A new completely integrable Liouville's system produced by the Kaup–Newell eigenvalue problem

Zhijun Qiao
Department of Mathematics, Liaoning University, Shenyang, Liaoning 110036, People's Republic of China

(Received 16 November 1992; accepted for publication 22 January 1993)

Under the constraint between the potentials and eigenfunctions, the Kaup–Newell eigenvalue problem is nonlinearized as a new completely integrable Hamiltonian system $(R^{2N},dp\wedge dq,H)$: $H=i\langle\Lambda^2p,q\rangle+\frac{1}{2}\langle\Lambda q,q\rangle\langle\Lambda p,p\rangle$. Furthermore, the involutive solution of the high-order Kaup–Newell equation is obtained. Specifically, the involutive solution of the well-known derivative Schrödinger equation $u_t=\frac{1}{2}iu_{xx}+\frac{1}{2}(u|u|^2)_x$ is developed.

I. INTRODUCTION

It is a very important task to find out new finite-dimensional completely integrable systems in soliton theory. It is a celebrated fact that the Hill–Schrödinger eigenvalue problem $-q_{xx}+uq=\lambda q$ is nonlinearized by the Mckean–Trublowitz identity $\langle q,q\rangle=1$ to be a famous mechanic system owing to the Neumann system

$$-q_{xx}+uq=\Lambda q,\quad u=\langle\Lambda q,q\rangle-\langle q_x,q_x\rangle,\quad \langle q,q\rangle=1,$$

which is completely integrable in the Liouville sense and can be regarded as a harmonic N-oscillator constraint on sphere S^N.[1,2] In light of this thought, some classical integrable systems generated through the nonlinearization of the eigenvalue problems are obtained.[3–5] In this article, we prove that the Kaup–Newell eigenvalue problem[6] is nonlinearized to be a new finite-dimensional completely integrable Hamiltonian system under the Bargmann constraint.

This article is divided into four sections. In the next section we present the commutator representation (or Lax representation) of the Kaup–Newell vector field. In Sec. III a new finite-dimensional involutive system $\{F_m\}$ is found out and moreover the nonlinearization of the Kaup–Newell eigenvalue problem under the Bargmann constraint is proven to be a new completely integrable Hamiltonian system. Section IV gives the description that the involutive solution of the compatible system $(H)=(F_0)$, (F_m) is mapped by $f:(u,v)^T=f(q,p)$ which is determined by the Bargmann constraint $u=-\langle\Lambda q,q\rangle$, $v=\langle\Lambda p,p\rangle$ into the solution of the $m+1$th Kaup–Newell equation and the involutive system $\{F_m\}$ is actually produced by the nonlinearized time part of the Lax pair of the high-order Kaup–Newell equation. Specifically, the involutive solution of the well-known derivative Schrödinger equation $u_t=\frac{1}{2}iu_{xx}+\frac{1}{2}(u|u|^2)_x$ is obtained.

II. COMMUTATOR REPRESENTATION OF THE KAUP–NEWELL VECTOR FIELD

Consider the Kaup–Newell eigenvalue problem[6]

$$y_x=My=\begin{pmatrix}-i\lambda^2 & \lambda u(x,t)\\ \lambda v(x,t) & i\lambda^2\end{pmatrix}y,\quad i^2=-1,\tag{2.1}$$

where λ is an eigenparameter, $y=(y_1,y_2)^T$, $y_x=\partial y/\partial x$, $u(x,t)$, and $v(x,t)$ are potential functions, $x\in\Omega$. The underlying interval Ω is $(-\infty,+\infty)$ or $(0,T)$ under the decaying conditions at infinity or periodic condition, respectively.

0022-2488/93/34(7)/3110/11/$6.00

Proposition 2.1: Let λ be an eigenvalue of Eq. (2.1). Then the functional gradient $\nabla\lambda$ of λ is

$$\nabla\lambda=\begin{pmatrix}\delta\lambda/\delta u\\ \delta\lambda/\delta v\end{pmatrix}=\begin{pmatrix}\lambda y_2^2\\ -\lambda y_1^2\end{pmatrix}\cdot\left(\int_\Omega (vy_1^2+4i\lambda y_1y_2-uy_2^2)dx\right)^{-1}. \tag{2.2}$$

Proof: See Ref. 7 Sec. II.

Proposition 2.2: Let λ be an eigenvalue of Eq. (2.1). Then $\nabla\lambda$ which is defined by Eq. (2.2) satisfies

$$K\nabla\lambda=\lambda^2\cdot J\nabla\lambda, \tag{2.3}$$

where

$$K=\frac{1}{2}\begin{pmatrix}\partial u\partial^{-1}u\partial & i\partial^2+\partial u\partial^{-1}v\partial\\ -i\partial^2+\partial v\partial^{-1}u\partial & \partial v\partial^{-1}v\partial\end{pmatrix},\quad J=\begin{pmatrix}0 & \partial\\ \partial & 0\end{pmatrix}. \tag{2.4}$$

$\partial=\partial/\partial x$, $\partial^{-1}\partial=\partial\partial^{-1}=1$. J is a symplectic operator. K and J are called the pair of Lenard's operators of Eq. (2.1).

