r-matrix and algebraic-geometric solution for integrable symplectic map

QIAO Zhijun

Institute of Mathematics and School of Mathematical Sciences, Peking University, Beijing 100871, *China*

Abstract A new Lax matrix is introduced for the integrable symplectic map (ISM), and the non-dynamical (i.e. constant) r-matrix of ISM is obtained. Moreover, an effective approach is systematically presented to construct the explicit solution (here, the explicit solution means algebraic-geometric solution expressed by the Riemann-Theta function) of a soliton system or nonlinear evolution equation from Lax matrix, r-matrix, and the theory of nonlinearization through taking the Toda lattice as an example. The given algebraic-geometric solution of the Toda lattice is almost-periodic and includes the periodic and finite-band solution.

Keywords: **symplectic map, r-matrix, algebraic-geometric solution.**

THE theory[1] of nonlinearization is quite an effective method to produce finite-dimensional completely integrable Hamiltonian system and integrable symplectic map[2]. In ref. [3], the dynamical r-matrix of integrable symplectic map was studied. This note is to present a new Lax matrix and a non-dynamical (i.e. constant) r-matrix of integrable symplectic map, and then to further study how to construct the explicit representation of solution of soliton system or nonlinear evolution equation by starting from a new Lax matrix of it, the non-dynamical r-matrix and the theory of nonlinearization and using the method of variables separation and modern algebraic-geometric tools.

Before displaying our main results, let us first give some conventions: $(\mathbb{R}^{2N}, \mathrm{d}p \wedge \mathrm{d}q)$ stands for the standard symplectic structure in Euclidean space $\mathbb{R}^{2N} = \{(p, q) \mid p = (p_1, \cdots, p_N), q = (q_1, \cdots, q_N)\}$, $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^N$, $[\cdot, \cdot]$ is the ordinary commutator of matrix, and in $(\mathbb{R}^{2N}, \mathrm{d}p \wedge \mathrm{d}q)$ the Poisson brackets of two Hamiltonian functions F, G are defined by[4]

$$\{F, G\} = \sum_{i=1}^{N}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right) = \left\langle\frac{\partial F}{\partial q}, \frac{\partial G}{\partial p}\right\rangle - \left\langle\frac{\partial F}{\partial p}, \frac{\partial G}{\partial q}\right\rangle.$$

$\lambda_1, \cdots, \lambda_N$ are N arbitrarily given distinct constants; λ and μ are the two different spectral parameters; $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$. Denote all infinitely times differentiable functions on real field $\mathbb{R}$ by $C^\infty(\mathbb{R})$.

The integer n stands for the spatial discrete variable. For the following Toda lattice the integer n is arbitrary, not being restricted to be finite or periodic.

1 New Lax matrix and r-matrix of Toda symplectic map

Introduce the following Lax matrix $L = L(\lambda, p, q)$:

$$L = \begin{pmatrix} -\frac{1}{2}\lambda & \langle p, q\rangle \\ -1 & \frac{1}{2}\lambda \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j}\begin{pmatrix} -p_jq_j & p_j^2 \\ -q_j^2 & p_jq_j \end{pmatrix} \equiv \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}. \tag{1}$$

Now, choose a 2×2 matrix $M = M(\lambda, p, q)$ as follows:

$$M = \begin{pmatrix} 0 & g \\ -\frac{1}{g} & \frac{\lambda - \langle q, q\rangle}{g} \end{pmatrix}, \quad g^2 = \langle \Lambda q, q\rangle - \langle p, q\rangle - \langle q, q\rangle^2. \tag{2}$$

Then by a direct calculation, we can get the following theorem.

Theorem 1. The discrete Lax equation

$$L'M = ML, \quad L' = L(\lambda, p', q') \tag{3}$$

is equivalent to a finite-dimensional symplectic map H: $(p, q)^{\mathrm{T}} \mapsto (p', q')^{\mathrm{T}}$

$$\begin{cases} p' = gq, \\ q' = \dfrac{\Lambda q - p - \langle q, q\rangle q}{g}. \end{cases} \tag{4}$$

