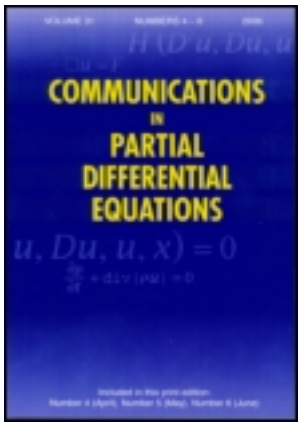

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Stability of Solitary Waves and Global Existence of a Generalized Two-Component Camassa–Holm System

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We study here the existence of solitary wave solutions of a generalized two-component Camassa–Holm system. In addition to those smooth solitary-wave solutions, we show that there are solitary waves with singularities: peaked and cusped solitary waves. We also demonstrate that all smooth solitary waves are orbitally stable in the energy space. We finally give a sufficient condition for global strong solutions to the equation in some special case.

Keywords Generalized two-component Camassa–Holm system; Global solutions; Solitary waves; Stability.

Mathematics Subject Classification 35B35; 35G25; 35A01.

1. Introduction

There are several classical models describing the motion of waves at the free surface of shallow water under the influence of gravity. Among these models, the best known is the Korteweg-de Vries (KdV) equation [43, 49]

$$u_t + 6uu_x + u_{xxx} = 0.$$

The KdV equation admits solitary wave solutions, i.e. solutions of the form $u(t, x) = \varphi(x - ct)$ which travel with fixed speed c, and that vanish at infinity. The KdV solitary waves are smooth and retain their individuality under interaction and eventually emerge with their original shapes and speeds [19]. Moreover, KdV is an integrable infinite-dimensional Hamiltonian system [41]. However, the KdV equation does not model the phenomenon of breaking for water waves. Instead, as

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soon as the initial profile $u_0 \in H^1(\mathbb{R})$, the solutions are global in time [35], whereas some shallow water waves break [49].

Another model, the Camassa–Holm (CH) equation [3]

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1.1}$$

arises as a model for the unidirectional propagation of shallow water waves over a flat bottom [3, 12, 20–22, 33], as well as water waves moving over an underlying shear flow [34]. Equation (1.1) is completely integrable with the Lax pair [3] and with infinitely many conservation laws as an bi-Hamiltonian system [25].

The CH equation has many remarkable properties that KdV does not have like solitary waves with singularities and breaking waves. The CH equation admits peaked solitary waves or "peakons" [1, 3, 38]: $u(t, x) = ce^{-|x-ct|}$, $c \neq 0$, which are smooth except at the crests, where they are continuous, but have a jump discontinuity in the first derivative. The peakons capture a feature that is characteristic for the waves of great height–waves of the largest amplitude that are exact solutions of the governing equations for water waves [8, 13, 48]. The CH equation also models wave breaking (i.e. the solution remains bounded while its slope becomes unbounded in finite time) (see [3, 5–7, 10, 41, 49]).

The CH equation also admits many multi-component generalizations. It is intriguing to know if the above two properties may persist in the systems. In this paper we consider the following generalized two-component CH system established in [4] which can be derived from shallow water theory with nonzero constant vorticity

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x - \sigma(2u_x u_{xx} + uu_{xxx}) + \rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0, \end{cases} \tag{1.2}$$

or equivalently, using the linear momentum $m = u - u_{xx}$,

$$\begin{cases} m_t + \sigma um_x - Au_x + 2\sigma mu_x + 3(1-\sigma)uu_x + \rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0, \end{cases}$$

where $u(t, x)$ is the horizontal velocity and $\rho(t, x)$ is related to the free surface elevation from equilibrium (or scalar density) with the boundary assumptions $u \to 0$, $\rho \to 1$ as $|x| \to \infty$. The scalar $A > 0$ characterizes a linear underlying shear flow and hence system (1.2) models wave-current interactions. The real dimensionless constant σ is a parameter which provides the competition, or balance, in fluid convection between nonlinear steepening and amplification due to stretching.

When $\sigma = 1$ it recovers the standard two-component CH system. Mathematical properties of the system have been also studied further in many works, cf. [11, 24, 27–30, 42, 44, 50] and references therein. The two-component CH system is completely integrable [11, 32, 45, 46] as it can be written as compatibility condition of two linear systems (Lax pair) with a spectral parameter ζ, that is,

$$\Psi_{xx} = \left[-\zeta^2\rho^2 + \zeta\left(m - \frac{A}{2} + \frac{1}{4}\right)\right]\Psi,$$

$$\Psi_t = \left(\frac{1}{2\zeta} - u\right)\Psi_x + \frac{1}{2}u_x\Psi.$$

In the case $\rho \equiv 0$, it becomes

$$u_t - u_{xxt} + 3uu_x = \sigma\,(2u_x u_{xx} + uu_{xxx})\,, \tag{1.3}$$

which models finite length, small amplitude radial deformation waves in cylindrical hyperelastic rods [18]. System (1.2) has the following two Hamiltonians

$$H_1 = \frac{1}{2}\int_{\mathbb{R}} \left(u^2 + u_x^2 + (\rho - 1)^2\right) dx,$$
$$H_2 = \frac{1}{2}\int_{\mathbb{R}} \left(u^3 + \sigma u u_x^2 + 2u(\rho - 1) + u(\rho - 1)^2 - Au^2\right) dx.$$

We study solitary wave solutions of (1.2), i.e. solutions of the form

$$(u(x, t), \rho(x, t)) = (\varphi(x - ct), \rho(x - ct)), \quad c \in \mathbb{R}$$

for some $\varphi, \rho : \mathbb{R} \to \mathbb{R}$ such that $\varphi \to 0$, $\rho \to 1$ as $|x| \to \infty$. In the study of the CH traveling waves it was observed through phase-plane analysis [40] that both peaked and cusped traveling waves exist. Subsequently, Lenells [38, 39] used a suitable framework for weak solutions to classify all weak traveling waves of the CH equation (1.1) and the hyperelastic rod equation (1.3).

