \partial^{-1} u^{-2} \partial^{-1} u^{-2},$$

  where $f(t_n), g(t_n), h(t_n)$ are three arbitrarily given functions with respect to the time variables t_n, but independent of x. They produce three iso-spectral ($\lambda_{t_k} = 0$) negative-order KdV hierarchies of eq. (5)

  $$v_{t_k} = J\mathcal{L}^{k+1} \cdot G_{-1}^l, \ l = 1, 2, 3, \quad k = -1, -2, \ldots. \tag{7}$$

  When $k = -1$, their representative equations are

  $$\left(-\frac{u_{xx}}{u}\right)_{t_{-1}} = 2f(t_n)\, u u_x, \tag{8}$$

  $$\left(-\frac{u_{xx}}{u}\right)_{t_{-1}} = g(t_n) \left(2 u u_x \partial^{-1} u^{-2} + 1\right), \tag{9}$$

  $$\left(-\frac{u_{xx}}{u}\right)_{t_{-1}} = h(t_n) \big(2 u u_x \partial^{-1} u^{-2} \partial^{-1} u^{-2} + \partial^{-1} u^{-2}\big). \tag{10}$$

Remark 1. *Apparently, the first one is differential and simpler and exactly recovers the eq.* (1) *after setting* $f(t_n) = 1$, *which we focus on in the current paper. Actually, these three representative equations* (8), (9), *and* (10) *come from* $v_{t_{-1}} = JG_{-1} = JK^{-1}0$ *with* $v = -\frac{u_{xx}}{u}$. *Clearly,* $v_{t_{-1}} = JK^{-1}0$ *is equivalent to* $KJ^{-1}v_t = 0$ $(t = t_{-1})$, *that is,*

$$\left(\frac{v_{txx}}{v_x}\right)_x + 4\left(\frac{v v_t}{v_x}\right)_x + 2v_t = 0. \tag{11}$$

This equation is exactly the one studied by Fuchssteiner [15] *(see eqs.* (7.1) *and* (7.22)*) there. Equations* (7.1) *in* [15] *has a typo and should be same as* (11)*. From* [15]*, the Camassa-Holm (CH) equation is gauge-equivalent to eq.* (11) *through some hodograph transformations* (7.11) *and* (7.12) *in* [15]*. In our paper, through using* $v = -\frac{u_{xx}}{u}$ *we further reduce eq.* (11) *to a more simple form (i.e. eq.* (1)*):*

$$\left(-\frac{u_{xx}}{u}\right)_t = 2 u u_x. \tag{12}$$

In other words, we found a very interesting fact that eq. (12) *can be viewed as a reduction form of the CH equation due to the above gauge-equivalence. In the next section, we will solve this form.*

In paper [18], the author dealt with equation $(\partial^2 + 4v + 2v_x \partial^{-1}) v_t = 0$ by using the positive KdV hierarchy approach, and all soliton solutions were given implicitly. This equation could be transformed to $(-u_{xx}/u)_t = 2uu_x$ through $v = -u_{xx}/u$, like we mentioned earlier in our paper, but, this is only one of three reductions. So, solutions of this equation cannot give all solitons of our equation $(-u_{xx}/u)_t = 2uu_x$. In our paper, we present all solitons and kink solutions in an explicit form. Also, there is the connection of the first negative KdV equation $(\partial^2 + 4v + 2v_x \partial^{-1}) v_t = 0$ with sine-Gordon [19]. But, the equation $(-u_{xx}/u)_t = 2uu_x$ we propose in the current paper is not equivalent to the sine-Gordon equation, because the sine-Gordon equation has only kink solution while our equation has both kink solutions and classical solitons.

Of course, we may generate higher-order nonlinear equations by selecting different members in the hierarchy. In the following, we will see that all equations in the KdV hierarchy (5) are integrable. Particularly, the above three eqs. (8), (9), and (10) are integrable.