Set

$$\begin{cases} u_n = \pm (\langle \Lambda q_n, q_n\rangle - \langle p_n, q_n\rangle - \langle p_n, q_n\rangle^2)^{\frac{1}{2}}, \\ v_n = \langle q_n, q_n\rangle, \end{cases} \tag{5}$$

or simply write it as f: $(p_n, q_n)^{\mathrm{T}} \mapsto (u_n, v_n)^{\mathrm{T}}$. Then (4) becomes the famous Toda spectral problem

$$L\psi_n \equiv (E^{-1}u_n + v_n + u_nE)\psi_n = \lambda\psi_n, \quad Ef_n = f_{n+1}, \quad E^{-1}f_n = f_{n-1}, \tag{6}$$

with $\lambda = \lambda_j$, $\psi = q_{n,j}$. Thus (4) is called the Toda symlectic map because the map H preserves the symplectic structure $\mathrm{d}p \wedge \mathrm{d}q$: $H^*(\mathrm{d}p \wedge \mathrm{d}q) = \mathrm{d}p \wedge \mathrm{d}q$. Theorem 1 shows that the Toda symplectic map H has the discrete Lax representation (3).

Equation (5) is a kind of discrete Bargmann constraint relation of the Toda spectral problem (6) (also see reference [2]).

Let $L_1(\lambda) = L(\lambda, p, q)\otimes I$, $L_2(\mu) = I\otimes L(\mu, p, q)$, I be the 2×2 unit matrix, and let $\{L_1(\lambda)\otimes, L_2(\mu)\}$ be the fundamental Poisson brackets[5]. Then we have the following result.

Theorem 2.

$$\{L_1(\lambda)\otimes, L_2(\mu)\} = [r_{12}(\lambda, \mu), L_1(\lambda)] - [r_{21}(\mu, \lambda), L_2(\mu)], \tag{7}$$

where the matrix $r_{12}(\lambda, \mu)$, $r_{21}(\mu, \lambda)$ is

$$r_{12}(\lambda, \mu) = \frac{2}{\mu - \lambda}P - S, \quad r_{21}(\mu, \lambda) = Pr_{12}(\lambda, \mu)P, \tag{8}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

It is easy to check that the matrix $r_{12}(\lambda, \mu)$ given by (8) satisfies the classical Yang-Baxter equation: $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] = 0$. So $r_{12}(\lambda, \mu) = \frac{2}{\mu - \lambda}P - S$ is an r-matrix of the Toda symplectic map (4).

Apparently, the r-matrix here found has no relation to the dynamical variables p, q, i.e. it is non-dynamical or constant. Additionally, the r-matrix $r_{12}(\lambda, \mu)$ satisfying the fundamental brackets (7) is not unique. Very interesting is that the present r-matrix $r_{12}=\frac{2}{\mu-\lambda}P-S$ of the discrete Toda symplectic map is the same as that of the continuous cKdV nonlinearized flow[6].

2 Integrability

Consider the determinant det L of the Lax matrix $L=L(\lambda, p, q)$

$$\det L=-\frac{1}{2}\operatorname{Tr}L^2=-\frac{1}{4}\lambda^2-\sum_{\alpha=1}^{N}\frac{E_\alpha}{\lambda-\lambda_\alpha}, \tag{10}$$

where

$$E_\alpha=\lambda_\alpha p_\alpha q_\alpha-p_\alpha^2-\langle p, q\rangle q_\alpha^2-\sum_{\beta\neq\alpha,\beta=1}^{N}\frac{(q_\alpha p_\beta-p_\alpha q_\beta)^2}{\lambda_\alpha-\lambda_\beta}, \quad \alpha=1, \cdots, N. \tag{11}$$

By (7) and (10), we can obtain

Theorem 3.

1)

$$\{E_\alpha, E_\beta\}=0, \quad \alpha, \beta=1, \cdots, N. \tag{12}$$

2) Let $F_s=\sum_{\alpha=1}^{N}\lambda_\alpha^s E_\alpha$, $s=0, 1, 2, \cdots$. Then

$$F_s=\langle\Lambda^{s+1}p, q\rangle-\langle\Lambda^s p, p\rangle-\langle p, q\rangle\langle\Lambda^s q, q\rangle - \sum_{j+k=s-1}(\langle\Lambda^j p, p\rangle\langle\Lambda^k q, q\rangle-\langle\Lambda^j p, q\rangle\langle\Lambda^k q, p\rangle), \quad s=0, 1, \cdots, \tag{13}$$

and $\{F_m, F_l\}=0$, $\forall\, m, l\in\mathbb{Z}^+$.

Theorem 4. The finite-dimensional Toda symplectic map H determined by (4) is completely integrable in Liouville's sense.