Using a natural weak formulation of the two-component CH system, we will establish exactly in what sense the peaked and cusped solitary waves are solutions. It was shown in [11, 42, 50] that when $\sigma = 1$ the two component system (1.2) has only smooth solitary waves, with a single crest profile and exponential decay far out. In [31], the authors considered a modified two-component CH equation which allows dependence on average density as well as pointwise density and a linear dispersion is added to the first equation of the system. They showed that the modified system admits peakon solutions in both u and ρ. However it is unclear whether the generalized two-component CH system (1.2) has solitary waves with singularities. We show here peaked solitary waves exist when $\sigma > 1$. We also provide an implicit formula for the peaked solitary waves. However whether these peaked solitary waves are solitons still remains open.

The stability of solitary waves is one of the fundamental qualitative properties of the solutions of nonlinear wave equations [47]. Due to the fact that the solitons hardly interact with each other at all it is reasonable to expect that they are stable. It has been proved that for the CH equation, the smooth solitary waves are orbitally stable [17]. Moreover, the peakons, whether solitary waves or periodic waves, are also orbitally stable [14, 15, 36, 37]. It was shown in [50] that when $\sigma = 1$ all solitary waves are orbitally stable. We prove in this paper that when $\sigma \leq 1$ all smooth solitary waves are orbitally stable. The proof of the stability basically follows the general approach in [26]. In comparison with the spectral arguments on the Hessian operator in [26], here we require more precise analysis on the spectrum of a linearized operator around the solitary waves for the system (1.2).

A special case of system (1.2) is when $\sigma = 0$. In the scalar equation case when $\sigma = 0$ it is the BBM equation [2]. The solutions are shown to be global in time. We show that the same results hold in the system case.

This paper is organized as follows. In Section 2 we classify the solitary waves of (1.2). In particular we show the existence of peaked solitary waves for $\sigma > 1$.

In Section 3 we prove that when $\sigma \leq 1$ all smooth solitary waves are nonlinearly stable. Finally in Section 4, we show that the system (1.2) is globally well-posed for $\sigma = 0$.

2. Solitary Waves

Let $X = H^1(\mathbb{R}) \times L^2(\mathbb{R})$ be a real Hilbert space with inner product $(,)$, and denote its element by $\vec{u} = (u, \eta)$. The dual of X is $X^* = H^{-1}(\mathbb{R}) \times L^2(\mathbb{R})$ and a natural isomorphism I from X to X^* can be defined by

$$I = \begin{pmatrix} 1 - \partial_x^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using the map I, the paring $\langle , \rangle$ between X and X^* can be represented as

$$\langle I\vec{u}, \vec{v} \rangle = \langle u, v \rangle_1 + \langle \eta, \xi \rangle_0, \quad \text{for } \vec{u} = (u, \eta) \in X, \quad \vec{v} = (v, \xi) \in X^*,$$

where $\langle , \rangle_s$ denotes the $H^s(\mathbb{R}) \times H^{-s}(\mathbb{R})$ dual pairing. We will identify the second dual X^{**} with X in a natural way.

Since $\rho \to 1$ as $|x| \to \infty$ in (1.2), we can let $\rho = 1 + \eta$ with $\eta \to 0$ as $|x| \to \infty$ and hence we can rewrite system (1.2) as

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x - \sigma(2u_x u_{xx} + uu_{xxx}) + (1 + \eta)\eta_x = 0, \\ \eta_t + ((1 + \eta)u)_x = 0. \end{cases} \tag{2.1}$$

The two Hamiltonians introduced in the Introduction define the following two functionals on X

$$E(\vec{u}) = \frac{1}{2} \int_{\mathbb{R}} \left(u^2 + u_x^2 + \eta^2 \right) dx, \tag{2.2}$$

$$F(\vec{u}) = \frac{1}{2} \int_{\mathbb{R}} \left(u^3 + \sigma u u_x^2 + 2u\eta + u\eta^2 - Au^2 \right) dx, \tag{2.3}$$

with $\vec{u} = (u, \eta) \in X$. The quantity $E(\vec{u})$ associates with the translation invariance of (2.1). Using functional $F(\vec{u})$, system (2.1) can be written in an abstract Hamiltonian form

$$\vec{u}_t = JF'(\vec{u}), \tag{2.4}$$

where J is a closed skew symmetric operator given by

$$J = \begin{pmatrix} -\partial_x(1 - \partial_x^2)^{-1} & 0 \\ 0 & -\partial_x \end{pmatrix}$$

and $F'(\vec{u}) : X \to X^*$ is the variational derivative of F in X at $\vec{u}$.

Note that if

$$p(x) := \frac{1}{2} e^{-|x|}, \quad x \in \mathbb{R}, \tag{2.5}$$

then $(1-\partial_x^2)^{-1}f = p * f$ for all $f \in L^2(\mathbb{R})$. We can then further rewrite system (2.1) in a weak form as

$$\begin{cases} u_t + \sigma u u_x + \partial_x p * \left(-Au + \dfrac{3-\sigma}{2}u^2 + \dfrac{\sigma}{2}u_x^2 + \dfrac{1}{2}(1+\eta)^2 \right) = 0, \\ \eta_t + ((1+\eta)u)_x = 0. \end{cases} \tag{2.6}$$

Definition 2.1. Let $0 < T \le \infty$. A function $\vec{u} = (u, \eta) \in C([0, T); X)$ is called a solution of (2.1) on $[0, T)$ if it satisfies (2.6) in the distribution sense on $[0, T)$ and $E(\vec{u})$ and $F(\vec{u})$ are conserved.