Proof:

$$J^{-1}K=\frac{1}{2}\begin{pmatrix}-i\partial+v\partial^{-1}u\partial & v\partial^{-1}v\partial\\ u\partial^{-1}u\partial & i\partial+u\partial^{-1}v\partial\end{pmatrix}.$$

Equation (2.1) implies that $uy_2y_{2x}-vy_1y_{1x}=i\lambda(y_1y_2)_x$. So we have

$$(-i\partial+v\partial^{-1}u\partial)(\lambda y_2^2)+v\partial^{-1}v\partial(-\lambda y_1^2)=2\lambda^2\cdot(\lambda y_2^2),$$

$$u\partial^{-1}u\partial(\lambda y_2^2)+(i\partial+u\partial^{-1}v\partial)(-\lambda y_1^2)=2\lambda^2\cdot(-\lambda y_1^2).$$

Thus $J^{-1}K\nabla\lambda=\lambda^2\cdot\nabla\lambda$.

Proposition 2.3: The eigenvalue problem (2.1) is equivalent to

$$L(u,v,\lambda)y=\lambda^2 y \tag{2.5}$$

in Eq. (2.5), the differential operator $L=L(u,v,\lambda)$ is

$$L(u,v,\lambda)=\begin{pmatrix}i\partial & -i\lambda u\\ -\lambda^{-1}v\partial & -i\partial+uv\end{pmatrix}. \tag{2.6}$$

Proof: Directly calculate.

Lemma 2.4: Let $L(u,v,\lambda)$ be expressed as (2.6), then the differential mapping of L is

$$L_{*w}(\xi)=\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}L(w+\epsilon\xi)=\begin{pmatrix}0 & -i\lambda\xi_1\\ -\lambda^{-1}\xi_2\partial & v\xi_1+u\xi_2\end{pmatrix} \tag{2.7}$$

and L_{*w} is an injective homomorphism. L_{*w} is simply written as L_* below. In Eq. (2.7), $w=(u,v)^T$, $\xi=(\xi_1,\xi_2)^T$, $\partial=\partial/\partial x$.

Consider the commutator $[V,L]$ of $V=V_1+V_2\partial$ and $L=L_1+L_2\partial$, here

$$V_1=\begin{pmatrix}0 & A\\ 0 & B\end{pmatrix},\quad V_2=\begin{pmatrix}C & 0\\ D & C\end{pmatrix};\quad L_1=\begin{pmatrix}0 & -i\lambda u\\ 0 & uv\end{pmatrix},\quad L_2=\begin{pmatrix}i & 0\\ -\lambda^{-1}v & -i\end{pmatrix}.$$

A, B, C, D are four undetermined functions.

$$
\begin{aligned}
[V,L]&=VL-LV\\
&=[V_1,L_1]-L_2V_{1x}+V_2L_{1x}+([V_1,L_2]-[L_1,V_2]+V_2L_{2x}-L_2V_{2x})\partial+[V_2,L_2]\partial^2\\
&=\begin{pmatrix}0 & i\lambda uB+uvA-iA_x-i\lambda u_xC\\ 0 & \lambda^{-1}vA_x+iB_x-i\lambda u_xD+(uv)_xC\end{pmatrix}\\
&\quad+\begin{pmatrix}-\lambda^{-1}vA+i\lambda uD-iC_x & -2iA\\ -\lambda^{-1}vB-(uv)D-\lambda^{-1}v_xC+\lambda^{-1}vC_x+iD_x & \lambda^{-1}vA-i\lambda uD+iC_x\end{pmatrix}\partial\\
&\quad+\begin{pmatrix}0 & 0\\ 2iD & 0\end{pmatrix}\partial^2.
\end{aligned}
\tag{2.8}
$$

We hope

$$[V,L]=L_*(KG)-L_*(JG)L. \tag{2.9}$$

In Eq. (2.9), K and J are the pair of Lenard's operators, $G=(G^{(1)}, G^{(2)})^T$. $G^{(1)}(x)$, $G^{(2)}(x)$ are two arbitrarily smooth functions on Ω.

