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Decomposition Method for the b-Balanced Shallow Water Equation

***Zhijun Qiao*[1]* and Jules Sadefo Kamdem*[2]†**
[1]Department of Mathematics, The University of Texas-Pan American
1201 W University Drive, Edinburg, TX 78541 - USA
[2]Laboratoire de mathématiques, CNRS UMR 6056,
BP 1039 moulin de la housse, 51687 Reims, France

Abstract

The Adomian decomposition approach is used for the b-balanced shallow water wave equation $m_t + m_x u + bmu_x = 0$, $m = u - cu_{xx}$. Approximate solutions are obtained for three smooth initial values. Compared with the existing method, our procedure just works with the polynomial and algebraic computations for this shallow water equation.

Key Words: Shallow water equation, adomian decomposition method, approximate solutions.

1. Introduction

Recently, the balanced water wave equation

$$m_t + m_x u + bmu_x = 0, \quad m = u - cu_{xx}, \tag{1}$$

where c is a constant, has arisen a lot of attractive attentions [6]. The family (1) is integrable only when $b = 2, 3$ [8], and it is reduced to the Camassa-Holm (CH) equation [2] as $b = 2$ and to the Degasperis-Procesi (DP) equation [3] as $b = 3$, respectively. The DP equation has bi-Hamiltonian structure and Lax pair [5], and can be also extended to a whole integrable hierarchy of equations [11] with parametric solutions under some constraints [12]. In addition, the CH equation also has new peaked and smooth solitons which are given in explicit forms [14].

In an earlier paper [7], we dealt with the decomposition of the CH equation using the Adomian method [1]. Very recently, Qiao [13] developed three new types of soliton solutions - M-shape peakon, dehisced soliton, and double dehisced 1-peak solions through studying the DP equation, and the most interesting is that a cuspon is a limit of the new peaked soliton solutions for the DP equation (see [13]). In the present paper, we discuss the

*E-mail address: qiao@utpa.edu
†E-mail address: sadefo@univ-reims.fr

b-balanced water wave equation (1) Using analysis procedure and the Adomian decomposition method, we are able to obtain the approximate solutions for three smooth initial values. Compared with the existing method, our procedure just works with the polynomial and algebraic computations for this family of water equations. Finally, we plot our approximate solutions in three dimensional space.

1.1. Analysis of the Decomposition Method

After an expansion, the b-balance water wave equation (1) for real $u(x,t)$ is rewritten as

$$L_t(u - c\,u_{xx}) = -m_x u \; - \; b m u_x, \tag{2}$$

where $L_t = \frac{\partial}{\partial t}$ is a linear operator.

Assuming the inverse operator L_t^{-1} exists and it can be taken as the definite integral with respect to t from t_0 to t, ie.

$$L^{-1} = \int_{t_0}^{t} (\cdot)\, dt$$

then applying the inverse operator L_t^{-1} on both sides of (2) yields

$$m(x,t) = m(x,t_0) \; + \; L_t^{-1}\left(-m_x u \; - \; b m u_x\right). \tag{3}$$

The ADM assumes that the unknown function $u(x,t)$ can be expressed by a sum of components defined by the decomposition series of the form:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

with u_0 defined as $u(x,0)$ where $u(x,t)$ will be determined recursively. The nonlinear operator

$$\begin{aligned} NL(u) \; &:= \; m_x u \; + \; b m u_x \\ &:= \; m[u_x] u \; + \; b m [u] u_x \end{aligned} \tag{4}$$

can also be decomposed by an infinite series of polynomials given by

$$NL(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n),$$

where $A_n(u_0, u_1, \ldots, u_n)$ are the Adomian's polynomials which are defined as

$$\begin{aligned} A_n \; &= \; \frac{1}{n!}\left[\frac{d}{d\lambda^n} NL\left(\sum_{i=0}^{\infty} \lambda^i u_i(x,t)\right)\right]_{\lambda=0}, \quad n = 0,1,2,\ldots \\ &= \; \frac{1}{n!}\left[\frac{d}{d\lambda^n}\left[m\left[\sum_{n=0}^{\infty} \lambda^j u_{jx}\right]\left(\sum_{n=0}^{\infty} \lambda^j u_j\right) \; + \; bm\left[\sum_{n=0}^{\infty} \lambda^j u_j\right]\left(\sum_{n=0}^{\infty} \lambda^j u_{jx}\right)\right]\right]_{\lambda=0} \\ &= \; \sum_{j=0}^{n} \left[m[u_{jx}] u_{n-j} \; + \; b m [u_{n-j}] u_{jx}\right], \end{aligned} \tag{5}$$