Applying the transport equation theory combined with the method of Besov spaces, one may follow the similar argument as in [29] to obtain the following local well-posedness result for the system (2.1).

Theorem 2.2. *If $(u_0, \eta_0) \in H^s \times H^{s-1}$, $s > 3/2$, then there exist a maximal time $T = T(\|(u_0, \eta_0)\|_{H^s \times H^{s-1}}) > 0$ and a unique solution (u, η) of (1.2) in $C([0, T); H^s \times H^{s-1}) \cap C^1([0, T); H^{s-1} \times H^{s-2})$ with $(u, \eta)|_{t=0} = (u_0, \eta_0)$. Moreover, the solution depends continuously on the initial data and T is independent of s.*

It is easily seen from the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ that $E(\vec{u})$ and $F(\vec{u})$ are both well defined in $H^s \times H^{s-1}$ with $s > 3/2$, and $E(\vec{u})$ is conserved, as suggested in the local wellposedness Theorem 2.2. From (2.4) we see that

$$\frac{d}{dt}F(\vec{u}) = \langle F'(\vec{u}), \vec{u}_t \rangle = \langle F'(\vec{u}), JF'(\vec{u}) \rangle = 0.$$

So $F(\vec{u})$ is also invariant.

Now we give the definition of *solitary waves* of (2.1).

Definition 2.3. A solitary wave of (2.1) is a nontrivial traveling wave solution of (2.1) of the form $\vec{\varphi}_c(t, x) = (\varphi_c(x - ct), \eta_c(x - ct)) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ with $c \in \mathbb{R}$ and φ_c, η_c vanishing at infinity.

For a solitary wave $\vec{\varphi} = (\varphi, \eta)$ with speed $c \in \mathbb{R}$, it satisfies

$$\begin{cases} \left[-c\varphi + \dfrac{\sigma}{2}\varphi^2 + p * \left(-A\varphi + \dfrac{3-\sigma}{2}\varphi^2 + \dfrac{\sigma}{2}\varphi_x^2 + \dfrac{1}{2}(1+\eta)^2 \right) \right]_x = 0, \\ [-c\eta + (1+\eta)\varphi]_x = 0, \end{cases} \quad \text{in } \mathscr{D}'(\mathbb{R}). \tag{2.7}$$

Integrating the above system and applying $(1-\partial_x^2)$ to the first equation we get

$$\begin{cases} -(c+A)\varphi + c\varphi_{xx} + \dfrac{3}{2}\varphi^2 = \sigma\varphi\varphi_{xx} + \dfrac{\sigma}{2}\varphi_x^2 - \dfrac{1}{2}(1+\eta)^2 + \dfrac{1}{2}, \\ -c\eta + (1+\eta)\varphi = 0, \end{cases} \quad \text{in } \mathscr{D}'(\mathbb{R}). \tag{2.8}$$

The fact that the second equation of the above holds in a strong sense comes from the regularity of φ and η.

Proposition 2.4. *If* (φ, η) *is a solitary wave of* (2.1) *for some* $c \in \mathbb{R}$, *then* $c \neq 0$ *and* $\varphi(x) \neq c$ *for any* $x \in \mathbb{R}$.

Proof. From the definition of solitary waves and the embedding theorem we know that φ and η are both continuous. If $c = 0$, then (2.8) becomes

$$\begin{cases} -A\varphi + \frac{3}{2}\varphi^2 = \sigma\varphi\varphi_{xx} + \frac{\sigma}{2}\varphi_x^2 - \frac{1}{2}(1+\eta)^2 + \frac{1}{2}, \\ (1+\eta)\varphi = 0. \end{cases} \tag{2.9}$$

Since η vanishes at infinity, the second equation of the above system indicates that $\varphi(x) = 0$ for $|x|$ large enough. Denote $x_0 = \max\{x : \varphi(x) \neq 0\}$. Hence $\varphi(x) = 0$ on $[x_0, \infty)$ and $\varphi \not\equiv 0$ on $(x_0 - \delta, x_0)$ for any $\delta > 0$. Consider now the first equation of (2.9) on $[x_0, \infty)$ we see that $\eta \equiv 0$ on $[x_0, \infty)$. Then the continuity of η implies that there exists a $\delta_1 > 0$ such that $1 + \eta(x) > 0$ on $(x_0 - \delta_1, x_0)$. This together with the second equation of (2.9) leads to $\varphi(x) \equiv 0$ on $(x_0 - \delta_1, x_0)$, which is a contradiction. Therefore $c \neq 0$.