According to Eq. (2.7) and $L=L_1+L_2\partial$, through calculating Eq. (2.9) and sorting it out, we have [note $\partial L=L_{1x}+(L_1+L_{2x})\partial+L_2\partial^2$]

$$
\begin{aligned}
[V,L]=&\begin{pmatrix}0 & -i\lambda(KG)^{(1)}\\ 0 & v(KG)^{(1)}+u(KG)^{(2)}\end{pmatrix}-\begin{pmatrix}0 & -i\lambda(JG)^{(1)}\\ 0 & v(JG)^{(1)}+u(JG)^{(2)}\end{pmatrix}-\begin{pmatrix}0 & 0\\ -\lambda^{-1}(JG)^{(2)} & 0\end{pmatrix}L_{1x}\\
&+\left[\begin{pmatrix}0 & 0\\ -\lambda^{-1}(KG)^{(2)} & 0\end{pmatrix}-\begin{pmatrix}0 & -i\lambda(JG)^{(1)}\\ 0 & v(JG)^{(1)}+u(JG)^{(2)}\end{pmatrix}L_2\right.\\
&\left.-\begin{pmatrix}0 & 0\\ -\lambda^{-1}(JG)^{(2)} & 0\end{pmatrix}(L_1+L_{2x})\right]\partial-\begin{pmatrix}0 & 0\\ -\lambda^{-1}(JG)^{(2)} & 0\end{pmatrix}L_2\partial^2.
\end{aligned}
\tag{2.10}
$$

Equation (2.10), i.e.,

$$
\begin{aligned}
[V,L]=&\begin{pmatrix}0 & -i\lambda(KG)^{(1)}+i\lambda uv(JG)^{(1)}\\ 0 & v(KG)^{(1)}+u(KG)^{(2)}-uv(v(JG)^{(1)}+u(JG)^{(2)})-iu_x(JG)^{(2)}\end{pmatrix}\\
&+\begin{pmatrix}-iv(JG)^{(1)} & -\lambda(JG)^{(1)}\\ \lambda^{-1}v(v(JG)^{(1)}+u(JG)^{(2)})-\lambda^{-1}(KG)^{(2)} & -i(JG)^{(1)}\end{pmatrix}\partial\\
&+\begin{pmatrix}0 & 0\\ i\lambda^{-1}(JG)^{(2)} & 0\end{pmatrix}\partial^2,
\end{aligned}
\tag{2.11}
$$

where $(\cdot)^{(i)}$ $(i=1,2)$ is the ith component of $(\cdot)$, KG and JG are

$$KG=\frac{1}{2}\begin{pmatrix}iG_{xx}^{(2)}+\partial u\partial^{-1}(uG_x^{(1)}+vG_x^{(2)})\\ -iG_{xx}^{(1)}+\partial v\partial^{-1}(uG_x^{(1)}+vG_x^{(2)})\end{pmatrix},\quad JG=\begin{pmatrix}G_x^{(2)}\\ G_x^{(1)}\end{pmatrix}. \tag{2.12}$$

Substitute Eq. (2.12) into Eq. (2.11) and compare the right-hand side of Eq. (2.8) with Eq. (2.11). We should choose

$$A=\tfrac{1}{2}i\lambda G_x^{(2)}, \quad B=-\tfrac{1}{2}uG_x^{(1)},$$
$$C=\tfrac{1}{2}\partial^{-1}(uG_x^{(1)}+vG_x^{(2)}), \quad D=\tfrac{1}{2}\lambda^{-1}G_x^{(1)}. \tag{2.13}$$

Thus, we have

Theorem 2.5: Let $G^{(1)}(x)$, $G^{(2)}(x)$ be two arbitrarily smooth functions, $G=(G^{(1)},G^{(2)})^T$. Let

$$V=V(G)=\frac{1}{2}\begin{pmatrix}0 & i\lambda G_x^{(2)}\\ 0 & -uG_x^{(1)}\end{pmatrix}+\frac{1}{2}\begin{pmatrix}\partial^{-1}(uG_x^{(1)}+vG_x^{(2)}) & 0\\ \lambda^{-1}G_x^{(1)} & \partial^{-1}(uG_x^{(1)}+vG_x^{(2)})\end{pmatrix}\partial. \tag{2.14}$$

Then

$$[V,L]=L_*(KG)-L_*(JG)L, \tag{2.15}$$

where λ is an eigenvalue of Eq. (2.6), K, J are expressed as (2.4).

Proof: We substitute Eq. (2.13) into the right-hand side of Eq. (2.8) and carefully calculate it. It is not difficult to find that the result is equal to the right-hand side of Eq. (2.11).

Define the Lenard's recursive sequence $\{G_j\}$: $G_{-1}=(1,0)^T$, $KG_j=JG_{j+1}$ $(j=-1,0,1,...)$. $G_j(x)$ is the polynomial of $u(x),v(x)$ and their derivatives[8] and is unique if its constant term is required to be zero. $X_j=JG_j$ is the Kaup–Newell vector field. The first few results of calculations are

$$X_{-1}=0, \quad X_0=\begin{pmatrix}u_x\\ v_x\end{pmatrix}, \quad X_1=\frac{1}{2}\begin{pmatrix}iu_{xx}+(u^2v)_x\\ -iv_{xx}+(v^2u)_x\end{pmatrix}.$$

The Kaup–Newell hierarchy of equations is produced by the Kaup–Newell vector field X_j, i.e.,

$$w_t=(u,v)_t^T=X_j(u,v), \quad j=0,1,... . \tag{2.16}$$

Equation (2.16) is reduced to be the well-known derivative Schrödinger equation if one lets $j=1$ and $v=u^*$.