Similar to the checking procedure in ref. [2], we can know that the following process:

$$\begin{pmatrix}p_0\\q_0\end{pmatrix}\xrightarrow{F_0}\begin{pmatrix}p_0(t)\\q_0(t)\end{pmatrix}\xrightarrow{H^n}\begin{pmatrix}p_n(t)\\q_n(t)\end{pmatrix}\xrightarrow{f}\begin{pmatrix}u_n(t)\\v_n(t)\end{pmatrix} \tag{14}$$

produces a solution of the Toda lattice

$$\dot{u}_n=u_n(v_{v+1}-v_n), \quad \dot{v}_n=2(u_n^2-u_{n-1}^2). \tag{15}$$

Thus, via the transformation $u_n=\mathrm{e}^{x_{n+1}-x_n}$, $v_n=\dot{x}_n$, (15) becomes the standard Toda lattice

$$\ddot{x}_n=2(\mathrm{e}^{2(x_{n+1}-x_n)}-\mathrm{e}^{2(x_n-x_{n-1})}), \tag{16}$$

which has the following form of solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}x_n(t)=\langle q_n(t), q_n(t)\rangle, \text{ i.e. } x_n(t)=\int\langle q_n(t), q_n(t)\rangle\,\mathrm{d}t. \tag{17}$$

3 Explicit (algebraic-geometric) solution

The method of variable separation is shown to be effective for continuous system by Sklyanin[7]. Now, we apply this method to discrete system, and concretely solve formula $\langle q_n(t), q_n(t)\rangle$ through using further mordern algebraic-geometric tools. In the following it shall be seen that $\langle q_n(t), q_n(t)\rangle$ can be expressed as an explicit form. In order to do that, we set $C(\lambda)\equiv-\frac{Q(\lambda)}{K(\lambda)}$, $K(\lambda)=\prod_{\alpha=1}^{N}(\lambda-\lambda_\alpha)$, and choose N distinct real zero points $\mu_1, \cdots, \mu_N$ of $Q(\lambda)$. Then we have

$$Q(\lambda)=\prod_{j=1}^{N}(\lambda-\mu_j), \quad \langle q, q\rangle=\sum_{\alpha=1}^{N}\lambda_\alpha-\sum_{j=1}^{N}\mu_j. \tag{18}$$

Let $\pi_j=A(\mu_j)$. Then π_j and μ_j are conjugated, and thus they are the separable variables[7].

Write $\det L=-\frac{P(\lambda)}{K(\lambda)}$, where $P(\lambda)$ is an $N+2$-order polynomial of λ whose first item's coefficient is $\frac{1}{4}$. Then $\pi_j^2=\frac{P(\mu_j)}{K(\mu_j)}$, $j=1, \cdots, N$. Now, we choose the generating function

$$W = \sum_{j=1}^{N} W_j(\mu_j, \{E_\alpha\}_{\alpha=1}^{N}) = \sum_{j=1}^{N}\int_{\mu_j(0)}^{\mu_j(n)} \sqrt{\frac{P(\lambda)}{K(\lambda)}}\mathrm{d}\lambda, \tag{19}$$

where $\mu_j(0)$ is an arbitrarily given constant. View $E_j (j = 1, \cdots, N)$ as actional variables. Then angle coordinates Q_j are chosen as $Q_j = \frac{\partial W}{\partial E_j}$, $j = 1, \cdots, N$, i.e.

$$Q_j = \sum_{k=1}^{N}\int_{\mu_k(0)}^{\mu_k(n)} \tilde{\omega}_j, \quad \tilde{\omega}_j = \frac{\prod\limits_{\alpha \neq j, \alpha=1}^{N}(\lambda - \lambda_\alpha)}{2\sqrt{K(\lambda)P(\lambda)}}\mathrm{d}\lambda, \quad j = 1, \cdots, N. \tag{20}$$

Hence, on the symplectic manifold $(\mathbb{R}^{2N}, \mathrm{d}E_\alpha \wedge \mathrm{d}Q_\alpha)$ the Hamiltonian function $F_0 = \sum_{\alpha=1}^{N} E_\alpha$ produces a linearized flow

$$\begin{cases} \dot{Q}_j = \dfrac{\partial F_0}{\partial E_j}, \\ \dot{E}_j = 0; \end{cases} \tag{21}$$

thus

$$\begin{cases} Q_j(n) = Q_j^0 + t + c_j n, \quad c_j = \sum\limits_{k=1}^{N}\int_{\mu_k(n)}^{\mu_k(n+1)} \tilde{\omega}_j, \\ E_j(n) = E_j(n-1), \end{cases} \tag{22}$$

where c_j are dependent on $E_1, \cdots, E_N$, and independent of t; Q_j^0 is an arbitrary fixed constant.