Next we show $\varphi \neq c$. If not and there is some $x_1 \in \mathbb{R}$ such that $\varphi(x_1) = c$. Then the second equation of (2.8) infers that

$$\varphi(x_1) = (c - \varphi(x_1))\,\eta(x_1) = 0,$$

so $c = 0$, which is a contraction. □

Using the above proposition we obtain from the second equation of (2.8) that

$$\eta = \frac{\varphi}{c - \varphi}. \tag{2.10}$$

Plugging this into the first equation of (2.8) we obtain an equation for the unknown φ only

$$-(c + A)\varphi + c\varphi_{xx} + \frac{3}{2}\varphi^2 = \sigma\varphi\varphi_{xx} + \frac{\sigma}{2}\varphi_x^2 - \frac{1}{2}\frac{c^2}{(c-\varphi)^2} + \frac{1}{2}, \quad \text{in } \mathscr{D}'(\mathbb{R}). \tag{2.11}$$

2.1. *The Case When* $\sigma = 0$

When $\sigma = 0$, (2.11) becomes

$$\varphi_{xx} = \frac{c + A}{c}\varphi - \frac{3}{2c}\varphi^2 + \frac{1}{2c} - \frac{1}{2}\frac{c}{(c-\varphi)^2}, \quad \text{in } \mathscr{D}'(\mathbb{R}). \tag{2.12}$$

Since $\varphi \in H^1(\mathbb{R})$ and $c - \varphi \neq 0$ we know that $|c - \varphi|$ is bounded away from 0. Hence from the standard local regularity theory to elliptic equation we see that $\varphi \in C^\infty(\mathbb{R})$ and so is η. Therefore in this case all solitary waves are smooth.

As for the existence, we may multiply (2.12) by ϕ_x and integrate on $(-\infty, x]$ to get

$$\varphi_x^2 = \frac{\varphi^2(c - \varphi - A_1)(c - \varphi - A_2)}{c(c - \varphi)} := G(\varphi), \tag{2.13}$$

where

$$A_1 = \frac{-A + \sqrt{A^2+4}}{2}, \quad A_2 = \frac{-A - \sqrt{A^2+4}}{2} \tag{2.14}$$

are the two roots of the equation $y^2 + Ay - 1 = 0$. Since $A > 0$, we know $A_1 > 0 > A_2$.

From the decay property of ϕ at infinity we know that a necessary condition for the existence is $c \geq A_1$ or $c \leq A_2$. But one may prove further that

Theorem 2.5. *When $\sigma = 0$, (2.1) admits a solitary wave solution if and only if*

$$c > A_1 \quad or \quad c < A_2. \tag{2.15}$$

All solitary waves are smooth in this case.

Proof. The regularity is discussed as above. So we will just focus on the existence part.

If $c = A_1$, then (2.13) becomes

$$\varphi_x^2 = \frac{-\varphi^3(A_1 - A_2 - \varphi)}{A_1(A_1 - \varphi)} := G_1(\varphi). \tag{2.16}$$

Hence we see that $\varphi(x) < 0$ near $-\infty$. Because $\varphi(x) \to 0$ as $x \to -\infty$, there is some x_0 sufficiently large negative so that $\varphi(x_0) = -\epsilon < 0$, with ϵ sufficiently small, and $\varphi_x(x_0) < 0$. From standard ODE theory, we can generate a unique local solution $\varphi(x)$ on $[x_0 - L, x_0 + L]$ for some $L > 0$. Since $A_1 > 0 > A_2$, we have

$$\left[\frac{-\varphi^3(A_1 - A_2 - \varphi)}{(A_1 - \varphi)}\right]' = \frac{\varphi^2\left[-3\varphi^2 + (6A_1 - 2A_2)\varphi - 3A_1(A_1 - A_2)\right]}{(A_1 - \varphi)^2} < 0, \tag{2.17}$$

for $\varphi < 0$. Therefore $G_1(\varphi)$ decreases for $\varphi < 0$. Because $\varphi_x(x_0) < 0$, φ decreases near x_0, so $G_1(\varphi)$ increases near x_0. Hence from (2.16), φ_x decreases near x_0, and then φ and φ_x both decreases on $[x_0 - L, x_0 + L]$. Since $\sqrt{G_1(\varphi)}$ is locally Lipschitz in φ for $\varphi \leq 0$, we can easily continue the local solution to all of $\mathbb{R}$ and obtain that $\phi(x) \to -\infty$ as $x \to \infty$, which fails to be in $H^1(\mathbb{R})$. Thus there is no solitary wave in this case.

Similarly we have that when $c = A_2$ there is no solitary wave. Therefore the theorem is proved. □

2.2. *The Case When $\sigma \neq 0$*

In this case we can rewrite (2.11) as

$$\left(\left(\varphi - \frac{c}{\sigma}\right)^2\right)_{xx} = \varphi_x^2 - \frac{2(c+A)}{\sigma}\varphi + \frac{3}{\sigma}\varphi^2 - \frac{1}{\sigma} + \frac{c^2}{\sigma(c-\varphi)^2}, \quad \text{in } \mathscr{D}'(\mathbb{R}). \tag{2.18}$$

The following lemma deals with the regularity of the solitary waves. The idea is inspired by the study of the traveling waves of Camassa–Holm equation [38].

Lemma 2.6. *Let $\sigma \neq 0$ and (φ, η) be a solitary wave of* (2.1). *Then*

$$\left(\varphi - \frac{c}{\sigma}\right)^k \in C^j\left(\mathbb{R}\backslash\varphi^{-1}(c/\sigma)\right), \quad \text{for } k \geq 2^j. \tag{2.19}$$

Therefore

$$\varphi \in C^\infty\left(\mathbb{R}\backslash\varphi^{-1}(c/\sigma)\right). \tag{2.20}$$

Proof. From Proposition 2.4 we know that $c \neq 0$ and $\varphi \neq c$ and thus φ satisfies (2.18). Let $v = \varphi - \frac{c}{\sigma}$ and denote

$$r(v) = \frac{3}{\sigma}\left(\frac{v+c}{\sigma}\right)^2 - \frac{2(c+A)}{\sigma}\left(\frac{v+c}{\sigma}\right) - \frac{1}{\sigma}.$$