Let c_j be constant. The equation

$$w_t=X_m+c_1X_{m-1}+\cdots+c_mX_0, \quad m=0,1,... \tag{2.17}$$

is called the high-order Kaup–Newell equation.

Theorem 2.6: Let $G_j=(G_j^{(1)},G_j^{(2)})^T$ be the Lenard's recursive sequence. Let $V_j=V(G_j)$, $W_m=\Sigma_{j=0}^{m} V_{j-1}L^{m-j}$. L is expressed as (2.6). Then

$$[W_m,L]=L_*(X_m), \quad m=0,1,... . \tag{2.18}$$

Proof:

$$[W_m,L]=\sum_{j=0}^{m}[V_{j-1},L]L^{m-j}$$

$$=\sum_{j=0}^{m}(L_*(KG_{j-1})L^{m-j}-L_*(JG_{j-1})L^{m-j+1})$$

$$=L_*(JG_m)-L_*(JG_{-1})L^{m+1}$$

$$=L_*(X_m).$$

Corollary 2.7: The Kaup–Newell equation $w_t=X_m(u,v)$ has the commutator representation

$$L_t=[W_m,L],\quad m=0,1,..., \tag{2.19}$$

i.e., $w_t=X_m$ is the natural compatible condition of $Ly=\lambda^2 y$ and $y_t=W_m y$.

Proof:

$$L_t=\begin{pmatrix}0 & -i\lambda u_t\\ -\lambda^{-1}v_t\partial & u_t v+v_t u\end{pmatrix}=L_*(w_t),$$

$$L_t-[W_m,L]=L_*(w_t)-L_*(X_m)=L_*(w_t-X_m).$$

L_* is injective, so this corollary is correct.

Corollary 2.8: The potential function $w(x)=(u(x),v(x))^T$ satisfies the stationary Kaup–Newell system

$$X_N+c_1X_{N-1}+\cdots+c_NX_0=0,\quad N=0,1,... \tag{2.20}$$

if and only if

$$[W_N+c_1W_{N-1}+\cdots+c_NW_0,L]=0, \tag{2.21}$$

where $c_1,...,c_N$ are constants.

III. NONLINEARIZATION OF EQ. (2.1) AND A FINITE-DIMENSIONAL INVOLUTIVE SYSTEM

Let $\lambda_1,\lambda_2,...,\lambda_N$ be N different eigenvalues of Eq. (2.1). Consider the Bargmann constraint[3]

$$G_0=\sum_{j=1}^{N}\gamma_j\cdot\nabla\lambda_j,\quad \gamma_j=\int_\Omega(vq_j^2+4i\lambda_jp_jq_j-up_j^2)dx, \tag{3.1}$$

i.e.,

$$u=-\langle\Lambda q,q\rangle,\quad v=\langle\Lambda p,p\rangle, \tag{3.2}$$

where $q=(q_1,...,q_N)^T$, $p=(p_1,...,p_N)^T$; $\Lambda=\mathrm{diag}(\lambda_1,...,\lambda_N)$; $\langle\cdot,\cdot\rangle$ is the standard inner product in R^N.

Under the Bargmann constraint (3.2), the nonlinearization of Eq. (2.1) gives the Bargmann system

$$\begin{cases} q_x=-i\Lambda^2 q-\langle\Lambda q,q\rangle\Lambda p=-\dfrac{\partial H}{\partial p}, \\ p_x=i\Lambda^2 p+\langle\Lambda p,p\rangle\Lambda q=\dfrac{\partial H}{\partial q}, \end{cases} \tag{3.3}$$

whose Hamiltonian function H is

$$H=i\langle\Lambda^2 p,q\rangle+\tfrac{1}{2}\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle. \tag{3.4}$$

The Poisson bracket of two functions in the symplectic space $(R^{2N},dp\wedge dq)$ is defined as[9]

$$(F,G)=\sum_{j=1}^{N}\left(\frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j}-\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j}\right)=\left\langle\frac{\partial F}{\partial p},\frac{\partial G}{\partial q}\right\rangle-\left\langle\frac{\partial F}{\partial q},\frac{\partial G}{\partial p}\right\rangle,$$

which is skew-symplectic, bilinear, and satisfies the Jacobi identity and Leibnitz rule: $(FG,H)=F(G,H)+G(F,H)$. F, G is called an involution,[9] if $(F,G)=0$.

Now we consider the function system $\{F_m\}$

$$F_m=i\langle\Lambda^{2m+2}p,q\rangle+\frac{1}{2}\langle\Lambda p,p\rangle\langle\Lambda^{2m+1}q,q\rangle+\frac{1}{2}\sum_{j=1}^{m}\begin{vmatrix}\langle\Lambda^{2(m-j)+1}q,q\rangle & \langle\Lambda^{2(m-j)+2}q,p\rangle\\ \langle\Lambda^{2j}q,p\rangle & \langle\Lambda^{2j+1}p,p\rangle\end{vmatrix},$$

$$m=0,1,2,... \tag{3.5}$$

specifically $F_0=H$.