Let us return to the spectral problem (2). Apparently, the Gateaux derivative matrix $L_*(\xi)$ of the spectral operator L in the direction $\xi \in C^\infty(\mathbb{R})$ at point v is

$$L_*(\xi) \triangleq \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} U(u + \epsilon\xi) = \xi, \tag{13}$$

which is obviously an injective homomorphism, *i.e.* $U_*(\xi) = 0 \Leftrightarrow \xi = 0$.

For any given C^∞-function G, one may consider the following operator equation [31] with respect to $V = V(G)$

$$[V, L] = L_*(KG) - L_*(JG)L. \tag{14}$$

Theorem 1. *For the spectral problem* (2) *and an arbitrary* C^∞-function G, *the operator equation* (14) *has the following solution*:

$$V = -\frac{1}{4}G_x + \frac{1}{2}G\partial, \tag{15}$$

where $\partial = \partial_x = \frac{\partial}{\partial x}$, *and subscripts stand for the partial derivatives in* x.

Proof: A direct substitution will complete the proof.

Theorem 2. *Let* $G_0 \in \mathrm{Ker}\ J$, $G_{-1} \in \mathrm{Ker}\ K$, *and let each* G_j *be given through the Lenard sequence* (4). *Then,*

1) *each new vector field* $X_k = JG_k$, $k \in \mathbb{Z}$ *satisfies the following commutator representation*

$$L_*(X_k) = [V_k, L],\ \forall k \in \mathbb{Z}; \tag{16}$$

2) *the entire KdV hierarchy* (5), *i.e.*

$$v_{t_k} = X_k = JG_k,\ \forall k \in \mathbb{Z}, \tag{17}$$

possesses the Lax representation

$$L_{t_k} = [V_k, L],\ \forall k \in \mathbb{Z}, \tag{18}$$

where

$$V_k = \sum V(G_j)L^{(k-j-1)},\ \sum = \begin{cases} \sum_{j=0}^{k-1}, & k>0,\\ 0, & k=0,\\ -\sum_{j=k}^{-1}, & k<0, \end{cases} \tag{19}$$

and $V(G_j)$ *is given by eq.* (15) *with* $G = G_j$.

Proof: Let us only prove the case for $k<0$. We have

$$\begin{aligned} [V_k, L] &= -\sum_{j=k}^{-1}[V(G_j), L]L^{k-j-1}\\ &= -\sum_{j=k}^{-1}(L_*(KG_j) - L_*(JG_j)L)L^{k-j-1}\\ &= -\sum_{j=k}^{-1} L_*(KG_j)L^{(k-j-1)} - L_*(KG_{j-1})L^{k-j}\\ &= L_*(KG_{k-1}) - L_*(KG_{-1})L^k\\ &= L_*(KG_{k-1}) = L_*(JG_k)\\ &= L_*(X_k). \end{aligned}$$

Noticing $L_{t_k} = L_*(v_{t_k})$, we have

$$L_{t_k} - [V_k, L] = L_*(v_{t_k} - X_k).$$

The injectiveness of L_* implies the second result holds.

So, the entire KdV hierarchy (5) has the Lax pair and all equations in the hierarchy are therefore integrable. In particular, the KdV equation (6) has the Lax pair $L_{t_1} = [W_1, L]$ with $L = \partial^2 + v$ and $W_1 = \partial^3 + \frac{3}{2}v\partial + \frac{3}{4}v_x$, which was well known in the literature [1]. An interesting thing is that the first member ($k=-1$) in the negative-order KdV hierarchy (7) has the standard Lax representation $L_{t_{-1}} = [V_{-1}^l, L]$ with $V_{-1}^l = \left(\frac{1}{4}G_{-1,x}^l - \frac{1}{2}G_{-1}^l\partial\right)L^{-1}$, $l = 1,2,3$, $L = \partial^2 + v = u^{-1}\partial u^2 \partial u^{-1}$, and $L^{-1} = u\partial^{-1}u^{-2}\partial^{-1}u$. In particular, the negative KdV equation (8) possesses the following Lax form: $L_{t_{-1}} = [V_{-1}^1, L]$ with

$$\begin{aligned} V_{-1}^1 &= \left(\frac{1}{4}G_{-1,x}^1 - \frac{1}{2}G_{-1}^1\partial\right)L^{-1}\\ &= \left(\frac{1}{2}uu_x - \frac{1}{2}u^2\partial\right)L^{-1}\\ &= -\frac{1}{2}u\partial^{-1}u. \end{aligned}$$

All of those negative members in the hierarchies (7) are integrable.