So $r(v)$ is a polynomial in v. From the fact that $\varphi - c \neq 0$ we know that

$$\frac{\sigma - 1}{\sigma}c - v \neq 0. \tag{2.21}$$

Then v satisfies

$$(v^2)_{xx} = v_x^2 + r(v) + \frac{c^2}{\sigma}\left(\frac{\sigma-1}{\sigma}c - v\right)^{-2}.$$

From the assumption we know that $(v^2)_{xx} \in L^1_{loc}(\mathbb{R})$. Hence $(v^2)_x$ is absolutely continuous and hence

$$v^2 \in C^1(\mathbb{R}), \quad \text{and then } v \in C^1\left(\mathbb{R}\backslash v^{-1}(0)\right).$$

So from (2.21) and that $v + \frac{c}{\sigma} \in H^1(\mathbb{R}) \subset C(\mathbb{R})$ we know

$$\left(\frac{\sigma-1}{\sigma}c - v\right)^{-2} \in C(\mathbb{R}) \cap C^1\left(\mathbb{R}\backslash v^{-1}(0)\right).$$

Moreover,

$$\begin{aligned}
(v^k)_{xx} &= (kv^{k-1}v_x)_x = \frac{k}{2}\left(v^{k-2}(v^2)_x\right)_x \\
&= k(k-2)v^{k-2}v_x^2 + \frac{k}{2}v^{k-2}(v^2)_{xx} \\
&= k(k-2)v^{k-2}v_x^2 + \frac{k}{2}v^{k-2}\left[v_x^2 + r(v) + \frac{c^2}{\sigma}\left(\frac{\sigma-1}{\sigma}c - v\right)^{-2}\right] \\
&= k\left(k - \frac{3}{2}\right)v^{k-2}v_x^2 + \frac{k}{2}v^{k-2}r(v) + \frac{kc^2}{2\sigma}v^{k-2}\left(\frac{\sigma-1}{\sigma}c - v\right)^{-2}.
\end{aligned} \tag{2.22}$$

For $k = 3$, the right-hand side of (2.22) is in $L^1_{loc}(\mathbb{R})$. Thus we deduce that

$$v^3 \in C^1(\mathbb{R}).$$

For $k \geq 4$ we see that (2.22) implies

$$(v^k)_{xx} = \frac{k}{4}\left(k - \frac{3}{2}\right) v^{k-4}\left[(v^2)_x\right]^2 + \frac{k}{2} v^{k-2} r(v) + \frac{kc^2}{2\sigma} v^{k-2}\left(\frac{\sigma - 1}{\sigma} c - v\right)^{-2} \in C(\mathbb{R}).$$

Therefore $v^k \in C^2(\mathbb{R})$ for $k \geq 4$.

For $k \geq 8$ we know from the above that

$$v^4,\ v^{k-4},\ v^{k-2},\ v^{k-2} r(v) \in C^2(\mathbb{R}), \quad \text{and} \quad v^{k-2}\left(\frac{\sigma - 1}{\sigma} c - v\right)^{-2} \in C^2\left(\mathbb{R} \backslash v^{-1}(0)\right).$$

Moreover we have

$$v^{k-2} v_x^2 = \frac{1}{4}(v^4)_x \frac{1}{k-4}(v^{k-4})_x \in C^1(\mathbb{R}).$$

Hence from (2.22) we conclude that

$$v^k \in C^3\left(\mathbb{R} \backslash v^{-1}(0)\right), \quad k \geq 8.$$

Applying the same argument to higher values of k we prove that $v^k \in C^j\left(\mathbb{R} \backslash v^{-1}(0)\right)$ for $k \geq 2^j$, and hence (2.19). □

Denote $\bar{x} = \min\{x : \varphi(x) = c/\sigma\}$ (if $\varphi \neq c/\sigma$ for all x then let $\bar{x} = +\infty$), then $\bar{x} \leq +\infty$. From Lemma 2.6, a solitary wave φ is smooth on $(-\infty, \bar{x})$ and hence (2.11) holds pointwise on $(-\infty, \bar{x})$. Therefore we may multiply by φ_x and integrate on $(-\infty, x]$ for $x < \bar{x}$ to get

$$\varphi_x^2 = \frac{\varphi^2 (c - \varphi - A_1)(c - \varphi - A_2)}{(c - \varphi)(c - \sigma\varphi)} := F(\varphi), \tag{2.23}$$

where A_1 and A_2 are defined in (2.14).

Applying the similar arguments as introduced in [38] we make the following conclusions.

1. When φ approaches a simple zero $m = c - A_1$ or $m = c - A_2$ of $F(\varphi)$ so that $F(m) = 0$ and $F'(m) \neq 0$. The solution φ of (2.23) satisfies

$$\varphi_x^2 = (\varphi - m) F'(m) + O((\varphi - m)^2) \quad \text{as } \varphi \to m,$$

where $f = O(g)$ as $x \to a$ means that $|f(x)/g(x)|$ is bounded in some interval $[a - \epsilon, a + \epsilon]$ with $\epsilon > 0$. Hence

$$\varphi(x) = m + \frac{1}{4}(x - x_0)^2 F'(m) + O((x - x_0)^4) \quad \text{as } x \to x_0, \tag{2.24}$$

where $\varphi(x_0) = m$.

2. If $F(\varphi)$ has a double zero at $\varphi = 0$, so that $F'(0) = 0$, $F''(0) > 0$, then

$$\varphi_x^2 = \varphi^2 F''(0) + O(\varphi^3) \quad \text{as } \varphi \to 0.$$

We get

$$\varphi(x) \sim \alpha \exp\left(-x\sqrt{F''(0)}\right) \quad \text{as } x \to \infty \tag{2.25}$$

for some constant α. thus $\varphi \to 0$ exponentially as $x \to \infty$.