Lemma 3.1: For F_m which is defined as Eq. (3.5), the inner-product $\langle\partial F_k/\partial p,\partial F_l/\partial q\rangle$, is symmetrical about k, l, i.e.,

$$\left\langle\frac{\partial F_k}{\partial p},\frac{\partial F_l}{\partial q}\right\rangle=\left\langle\frac{\partial F_l}{\partial p},\frac{\partial F_k}{\partial q}\right\rangle,\quad \forall k,l\in Z^+. \tag{3.6}$$

Proof:

$$\frac{\partial F_k}{\partial p}=i\Lambda^{2k+2}q+\langle\Lambda q,q\rangle\Lambda^{2k+1}p+\sum_{j=1}^{k}(\langle\Lambda^{2j+1}q,q\rangle\Lambda^{2(k-j)+1}p-\langle\Lambda^{2j}q,p\rangle\Lambda^{2(k-j)+2}q),$$

$$\frac{\partial F_l}{\partial q}=i\Lambda^{2l+2}p+\langle\Lambda p,p\rangle\Lambda^{2l+1}q+\sum_{s=1}^{l}(\langle\Lambda^{2s+1}p,p\rangle\Lambda^{2(l-s)+1}q-\langle\Lambda^{2s}p,q\rangle\Lambda^{2(l-s)+2}p).$$

Calculate the inner product of the left-hand and right-hand sides of the above two equalities, respectively. Through a series of careful calculations, it is easy to find that $\langle\partial F_k/\partial p,\partial F_l/\partial q\rangle$ is expressed as the sum of the symmetrical items about k, l. So the required result is right.

Theorem 3.2: The functions defined as Eq. (3.5) are in involution in pairs

$$(F_k,F_l)=0,\quad \forall k,l\in Z^+$$

specifically $(H,F_m)=0$, $\forall m\in Z^+$.

Proof:

$$(F_k,F_l)=\left\langle\frac{\partial F_k}{\partial p},\frac{\partial F_l}{\partial q}\right\rangle-\left\langle\frac{\partial F_k}{\partial q},\frac{\partial F_l}{\partial p}\right\rangle=0.$$

Theorem 3.3: Under the Bargmann constraint (3.2), the Hamiltonian system $(R^{2N}, dp \wedge dq, H=F_0)$ which is given by Eq. (3.3) is completely integrable in the Liouville's sense and its involutive system is composed of $F_m(\forall m \in Z^+)$.

Remark: The finite-dimensional involutive systems $\{F_m\}$ are the stationary points of the corresponding higher-order flows (see Refs. 10 and 11) and therefore special cases of the systems considered in Refs. 10 and 11.

Theorem 3.4: Let $(q,p)^T$ be a solution of the Bargmann system (3.3). Then $u=-\langle \Lambda q,q\rangle$, $v=\langle \Lambda p,p\rangle$ satisfy a stationary Kaup–Newell equation

$$X_N+\alpha_1 X_{N-1}+\cdots+\alpha_N X_0=0, \tag{3.7}$$

with suitably chosen constants α_j $(j=1,\ldots,N)$.

Proof: Let the operator $(J^{-1}K)^k$ act upon two sides of Eq. (3.1). In virtue of Eq. (2.3) and $J^{-1}K$: $G_j \mapsto G_{j+1}$ with an extra term const $\cdot\, G_{-1}$ $(j=-1,0,1,\ldots,)$, we obtain

$$G_k+\beta_2 G_{k-2}+\cdots+\beta_k G_0+\beta_{k+1}G_{-1}=\sum_{j=1}^{N}\lambda_j^{2k}\nabla\lambda_j, \tag{3.8}$$

where $\beta_2,\ldots,\beta_{k+1}$ are constants.

Consider the polynomial

$$P(\lambda)=\prod_{j=1}^{N}(\lambda-\lambda_j^2)=p_0\lambda^N+p_1\lambda^{N-1}+\cdots+p_N, \quad p_0=1. \tag{3.9}$$

Equation (3.7) is obtained as the operator $J\Sigma_{k=0}^{N} p_{N-k}$ acts upon two sides of Eq. (3.8).