since $m[\cdot]$ is a linear operator. Recursively, we then generates the formula of u_n:

$$\begin{cases} u_0 = u(x,t_0) - c\, u_{xx}(x,t_0) & \text{n=0} \\ u_{n+1} = c\, u_{nxx} \;-\; \int_{t_0}^{t} A_n\, ds & \text{if} n \neq 0. \end{cases} \tag{6}$$

As we see, it is easy to write a computer code for generating the Adomian polynomials. We summarize the whole procedure in the following algorithm

Algorithm

- **Input data**:
  t_0 – initial time
  $u(x,t_0)$ – initial condition,
  c – positive constant parameter, i.e: $m[u](x,t_0) = u(x,t_0) - c\, u_{xx}(x,t_0) = J(x)$,
  b – constant parameter of (1),
  k – number of terms in the approximation.

- **Output**: $u_{approx}(x,t)$: the approximate solution

  – Step 1: Set $u_0 = J(x)$ and $u_{approx}(x,t) = u_0$.

  – Step 2: For $k = 0$ to $n-1$, do Step 3, Step 4, and Step 5.

  – Step 3: Compute

  $$A_k \;=\; \sum_{j=0}^{k} \left[m\left[u_{jx}\right] u_{k-j} \;+\; b\, m\left[u_{k-j}\right] u_{jx} \right].$$

  – Step 4: Compute

  $$u_{k+1} = c\, u_{kxx} \;-\; \int_{t_0}^{t} A_k\, ds \;\; \text{if} k \neq 0.$$

  – Step 5: Compute $u_{approx} = u_{approx} \;+\; u_{k+1}$.

  – Stop

2. Convergence Analysis

In this section, we discuss the convergence property of the approximated solution for the b-balanced wave equation.

Let us consider the b-balanced wave equation in the Hilbert space $H = L^2((\alpha,\beta) \times [0,T])$:

$$H = \left\{ v : (\alpha,\beta) \times [0,T] \quad \text{with} \int_{(\alpha,\beta)\times[0,T]} v^2(x,s)\,ds\,d\tau < +\infty \right\}. \tag{7}$$

Then the operator is of the form

$$\begin{aligned} T(u) &= L_t(u - c\,u_{xx}) \\ &= -(1+b)\,u_x u + u_{xxx}u + b\,u_{xx}u_x. \end{aligned} \tag{8}$$

The Adomian decomposition method is convergent if the following two hypotheses are satisfied[1] and some references therein for more details.:

- (Hyp1): There exists a constant $k > 0$ such that the following inner product holds in H:
$$(T(u) - T(v), u - v) \geq k\|u - v\|, \quad \forall u, v \in H; \tag{9}$$
- (Hyp2): As long as both $u \in H$ and $v \in H$ are bounded (i.e. there is a positive number M such that $\|u\| \leq M$, $\|v\| \leq M$), there exists a constant $\theta(M) > 0$ such that
$$(T(u) - T(v), u - v) \leq \theta(M)\|u - v\|\,\|w\|, \quad \forall w \in H. \tag{10}$$

Theorem 2.1. *(***Sufficient conditions of convergence for the** ***b*****-balanced wave equation***)* *Let*

$$\begin{aligned} T(u) &= L_t(u - c u_{xx}) \\ &= -(1+b)\,u_x u + u_{xxx}u + b\,u_{xx}u_x, \\ &\quad \text{with} \quad b \leq 3, \quad L_t = \frac{\partial}{\partial t}, \end{aligned}$$

and consider the free initial and boundary conditions for the b-balanced equation. Then the Adomian decomposition method leads to a special solution of the b-balanced wave equation.