3. If φ approaches a simple pole $\varphi(x_0) = c/\sigma$ of $F(\varphi)$ (when $\sigma \neq 1$). Then

$$\varphi(x) - \frac{c}{\sigma} = \beta|x - x_0|^{2/3} + O((x - x_0)^{4/3}) \quad \text{as } x \to x_0, \tag{2.26}$$

$$\varphi_x = \begin{cases} \frac{2}{3}\beta|x - x_0|^{-1/3} + O((x - x_0)^{1/3}) & \text{as } x \downarrow x_0, \\ -\frac{2}{3}\beta|x - x_0|^{-1/3} + O((x - x_0)^{1/3}) & \text{as } x \uparrow x_0, \end{cases} \tag{2.27}$$

for some constant β. In particular, when $F(\varphi)$ has a pole, the solution φ has a cusp.

4. Peaked solitary waves occur when φ suddenly changes direction: $\varphi_x \mapsto -\varphi_x$ according to (2.23).

Now we give the following theorem on the existence of solitary waves of (2.1) for $\sigma \neq 0$.

Theorem 2.7. *For $\sigma \neq 0$, we have*

(1) *If $0 < \sigma \leq 1$, a solitary wave (φ, η) of (2.1) exists if and only if condition (2.15) holds.*
If $c > A_1$ then $\varphi > 0$ and $\max_{x\in\mathbb{R}} \varphi(x) = c - A_1$. If $c < A_2$ then $\varphi < 0$ and $\min_{x\in\mathbb{R}} \varphi(x) = c - A_2$.

(2) *If $\sigma < 0$, then*

- *if $c > A_1$ then there is a smooth solitary wave $\varphi > 0$ with $\max_{x\in\mathbb{R}} \varphi(x) = c - A_1$, and an anticusped solitary wave (the solution profile has a cusp pointing downward) $\varphi < 0$ with $\min_{x\in\mathbb{R}} \varphi(x) = c/\sigma$;*
- *if $c < A_2$ then there is a smooth solitary wave $\varphi < 0$ with $\min_{x\in\mathbb{R}} \varphi(x) = c - A_2$, and a cusped solitary wave $\varphi > 0$ with $\max_{x\in\mathbb{R}} \varphi(x) = c/\sigma$;*
- *if $c = A_1$ then there is an anticusped solitary wave $\varphi < 0$ with $\min_{x\in\mathbb{R}} \varphi(x) = c/\sigma$;*
- *if $c = A_2$ then there is a cusped solitary wave $\varphi > 0$ with $\max_{x\in\mathbb{R}} \varphi(x) = c/\sigma$.*

(3) *If $\sigma > 1$, a solitary wave exists if and only if c satisfies (2.15). If $c > A_1$ then $\varphi > 0$. If $c < A_2$ then $\varphi < 0$. Moreover,*

- *If $A_1 < c < \frac{\sigma}{\sigma-1}A_1$, then the solitary waves are smooth and unique up to translation with $\max_{x\in\mathbb{R}} \varphi(x) = c - A_1$;*
- *If $c = \frac{\sigma}{\sigma-1}A_1$ then the solitary wave is peaked with $\max_{x\in\mathbb{R}} \varphi(x) = c - A_1 = c/\sigma$;*
- *If $c > \frac{\sigma}{\sigma-1}A_1$ then the solitary waves are cusped with $\max_{x\in\mathbb{R}} \varphi(x) = c/\sigma$;*
- *If $\frac{\sigma}{\sigma-1}A_2 < c < A_2$ then the solitary waves are smooth and unique up to translation with $\min_{x\in\mathbb{R}} \varphi(x) = c - A_2$;*
- *If $c = \frac{\sigma}{\sigma-1}A_2$ then the solitary wave is antipeaked (the solution profile has a peak pointing downward) with $\min_{x\in\mathbb{R}} \varphi(x) = c - A_2 = c/\sigma$;*
- *If $c < \frac{\sigma}{\sigma-1}A_2$ then the solitary waves are anticusped with $\min_{x\in\mathbb{R}} \varphi(x) = c/\sigma$.*

Moreover, each kind of the above solitary waves is unique and even up to translations. When $c > A_1$ or $c < A_2$, all solitary waves decay exponentially to zero at infinity.

Proof. First from (2.23) and the decay of $\varphi(x)$ at infinity we know that a necessary condition for the existence of solitary wave is that $c \geq A_1$ or $c \leq A_2$.

If $c = A_1$ then (2.23) becomes

$$\varphi_x^2 = \frac{-\varphi^3(A_1 - A_2 - \varphi)}{(A_1 - \varphi)(A_1 - \sigma\varphi)} := F_1(\varphi). \tag{2.28}$$

Hence we see that $\varphi(x) < 0$ near $-\infty$. Similarly as in the proof of Theorem 2.5 , we can find some x_0 sufficiently large negative with $\varphi(x_0) = -\epsilon < 0$ and $\varphi_x(x_0) < 0$, and we can construct a unique local solution $\varphi(x)$ on $[x_0 - L, x_0 + L]$ for some $L > 0$.