In addition, according to the proof of Theorem 3.4, we have

Lemma 3.5: Let $(q,p)^T$ be a solution of Eq. (3.3) and G_j be the Lenard's recursive sequence, then there exist constants $c_2,\ldots,c_{m+1}$ such that

$$A_m=\begin{pmatrix} A_m^{(1)} \\ A_m^{(2)} \end{pmatrix}=\begin{pmatrix} \langle \Lambda^{2m+1}p,p\rangle \\ -\langle \Lambda^{2m+1}q,q\rangle \end{pmatrix}=G_m+\sum_{s=2}^{m+1} c_s G_{m-s}, \quad m=0,1,\ldots \tag{3.10}$$

or

$$A_m=\sum_{s=0}^{m+1} c_s G_{m-s}, \quad c_0=1, \quad c_1=0. \tag{3.11}$$

IV. THE INVOLUTIVE SOLUTIONS OF THE KAUP–NEWELL HIERARCHY

Consider the canonical system of F_m-flow

$$(F_m)\begin{pmatrix} q_{t_m} \\ p_{t_m} \end{pmatrix}=\begin{pmatrix} -\dfrac{\partial F_m}{\partial p} \\ \dfrac{\partial F_m}{\partial q} \end{pmatrix}=I\nabla F_m, \quad I=\begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}, \tag{4.1}$$

where I_N is the $N\times N$ unit matrix. Let $g_m^{t_m}$ be defined as the solution operator of the initial value problem (4.1), then the solution of Eq. (4.1) is expressed as

$$\begin{pmatrix} q(t_m) \\ p(t_m) \end{pmatrix}=g_m^{t_m}\begin{pmatrix} q(0) \\ p(0) \end{pmatrix}. \tag{4.2}$$

Since any two F_k, F_l are in involution, we have (see Ref. 9).

Proposition 4.1: (1) Any two canonical systems $(F_k),(F_l)$ are compatible; (2) the Hamiltonian phase-flow $g_k^{t_k}, g_l^{t_l}$ commute.

Denote the flow variables of (F_0) and (F_m) by $x=t_0$, $t=t_m$, respectively. Define

$$\begin{pmatrix} q(x,t_m) \\ p(x,t_m) \end{pmatrix} = g_0^x g_m^{t_m} \begin{pmatrix} q(0,0) \\ p(0,0) \end{pmatrix}. \tag{4.3}$$

The commutativity of $g_0^x, g_m^{t_m}$ implies that it is a smooth function of (x,t), which is called the involutive solution of the consistent systems of equations $(F_0),(F_m)$.

Theorem 4.2: Let $(q(x,t_m),p(x,t_m))^T$ be an involutive solution of the consistent system $(F_0),(F_m)$. Let $u(x,t_m)=-\langle \Lambda q,q\rangle$, $v(x,t_m)=\langle \Lambda p,p\rangle$. Then
(1) the flow equations $(F_0),(F_m)$ are reduced to be the spatial part and the time part, respectively, of the Lax pair for the high-order Kaup–Newell equation with u,v as their potentials ($c_1,...,c_m$ are independent of x)

$$\begin{pmatrix} q_x \\ p_x \end{pmatrix} = \begin{pmatrix} -i\Lambda^2 & u\Lambda \\ v\Lambda & i\Lambda^2 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \tag{4.4}$$

$$\begin{pmatrix} q_{t_m} \\ p_{t_m} \end{pmatrix} = (W_m + c_1 W_{m-1} + \cdots + c_m W_0) \begin{pmatrix} q \\ p \end{pmatrix}. \tag{4.5}$$

(2) $u(x,t_m)=-\langle \Lambda q,q\rangle$, $v(x,t_m)=\langle \Lambda p,p\rangle$ satisfy the high-order Kaup–Newell equation

$$(u_{t_m}, v_{t_m})^T = X_m + c_1 X_{m-1} + \cdots + c_m X_0. \tag{4.6}$$

Proof: From Eq. (3.3), we immediately know (F_0) is Eq. (4.4). Through careful calculation we have (here order $A_{-1}=0$, $\partial^{-1}0=2$)

$$\begin{aligned} &\frac{1}{2}\sum_{j=0}^{m} \{\partial^{-1}(uA_{j-1,x}^{(1)} + vA_{j-1,x}^{(2)})\Lambda^{2(m-j)} q_x + iA_{j-1,x}^{(2)} \Lambda^{2(m-j)+1} p\} \\ &= -i\Lambda^{2m+2} q - \langle \Lambda q,q\rangle \Lambda^{2m+1} p + \sum_{j=1}^{m} (\langle \Lambda^{2j} q,p\rangle \Lambda^{2(m-j)+2} q - \langle \Lambda^{2j+1} q,q\rangle \Lambda^{2(m-j)} p) \\ &= -\frac{\partial F_m}{\partial p}, \end{aligned} \tag{4.7}$$

$$\begin{aligned} &\frac{1}{2}\sum_{j=0}^{m} \{\partial^{-1}(uA_{j-1,x}^{(1)} + vA_{j-1,x}^{(2)})\Lambda^{2(m-j)} p_x + A_{j-1,x}^{(1)} \Lambda^{2(m-j)-1} q_x - uA_{j-1,x}^{(1)} \Lambda^{2(m-j)} p\} \\ &= i\Lambda^{2m+2} p + \langle \Lambda p,p\rangle \Lambda^{2m+1} q + \sum_{j=1}^{m} (\langle \Lambda^{2j+1} p,p\rangle \Lambda^{2(m-j)+1} q - \langle \Lambda^{2j} q,p\rangle \Lambda^{2(m-j)+2} p) \\ &= \frac{\partial F_m}{\partial q}. \end{aligned} \tag{4.8}$$

In the calculation of Eqs. (4.7) and (4.8), Eq. (4.4), $u(x,t_m)=-\langle \Lambda q,q\rangle$, $v(x,t_m)=\langle \Lambda p,p\rangle$, and the equality $\frac{1}{2}\partial^{-1}(uA_{j-1,x}^{(1)} + vA_{j-1,x}^{(2)}) = i\langle \Lambda^{2j} q,p\rangle$ are used.