Proof. To prove the theorem, we just verify the conditions (Hyp1) and (Hyp2). For $\forall u, v \in H$, let us calculate:

$$\begin{aligned} T(u) - T(v) &= -(1+b)\,(u_x u - v_x v) + (u_{xxx}u - v_{xxx}v) + b\,(u_{xx}u_x - v_{xx}v_x) \\ &= -\frac{1+b}{2}(u^2 - v^2)_x + \frac{1}{2}\left((u^2 - v^2)_{xxx} - 3\,(u_x^2 - v_x^2)_x\right) + \frac{b}{2}\,(u_x^2 - v_x^2)_x \\ &= -\frac{1+b}{2}(u^2 - v^2)_x + \frac{1}{2}\,(u^2 - v^2)_{xxx} + \frac{b-3}{2}\,(u_x^2 - v_x^2)_x \end{aligned}$$

[1] See [7]

Therefore, we have the inner product

$$
\begin{aligned}
(T(u)-T(v),u-v) &= \frac{1+b}{2}\left(-\frac{\partial}{\partial x}(u^2-v^2),u-v\right)-\frac{1}{2}\left(-\frac{\partial}{\partial x}(u_x^2-v_x^2),u-v\right) \\
&\quad +\frac{3-b}{2}\left(-\frac{\partial^3}{\partial x^3}(u^2-v^2),u-v\right).
\end{aligned} \tag{11}
$$

Let us assume that u, v are bounded and there is a constant $M>0$ such that $(u,u),(v,v)<M^2$. By using Schwartz inequality

$$
\left(\frac{\partial}{\partial x}(u^2-v^2),u-v\right)\le\|(u^2-v^2)_x\|\,\|u-v\|, \tag{12}
$$

and since there exist θ_1 and θ_2 such that $\|(u-v)_x\|\le\theta_1\|u-v\|$, $\|(u+v)_x\|\le\theta_2\|u-v\|$ and $\|u+v\|\le 2M$, we have

$$
\left(\frac{\partial}{\partial x}(u^2-v^2),u-v\right)\le 2M\theta_1\theta_2\|u-v\|^2.
$$

$\Leftrightarrow$

$$
\left(-\frac{\partial}{\partial x}(u^2-v^2),u-v\right)\ge 2M\theta_1\theta_2\|u-v\|^2. \tag{13}
$$

Following the preceding procedure, we can calculate:

$$
\begin{aligned}
\left(\frac{\partial}{\partial x}(u_x^2-v_x^2),u-v\right) &\le \|(u_x^2-v_x^2)_x\|\,\|u-v\| \\
&\le \theta_3\|u_x+v_x\|\,\|u_x-v_x\|\,\|u-v\| \\
&\le 2M\theta_3\theta_4\theta_5\|u-v\|^2,
\end{aligned}
$$

$\Leftrightarrow$

$$
\left(-\frac{\partial}{\partial x}(u_x^2-v_x^2),u-v\right) \ge 2M\theta_3\theta_4\theta_5\|u-v\|^2, \tag{14}
$$

where θ_i (i=3,4,5) are positive constants.

Moreover, the Cauchy-Schwartz-Buniakowski inequality yields

$$
\left(\frac{\partial^3}{\partial x^3}(u^2-v^2),u-v\right)\le\|(u^2-v^2)_{xxx}\|\,\|u-v\|, \tag{15}
$$

then by using the mean value theorem, we have

$$
\begin{aligned}
\left(\frac{\partial^3}{\partial x^3}(u^2-v^2),u-v\right) &\le \theta_6\theta_7\theta_8\|u^2-v^2\|\|u-v\| \\
&\le 2M\theta_6\theta_7\theta_8\|u-v\|^2
\end{aligned}
$$

$\Leftrightarrow$

$$
\left(-\frac{\partial^3}{\partial x^3}(u^2-v^2),u-v\right) \ge 2M\theta_6\theta_7\theta_8\|u-v\|^2 \tag{16}
$$

where θ_j $(j = 6,7,8)$ are three positive constants, and $\|(u^2 - v^2)_{xxx}\| \leq \theta_6 \|(u^2 - v^2)_{xx}\|$, $\|(u+v)_{xx}\| \leq \theta_7 \|(u+v)_x\|$ and $\|(u+v)_x\| \leq \theta_8 \|u+v\|$.