If $\sigma < 0$, we see that $\frac{1}{A_1 - \sigma\varphi}$ is decreasing when $\varphi < 0$. Together with (2.17) we see that $F_1(\varphi)$ decreases for $\varphi < 0$. Because $\varphi_x(x_0) < 0$, φ decreases near x_0, so $F_1(\varphi)$ increases near x_0. Hence from (2.28), φ_x decreases near x_0, and then φ and φ_x both decreases on $[x_0 - L, x_0 + L]$. Since $\sqrt{F_1(\varphi)}$ is locally Lipschitz in φ for $A_1/\sigma < \varphi \leq 0$, we can easily continue the local solution to $(-\infty, x_0 - L]$ with $\varphi(x) \to 0$ as $x \to -\infty$. As for $x \geq x_0 + L$, we can solve the initial valued problem

$$\begin{cases} \psi_x = -\sqrt{F_1(\psi)}, \\ \psi(x_0 + L) = \varphi(x_0 + L) \end{cases}$$

all the way until $\psi = A_1/\sigma$, which is a simple pole of $F_1(\psi)$. From (2.26) and (2.27) we know that we can construct an anticusped solution with a cusp singularity at $\varphi = A_1/\sigma = c/\sigma$.

If $\sigma > 0$, a direct computation shows that

$$F_1'(\varphi) < 0, \quad \text{for } \varphi < 0.$$

Therefore the same argument indicates that $\varphi(x) \to -\infty$ as $x \to +\infty$, which fails to be in $H^1(\mathbb{R})$. Hence in this case there is no solitary wave.

Similarly we conclude that when $c = A_2$, there is no solitary wave when $\sigma > 0$. When $\sigma < 0$, there is a solitary wave with a cusp of height c/σ.

Now we consider $c > A_1$ or $c < A_2$. Again we will only look at $c > A_1$. The other case $c < A_2$ can be handled in a very similar way. From (2.23) we see that φ cannot oscillate around zero near infinity. Let us consider the following two cases.

Case 1. $\varphi(x) > 0$ near $-\infty$. Then there is some x_0 sufficiently large negative so that $\varphi(x_0) = \epsilon > 0$, with ϵ sufficiently small, and $\varphi_x(x_0) > 0$.

(i) When $\sigma \leq 1$, $\sqrt{F(\varphi)}$ is locally Lipschitz in φ for $0 \leq \varphi \leq c - A_1$. Hence there is a local solution to

$$\begin{cases} \varphi_x = \sqrt{F(\varphi)}, \\ \varphi(x_0) = \epsilon \end{cases}$$

on $[x_0 - L, x_0 + L]$ for some $L > 0$. Therefore from (2.24) and (2.25) we see that in this case we can obtain a smooth solitary wave with maximum height $\varphi = c - A_1$ and an exponential decay to zero at infinity

$$\varphi(x) = O\left(\exp\left(-\frac{\sqrt{c^2 + Ac - 1}}{c}|x|\right)\right) \quad \text{as } |x| \to \infty. \tag{2.29}$$

(ii) When $\sigma > 1$, $\sqrt{F(\varphi)}$ is locally Lipschitz in φ for $0 \le \varphi < c/\sigma$. Thus if $c - A_1 < c/\sigma$, i.e., $c < \frac{\sigma}{\sigma - 1}A_1$, it becomes the same as (i) and hence we obtain smooth solitary waves with exponential decay.

If $c - A_1 = c/\sigma$ then the smooth solution can be constructed until $\varphi = c - A_1 = c/\sigma$. However at $\varphi = c - A_1 = c/\sigma$ it can make a sudden turn and so give rise to a peak. Since $\varphi = 0$ is still a double zero of $F(\varphi)$, we still have the exponential decay here.

Lastly if $c - A_1 > c/\sigma$, then $\varphi = c/\sigma$ becomes a pole of $F(\varphi)$. Hence from (2.26) and (2.27) we see that we obtain a solitary wave with a cusp at $\varphi = c/\sigma$ and decays exponentially.

Case 2. $\varphi(x) < 0$ near $-\infty$. In this case we are solving

$$\begin{cases} \varphi_x = -\sqrt{F(\varphi)}, \\ \varphi(x_0) = -\epsilon \end{cases}$$

for some x_0 sufficiently large negative and $\epsilon > 0$ sufficiently small.

When $\sigma > 0$ we see that $F'(\varphi) < 0$ for $\varphi < 0$. Thus in this case there is no solitary wave.

When $\sigma < 0$, $\varphi = c/\sigma < 0$ is a pole of $F(\varphi)$. Hence from similar argument as before, we obtain an anticusped solitary wave with $\min_{x \in \mathbb{R}} = c/\sigma$, which decays exponentially.

Finally, from the standard ODE theory and the fact that the equation (2.11) is invariant under the transformations $x \mapsto x + d$ for any constant d, and $x \mapsto -x$, we conclude that the solitary waves obtained above are unique and even up to translations. □

Though there is no explicit expression for φ, and so η in view of (2.10), as in [50], the effects of the traveling speed c on the function φ can be analyzed to provide some general description of its profile. Similarly to the case in [50] we have

Proposition 2.8. *Let $c > A_1$ or $c < A_2$, and φ is a smooth solitary wave of* (2.1) *as obtained in Theorem* 2.7. *Then $\partial_c \varphi$ decays exponentially to zero at infinity and has at most two zeros on* $\mathbb{R}$. *In particular, if $A_1 < c < \frac{2}{A}$, the $\partial_c \varphi$ has exactly two zeros on* $\mathbb{R}$.

Proof. Again we only discuss the case $c > A_1$. The other case $c < A_2$ can be handled in the same way.