Substituting Eq. (3.11) into Eqs. (4.7) and (4.8), respectively, we obtain

$$\frac{\partial q}{\partial t_m}=-\frac{\partial F_m}{\partial p}$$

$$=\frac{1}{2}\sum_{j=0}^{m}\sum_{s=0}^{j}c_s\{\partial^{-1}(uG^{(1)}_{j-1-s,x}+vG^{(2)}_{j-1-s,x})\Lambda^{2(m-j)}q_x+iG^{(2)}_{j-1-s,x}\Lambda^{2(m-j)+1}p\}$$

$$=\frac{1}{2}\sum_{s=0}^{m}c_s\sum_{k=0}^{m-s}\{\partial^{-1}(uG^{(1)}_{k-1,x}+vG^{(2)}_{k-1,x})\Lambda^{2(m-s-k)}q_x+iG^{(2)}_{k-1,x}\Lambda^{2(m-s-k)+1}p\}, \quad (4.9)$$

$$\frac{\partial p}{\partial t_m}=\frac{\partial F_m}{\partial q}$$

$$=\frac{1}{2}\sum_{j=0}^{m}\sum_{s=0}^{j}c_s\{\partial^{-1}(uG^{(1)}_{j-1-s,x}+vG^{(2)}_{j-1-s,x})\Lambda^{2(m-j)}p_x+G^{(1)}_{j-1-s,x}\Lambda^{2(m-j)-1}q_x$$

$$-uG^{(1)}_{j-1-s,x}\Lambda^{2(m-j)}p\}$$

$$=\frac{1}{2}\sum_{s=0}^{m}c_s\sum_{k=0}^{m-s}\{\partial^{-1}(uG^{(1)}_{k-1,x}+vG^{(2)}_{k-1,x})\Lambda^{2(m-s-k)}p_x+G^{(1)}_{k-1,x}\Lambda^{2(m-k-s)-1}q_x$$

$$-uG^{(1)}_{k-1,x}\Lambda^{2(m-k-s)}p\}. \quad (4.10)$$

In virtue of Eqs. (4.9) and (4.10), we get

$$\begin{pmatrix} q_{t_m} \\ p_{t_m} \end{pmatrix}=\sum_{s=0}^{m}c_s\sum_{k=0}^{m-s}V_{k-1}\begin{pmatrix}\Lambda^{2(m-s-k)}q \\ \Lambda^{2(m-s-k)}p\end{pmatrix}$$

$$=\sum_{s=0}^{m}c_s\sum_{k=0}^{m-s}V_{k-1}L^{m-s-k}\begin{pmatrix} q \\ p\end{pmatrix}$$

$$=\sum_{s=0}^{m}c_sW_{m-s}\begin{pmatrix} q \\ p\end{pmatrix}$$

$$=(W_m+c_1W_{m-1}+\cdots+c_mW_0)\begin{pmatrix} q \\ p\end{pmatrix},$$

where $W_{m-s}=\Sigma_{k=0}^{m-s}V_{k-1}L^{m-s-k}$, V_{k-1} is expressed as one in Theorem 2.6.

$$\frac{\partial u}{\partial t_m}=-2\left\langle\Lambda q,\frac{\partial q}{\partial t_m}\right\rangle=2\left\langle\Lambda q,-\frac{\partial F_m}{\partial p}\right\rangle=2i\langle\Lambda^{2m+3}q,q\rangle+2\langle\Lambda q,q\rangle\langle\Lambda^{2m+2}q,p\rangle=A^{(2)}_{m,x}, \quad (4.11)$$

$$\frac{\partial v}{\partial t_m}=2\left\langle\Lambda p,\frac{\partial p}{\partial t_m}\right\rangle=2\left\langle\Lambda p,\frac{\partial F_m}{\partial q}\right\rangle=2i\langle\Lambda^{2m+3}p,p\rangle+2\langle\Lambda p,p\rangle\langle\Lambda^{2m+2}q,p\rangle=A^{(1)}_{m,x}. \quad (4.12)$$

By using Eqs. (3.11), (4.11), (4.12), and $X_k=JG_k$, we have (note $JG_{-1}=0$)

$$\begin{pmatrix} u_{t_m} \\ v_{t_m} \end{pmatrix} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \begin{pmatrix} A_m^{(1)} \\ A_m^{(2)} \end{pmatrix} = JA_m = J\left(\sum_{s=0}^{m+1} c_s G_{m-s} \right) = X_m + c_1 X_{m-1} + \cdots + c_m X_0.$$

As a special case of Theorem 4.2, we can get the involutive solution of the well-known derivative Schrödinger equation (DSE)

$$u_t = \tfrac{1}{2} i u_{xx} + \tfrac{1}{2}(u|u|^2)_x \tag{4.13}$$

if we choose $m=1$, $u=v^*$.