Choose closed paths α_i, β_i, $i = 1, \cdots, N$ of Riemann surface Γ: $\mu^2 = P(\lambda)K(\lambda)$ with N handles. Then $\tilde{\omega}_j$ can be normed as $\omega_j = \sum_{l=1}^{N} r_{j,l}\tilde{\omega}_l$, i.e. ω_j satisfy $\oint_{\alpha_i}\omega_j = \delta_{ij}$, $\oint_{\beta_i}\omega_j = B_{ij}$, where $B = (B_{ij})_{N\times N}$ is symmetric and the imaginary part ImB of B is a positive definite matrix.

By Riemann Theorem[8] through a lengthy calculation, we can get

$$\langle q_n(t), q_n(t)\rangle = \sum_{\alpha=1}^{N}\lambda_\alpha - \tilde{C} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\ln\frac{\Theta(\phi(n, t) + K + \eta_+)}{\Theta(\phi(n, t) + K + \eta_-)}\right), \tag{23}$$

where constant $\tilde{C}$ is independent of $\phi(n, t)$[9], Θ is the Riemann-Theta function on Riemann surface Γ, $\phi(n, t) = (\phi_1(n, t), \cdots, \phi_N(n, t))^{\mathrm{T}} \equiv \left(\sum_{l=1}^{N} r_{1,l}(Q_l^0 + t + c_l n), \cdots, \sum_{l=1}^{N} r_{N,l}(Q_l^0 + t + c_l n)\right)^{\mathrm{T}}$, and $K \in C^N$ is the Riemann constant vector. The j-th component of $\eta_\pm$ is $\eta_{\pm,j} = \int_{\infty_\pm}^{P_0}\omega_j$, here $\infty_\pm = (0, \pm\sqrt{P(z^{-1})K(z^{-1})}\,|_{z=0})$. P_0 is arbitrarily chosen on Riemann surface Γ. So the standard Toda lattice (16) has the following representation of explicit solution, called algebraic-geometric solution:

$$x_n(t) = \ln\frac{\Theta(Un + Vt + Z)}{\Theta(U(n+1) + Vt + Z)} + Cn + Rt + \text{const.}, \tag{24}$$

where $U = \hat{R}\hat{C}$, $V = \hat{R}\hat{J}$, $Z = \hat{R}Q^0 + K + \eta_1$. Here $\hat{C} = (c_1, \cdots, c_N)^{\mathrm{T}}$, $\hat{J} = (1, \cdots, 1)^{\mathrm{T}}$, $Q^0 = (Q_1^0, \cdots, Q_N^0)^{\mathrm{T}}$, matrix $\hat{R} = (r_{j,l})_{N\times N}$ is determined by the relation equality $\sum_{l=1}^{N} r_{j,l}\oint_{\alpha_i}\tilde{\omega}_l = \delta_{ij}$, the symmetric matrix $B = (B_{ij})_{N\times N}$ in Θ function is given by $\sum_{l=1}^{N} r_{j,l}\oint_{\beta_i}\tilde{\omega}_l = B_{ij}$; $R = \sum_{\alpha=1}^{N}\lambda_\alpha - \tilde{C}$, and C is a certain constant to be determined by the algebraic-geometric attributes on the Riemann surface Γ[10].

Thus, the algebraic-geometric solution of the Toda lattice (15) is

$$\begin{cases} u_n(t) = \mathrm{e}^{x_{n+1}-x_n} = \mathrm{e}^C\dfrac{\Theta^2(U(n+1) + Vt + Z)}{\Theta(U(n+2) + Vt + Z)\Theta(Un + Vt + Z)}, \\ v_n(t) = \dot{x}_n = R + \dfrac{\mathrm{d}}{\mathrm{d}t}\ln\dfrac{\Theta(Un + Vt + Z)}{\Theta(U(n+1) + Vt + Z)}. \end{cases} \tag{25}$$

Obviously, $u_n(t)$ and $v_n(t)$ are almost periodic functions, and they are periodic if $U = \frac{M}{N}$, where M is an N-dimensional integer column vector. Apparently they are the finite-band solution of (15).

The method described above can be also applied to other soliton systems.

Acknowledgement The author would like to express his sincerest thanks to Profs. Gu Chaohao and Hu Hesheng for their enthusiastic instructions and helps. This work was supported by the National Natural Science Foundation of China (Grant No. 19501019), China Postdoctoral Science Foundation, and Science and Technology Foundation of Liaoning Province.

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(*Received January* 29, 1998)