All traveling-wave solutions of eq. (1). – Let us now consider the traveling-wave solution of eq. (1) through a generic setting $u(x,t) = U(x-ct)$, where c is the wave speed. Let $\xi = x - ct$, then $u(x,t) = U(\xi)$. Substituting it into eq. (1) yields

$$c\left(\frac{U''}{U}\right)' = 2UU'. \tag{20}$$

Integrating it once, we obtain the following standard cubic Hamiltonian system for $c \neq 0$:

$$\frac{\mathrm{d}U}{\mathrm{d}\xi} = y = \frac{\partial H}{\partial y}, \qquad \frac{\mathrm{d}y}{\mathrm{d}\xi} = gU + \frac{1}{c}U^3 = -\frac{\partial H}{\partial U}, \tag{21}$$

where g is an integral constant, and the Hamiltonian function is

$$H(U,y) = \frac{1}{2}y^2 - \frac{1}{2}gU^2 - \frac{1}{4c}U^4. \tag{22}$$

When $gc \geqslant 0$, (21) has only one equilibrium point $O(0,0)$. When $gc<0$, (21) has three equilibrium points $O(0,0)$ and $E_{1,2}(\pm\sqrt{|cg|}, 0)$. Write that

$$h_0 = H(0,0) = 0, \quad h_1 = H(\pm\sqrt{|cg|}, 0) = \frac{1}{4}cg^2.$$

By qualitative analysis, we have the bifurcations of phase portraits of (21) in the (c,g) parametric plane shown in fig. 1(1-1)–(1-6).

Next, we present the exact traveling-wave solutions of (1) in an explicit form.

Case 1: $c>0, g=0$ (see fig. 1(1-1)).

In this case, corresponding to the saddle point $O(0,0)$ (21) reads as $y = \pm\frac{U^2}{\sqrt{2c}}$. Using the first equation of (21) and taking integration, we obtain

$$U(\xi) = \mp\frac{\sqrt{2c}}{\xi + \xi_0}, \quad \xi = x - ct, \tag{23}$$

where ξ_0 is an initial value of ξ. Clearly, when $\xi \to -\xi_0, U(\xi) \to \infty$. *i.e.*, $U(\xi)$ is unbounded at $\xi = \xi_0$. Thus,