Corollary 4.3: Let $(q(x,t_1),p(x,t_1))^T$ be the involution of the compatible system $(F_0)=(H),(F_1)$. Let $u(x,t_1)=-\langle \Lambda q,q\rangle$, $v(x,t_1)=\langle \Lambda p,p\rangle$, and $u=v^*$. Then (1) the flow equations $(F_0),(F_1)$ are reduced to the spatial part and the time part, respectively, of the Lax pair for the derivative Schrödinger equation (4.13) with u as their potential

$$\begin{pmatrix} q_x \\ p_x \end{pmatrix} = \begin{pmatrix} -i\Lambda^2 & u\Lambda \\ u^*\Lambda & i\Lambda^2 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \tag{4.14}$$

$$\begin{pmatrix} q_{t_1} \\ p_{t_1} \end{pmatrix} = \begin{pmatrix} -i\Lambda^3 - \tfrac{1}{2} i|u|^2\Lambda^2 & \tfrac{1}{2}(iu_x + |u|^2 u)\Lambda \\ \tfrac{1}{2}(-iu_x^* + |u|^2 u^*)\Lambda & i\Lambda^3 + \tfrac{1}{2} i|u|^2\Lambda^2 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}. \tag{4.15}$$

(2) $u(x,t_1)$ satisfies the DSE (4.13).

Proof: According to $u=v^*$, Eq. (4.4) is evidently Eq. (4.14). Choosing $m=1$, $c_1=0$ in Eqs. (4.5) and (4.6), we have

$$\begin{pmatrix} q_{t_1} \\ p_{t_1} \end{pmatrix} = W_1 \begin{pmatrix} q \\ p \end{pmatrix} = (V_{-1}\Lambda + V_0) \begin{pmatrix} q \\ p \end{pmatrix}, \tag{4.16}$$

where

$$V_{-1} = \begin{pmatrix} \partial I_N & 0 \\ 0 & \partial I_N \end{pmatrix}, \quad V_0 = \frac{1}{2} \begin{pmatrix} uv\partial I_N & i\Lambda u_x I_N \\ \Lambda^{-1} v_x \partial I_N & -uv_x I_N + uv\partial I_N \end{pmatrix}.$$

I_N is the $N \times N$ unit matrix.

Substituting the expression of V_{-1} and V_0 into Eq. (4.16), through some calculation we know Eq. (4.16) implies Eq. (4.15).

The required result (2) is obtained because of (2) of Theorem 4.2 and

$$\begin{pmatrix} u_{t_1} \\ v_{t_1} \end{pmatrix} = X_1, \quad X_1 = \frac{1}{2} \begin{pmatrix} iu_{xx} + (u^2 v)_x \\ -iv_{xx} + (v^2 u)_x \end{pmatrix}, \quad u=v^*.$$

ACKNOWLEDGMENTS

The author is very indebted to Professor Cao Cewen for his instruction and expresses his sincere thanks to the referee for the advised opinions.

This work has been supported by the Natural Science Foundation of Education Committee, Liaoning Province, China.

[1] H. Knorrer, J. Rein. Angew. Math. **334**, 69 (1982).

[2] J. Moser, *Integrable Hamiltonian System and Spectral Theory*, Proceedings of the 1983 Beijing Symposium on Differential Geometry and Differential Equations (Science, Beijing, 1986), pp. 157–229.

[3] C. Cao and X. Geng, *Classical integrable systems generated through nonlinearization of eigenvalue problems*, in Nonlinear Physics, Research Reports in Physics, edited by C. Gu *et al.* (Springer-Verlag, Berlin, 1990), pp. 68–78.
[4] C. Cao and X. Geng, J. Phys. A **23**, 4117 (1990).
[5] Z. Qiao, Master paper, Zhengzhou University, PRC, 1989.
[6] J. D. Kaup and A. C. Newell, J. Math. Phys. **19**, 789 (1978).
[7] C. Cao, Sci. China A **33**, 528 (1990).
[8] P. D. Lax, SIAM Rev. **18**, 351 (1976).
[9] V. I. Arnol'd, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, Berlin, 1978).
[10] A. G. Reyman and M. A. Semenov-Tian-Shansky, Inv. Math. **54**, 81 (1979).
[11] A. G. Reyman and M. A. Semenov-Tian-Shansky, Inv. Math. **63**, 423 (1981).