Substituting (13), (14), (16) into (11) generates the following inner product:

$$\begin{aligned}(T(u)-T(v),u-v) &= \left(-b\frac{\partial}{\partial x}(u^2-v^2),u-v\right)\\ &\quad -(c-d)\left(\frac{\partial}{\partial x}(u_x^2-v_x^2),u-v\right)-\frac{d}{2}\left(\frac{\partial^3}{\partial x^3}(u^2-v^2),u-v\right)\\ &\geq k\|u-v\|^2,\end{aligned}$$

where $k = ((1+b)\theta_1\theta_2 - \theta_3\theta_4\theta_5 + (3-b)\theta_6\theta_7\theta_8)\, M$. So, (Hyp1) is true for the b-balanced wave equation.

Let us now verify the hypotheses (Hyp2) for the operator $T(u)$. We directly compute:

$$\begin{aligned}(T(u)-T(v),w) &= \frac{1+b}{2}\left(-b\frac{\partial}{\partial x}[u^2-v^2],w\right)-\frac{1}{2}\left(-\frac{\partial}{\partial x}[u_x^2-v_x^2],w\right)\\ &\quad \frac{3-b}{2}\left(-\left[\frac{\partial^3}{\partial x^3}(u^2-v^2)\right],w\right)\\ &\leq \theta(M)\|u-v\|\|w\|\end{aligned}$$

where $\theta(M) = 3M/2$. Therefore, (Hyp2) is correct as well. QED.

3. Implementation of the Method and Approximate Solutions

In this section, we take some examples to show the procedure and present some approximate solutions $u(x,t) \simeq \sum_{j=0}^{10} u_n(x,t)$, for the b-balanced equation.

Example 3.1.

$$\begin{cases} m_t + m_x u + b m u_x = 0, \quad m = u - c u_{xx} \\ u_0 = u(x,0) = -3\, e^{4x} + 2\, e^{5x}. \end{cases} \tag{17}$$

Example 3.2.

$$\begin{cases} m_t + m_x u + b m u_x = 0, \quad m = u - c u_{xx} \\ u_0 = u(x,0) = \cosh(g\, x^2). \end{cases} \tag{18}$$

Example 3.3.

$$\begin{cases} m_t + m_x u + b m u_x = 0, \quad m = u - c u_{xx} \\ u_0 = u(x,0) = 3\, \cosh(c_1\, x) + 2\, \sin(c_2\, x). \end{cases} \tag{19}$$

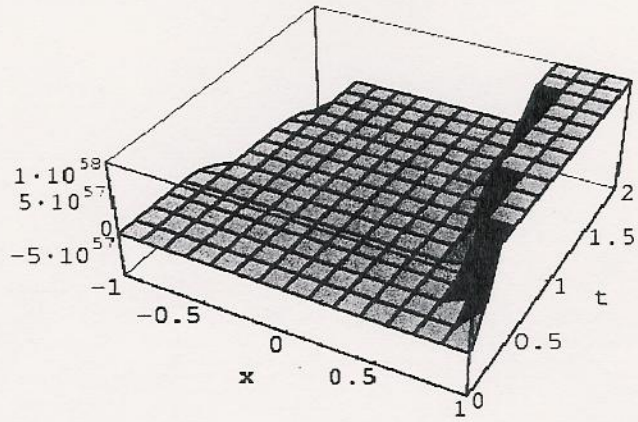

Figure 1. Approximate solution for equation (17) with $b = -1$.

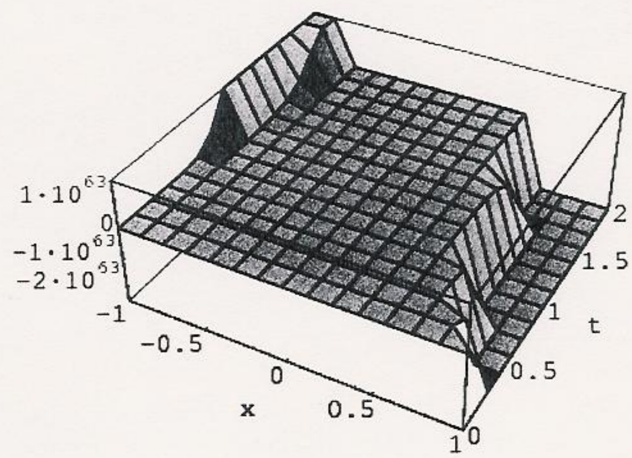

Figure 2. Approximate solution for equation (17) with $b = 0$.