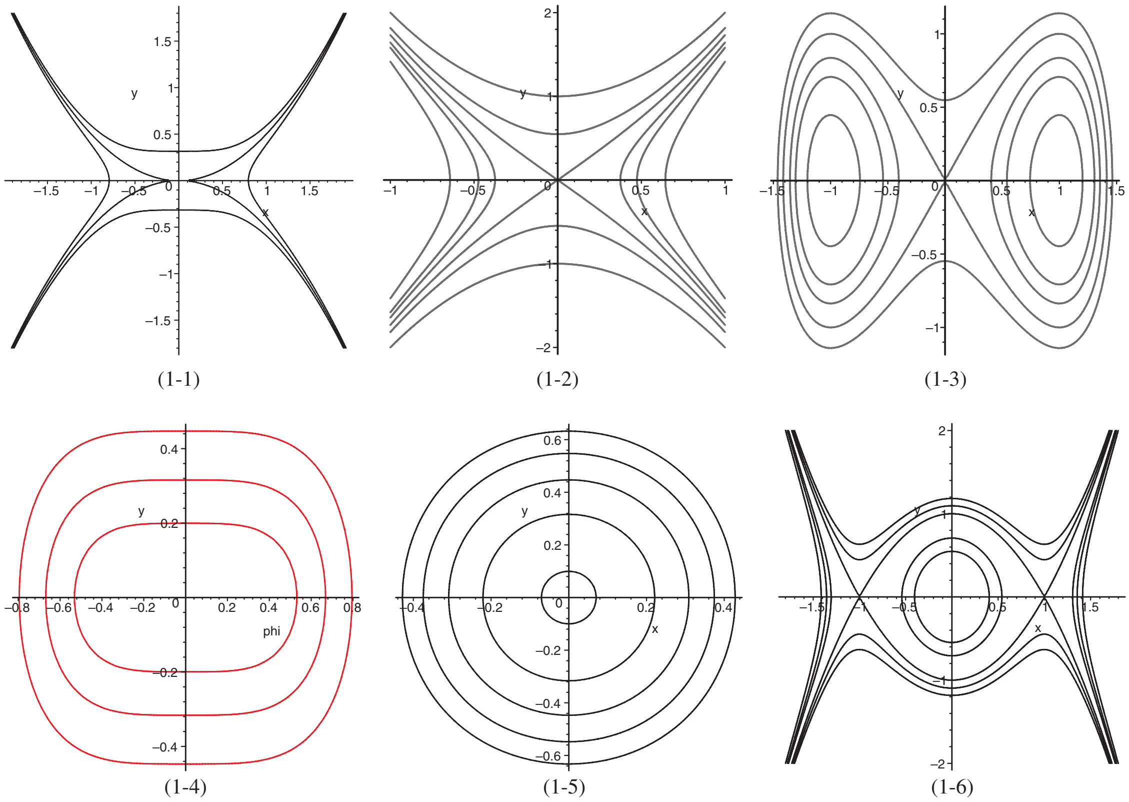

Fig. 1: (Colour on-line) The change of phase portraits of (21) in the (c, g) parameter plane. (1-1) $g = 0$, $c > 0$. (1-2) $g > 0$, $c > 0$. (1-3) $g > 0$, $c < 0$. (1-4) $g = 0$, $c < 0$. (1-5) $g < 0$, $c > 0$. (1-6) $g < 0$, $c < 0$.

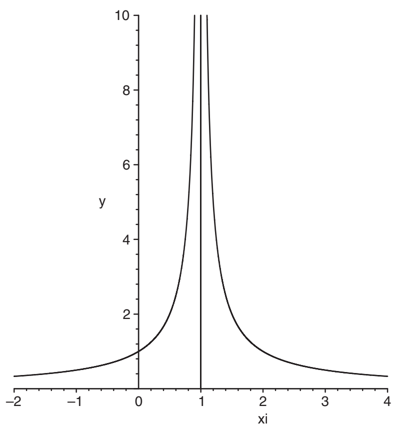

Fig. 2: The profiles of the functions (23) where $\xi_0 = -1$.

we have two unbounded breaking wave solutions shown in fig. 2.

Case 2: $c > 0, g > 0$ (see fig. 1(1-2)).

Corresponding to the saddle $O(0, 0)$ (21) reads as $y^2 = gU^2 + \frac{1}{2c}U^4$. Using the first equation of (21) to take integration, we obtain

$$U(\xi) = \pm \frac{8Acg}{A^2 e^{\sqrt{g}(\xi+\xi_0)} - 8cge^{-\sqrt{g}(\xi+\xi_0)}}, \qquad (24)$$

where A is an integrant constant. When $A^2 = 8cg$, the functions defined by (24) are discontinuous at $\xi = -\xi_0$. The profiles of (24) like fig. 2.

Case 3: $g > 0, c < 0$ (see fig. 1(1-3)).

There exist three equilibrium points of (21) at $E_{1,2}$ and $O(0, 0)$. O is a saddle point, $E_{1,2}$ are center points.

Corresponding to two homoclinic orbits, defined by $H(U, y) = 0$, we have the parametric representations

$$U(\xi) = \pm\sqrt{2|c|g}\ \mathrm{sech}\sqrt{\frac{|c|g}{2}}\xi. \qquad (25)$$

They give two soliton solutions of (1).

For $h \in (\frac{1}{4}cg^2, 0)$, corresponding to two families of periodic orbits of (21), defined by $H(U, y) = h$, *i.e.*, $y^2 = \frac{1}{2|c|}(4|c|h + 2g|c|U^2 - U^4) = \frac{1}{2|c|}(r_1^2 - U^2)(U^2 - r_2^2)$, where $r_1^2 = g|c| + \sqrt{g^2c^2 - 4|c|h}, r_2^2 = g|c| - \sqrt{g^2c^2 - 4|c|h}$, we obtain the parametric representations of periodic wave solutions of (1) as follows:

$$U(\xi) = \pm r_1 \mathrm{dn}\left(\frac{r_1}{\sqrt{2|c|}}\xi, \frac{\sqrt{r_1^2 - r_2^2}}{r_1}\right). \qquad (26)$$

For $h \in (0, \infty)$, corresponding to the family of periodic orbits of (21), enclosing three equilibrium points defined by $H(U,y)=h$, we have the following parametric representation of periodic wave solutions of (1):

$$u(\xi)=r_1\mathrm{cn}\left(\sqrt{\frac{(r_1^2-r_2^2)}{2|c|}}\xi,\frac{r_1}{\sqrt{r_1^2-r_2^2}}\right). \tag{27}$$

Case 4: $g\leqslant 0, c<0$ (see fig. 1(1-4), (1-5)).

In this case, the origin $O(0,0)$ of (21) is an unique equilibrium point, which is a center. There exists a family of periodic orbits of (21), enclosing the origin. Equation (1) has the same parametric representation of periodic wave solutions as (27).

Case 5: $g<0, c>0$ (see fig. 1(1-6)).

In this case, there exist three equilibrium points of (21) at $E_{1,2}$ and $O(0,0)$. O is a center, $E_{1,2}$ are saddle points. The heteroclinic orbits, defined by $H(u,y)=h_1$, have the parametric representations

$$U(\xi)=\pm\sqrt{c|g|}\tanh\left(\frac{\xi}{\sqrt{2|g|}}\right), \tag{28}$$

which gives a kink wave solution and an antikink wave solution of (1).

We see from (22) that $y^2=\frac{1}{2c}(4ch+2cgU^2+U^4$. For $h\in(0,h_1)$, it can be written as $y^2=\frac{1}{2c}((z_1^2-U2)(z_2^2-U^2)$, where $z_1^2=|g|c+\sqrt{g^2c^2-4ch}, z_2^2=|g|c-\sqrt{g^2c^2-4ch}$, Thus, the family of periodic orbits, defined by $H(u,y)=h$, has the parametric representation

$$u(\xi)=Z_2\mathrm{sn}\left(\frac{z_1\xi}{\sqrt{2c}},\frac{z_2}{z_1}\right), \tag{29}$$

which gives a family of periodic wave solutions of (1).

Conclusions. – In this paper, we reported an interesting property of integrable system: solitons and kink solutions can occur in the same integrable equation, and those solutions are given explicitly. Within our knowledge, this is probably the first integrable example possessing such property. We found this equation in the negative-order KdV hierarchy, which is gauge-equivalent to the CH equation. Since eq. (1) has the Lax pair, we may try to get the r-matrix structure of the constrained system of Lax equations, and parametric and algebro-geometric solutions [25], but that is beyond the scope of this paper. The symmetry of eqs. (8), (9), and (10) were already discussed in [32]. Recently, a twofold integrable hierarchy associated with the KdV equation was given in [33]. About other negative-order integrable hierarchies, such as the AKNS, the Kaup-Newell, the Harry-Dym, the Toda etc., one may see the literature [26].

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