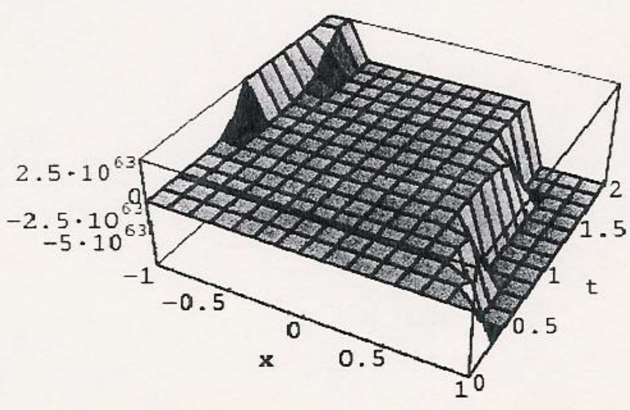

Figure 3. Approximate solution for for equation (17) with $b = 0.5$.

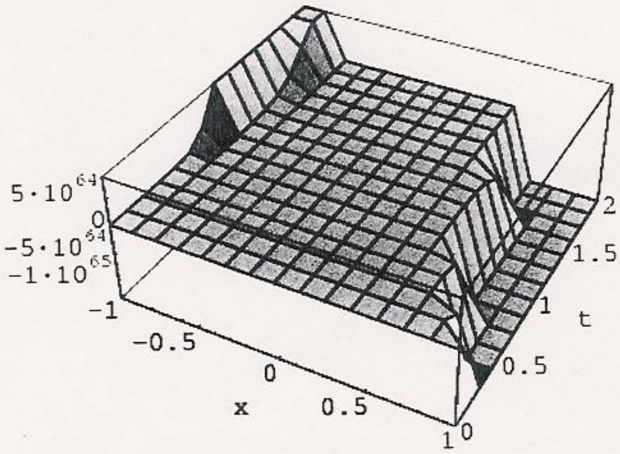

Figure 4. Approximate solution for for equation (17) with $b = 3$.

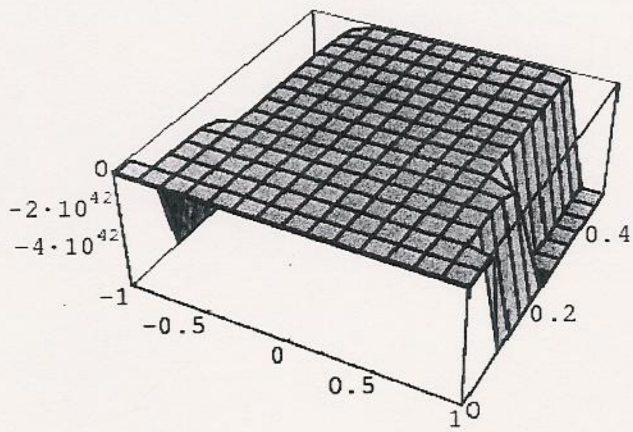

Figure 5. Approximate solution for equation (18) with $b = 1, c = 1, g = 2$.

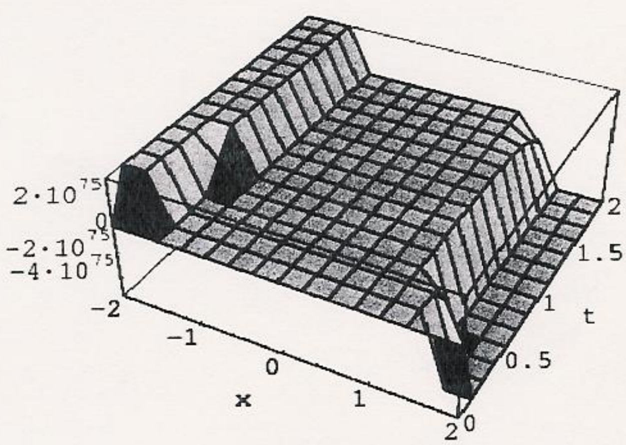

Figure 6. Approximate solution for equation (19) with $b = 0$, $c_1 = 4$, $c_2 = 5$.

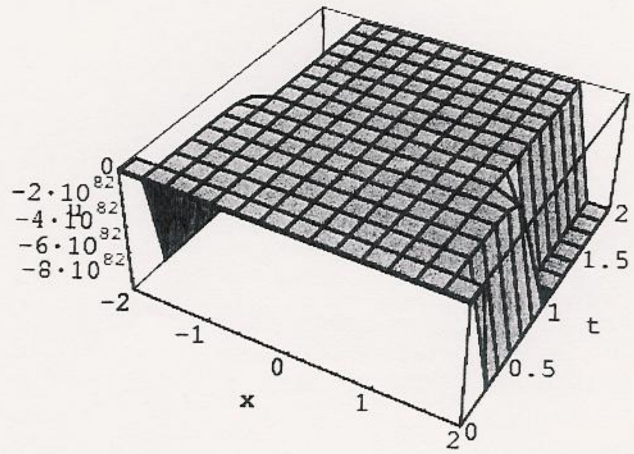

Figure 7. Approximate solution for equation (19) with $b=-1, c_1=4, c_2=5$.

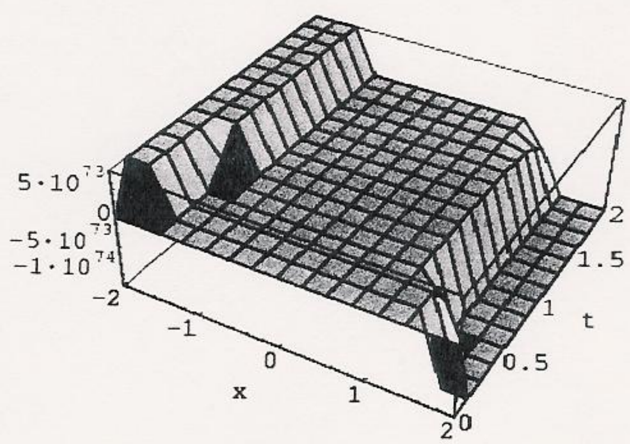

Figure 8. Approximate solution for equation (19) with $b=-2, c_1=4, c_2=5$.

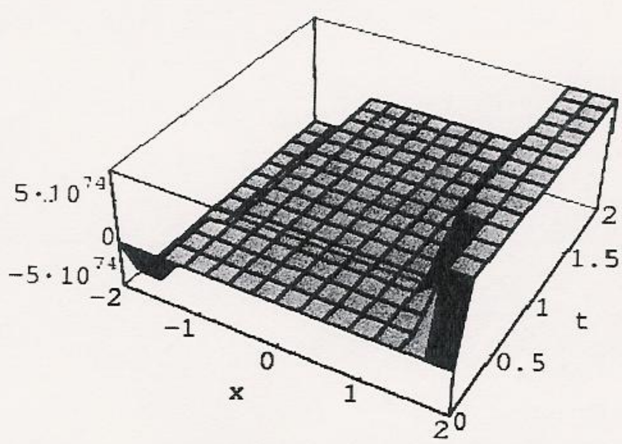

Figure 9. Approximate solution for equation (19) with $b=-3, c_1=4, c_2=5$.

4. Conclusions

In this paper, we apply the Adomian polynomial decomposition method to solve the family of b-balanced shallow water equations in an explicitly approximate form. The initial values we adopted are smooth, but the more interesting is: approximate solutions are non-smooth (see figures 1 - 9). In comparison with the existing method to obtain peaked solitary solutions, our procedure just works on the polynomial and algebraic computations. In the recent literatures, there are also other methods [4, 9, 10] to deal with nonlinear partial differential equations, where smooth solutions were obtained. Our paper presents some continuous but non-smooth solutions for the b-balanced equation (1).

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