Denote $\omega = \partial_c \varphi$. The exponential decay of ω can be inferred from (2.29). Since φ is unique and even up to translations, we may assume that $\varphi(0) = c - A_1$. Hence $\omega(0) = 1$ and ω is even. Assume $\omega(x_0) = 0$ for some $x_0 > 0$. Differentiating (2.23)

with respect to c and evaluating at $x = x_0$ we get

$$2\varphi_x\omega_x = \frac{\varphi^2}{c-\sigma\varphi}\left[1 + \frac{1}{(c-\varphi)^2} + \frac{(c-\varphi)^2 + A(c-\varphi) - 1}{(c-\varphi)(c-\sigma\varphi)}\right]$$
$$= \frac{\varphi^2}{c-\sigma\varphi}\left[1 + \frac{1}{(c-\varphi)^2} + \frac{\varphi_x^2}{\varphi^2}\right] > 0,$$

since $c - \sigma\varphi > 0$. Because $\varphi_x(x_0) < 0$, we see from the above inequality that $\omega_x(x_0) < 0$. So ω is strictly decreasing near x_0. It is then deduced from the continuity of ω that it has at most two zeros on $\mathbb{R}$.

If $A_1 < c < \frac{2}{A}$, then from the decay estimate (2.29) we see that φ decays faster at infinity as c gets larger, since

$$\partial_c\left(\frac{\sqrt{c^2 + Ac - 1}}{c}\right) = \frac{2 - Ac}{2c^2\sqrt{c^2 + Ac - 1}} > 0.$$

Hence $\omega(x) < 0$ at infinity. Therefore ω has at least two zeros. Thus combining the above argument we proved that $\omega(x)$ has exactly two zeros $\pm x_0$ in this case. □

Next we try to find an implicit formula for the peaked solitary waves. Let us consider only the case $c > A_1$. Then from Theorem 2.7 we know that peaked solitary waves exist only when $c = \frac{\sigma}{\sigma-1}A_1$. In this case we have

$$\varphi_x^2 = \frac{\varphi^2(c - A_2 - \varphi)}{c - \varphi}.$$

Since φ is positive, even with respect to some x_0 and decreasing on (x_0, ∞), so for $x > x_0$ we have

$$\varphi_x = -\varphi\sqrt{1 - \frac{A_2}{c-\varphi}}.$$

Hence from the separation of variables we get

$$-(x - x_0) = \int_{c-A_1}^{\varphi} \frac{dt}{t\sqrt{1 - \frac{A_2}{c-t}}}.$$

Letting $w = 1 - \frac{A_2}{c-t}$ and the above becomes

$$-(x - x_0) = \int_{1-\frac{A_2}{A_1}}^{1-\frac{A_2}{c-\varphi}} \frac{-A_2}{[cw - (c - A_2)](w-1)\sqrt{w}}dw$$
$$= \int_{1-\frac{A_2}{A_1}}^{1-\frac{A_2}{c-\varphi}} \frac{1}{\sqrt{w}}\left[\frac{c}{cw - (c - A_2)} - \frac{1}{w-1}\right]dw$$
$$= \left(\sqrt{\frac{c}{c - A_2}}\ln\left|\frac{\sqrt{cw} - \sqrt{c - A_2}}{\sqrt{cw} + \sqrt{c - A_2}}\right| - \ln\left|\frac{\sqrt{w} - 1}{\sqrt{w} + 1}\right|\right)\Bigg|_{1-\frac{A_2}{A_1}}^{1-\frac{A_2}{c-\varphi}}.$$

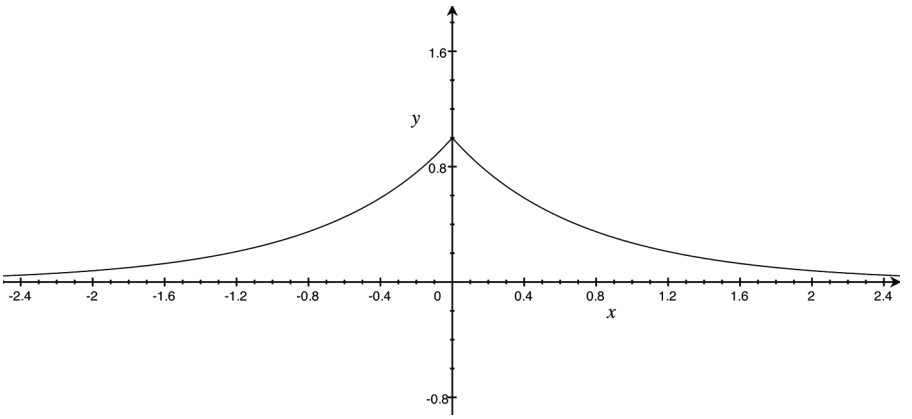

Figure 1. A peaked solitary wave with $x_0 = 0$, $A = 0$, and $c = \sigma = 2$.

Therefore we obtain an implicit formula for the peaked solitary waves.

$$-|x - x_0| = \left(\sqrt{\frac{c}{c - A_2}} \ln \left| \frac{\sqrt{cw} - \sqrt{c - A_2}}{\sqrt{cw} + \sqrt{c - A_2}} \right| - \ln \left| \frac{\sqrt{w} - 1}{\sqrt{w} + 1} \right| \right) \Bigg|_{w=1-\frac{A_2}{A_1}}^{1-\frac{A_2}{c-\varphi}}. \tag{2.30}$$

Figure 1 shows such.

3. Stability

In this section, we want to discuss the stability of the smooth solitary waves of (2.1). For fixed c and $\epsilon > 0$, we define the "ϵ-tube" of a solitary wave $\vec{\varphi}_c$ to be

$$U_\epsilon = \left\{ \vec{u} \in X : \inf_{s \in \mathbb{R}} \|\vec{u} - \vec{\varphi}_c(\cdot - s)\|_X < \epsilon \right\}. \tag{3.1}$$

According to Theorem 2.7, the solitary waves for (2.1) travel with speeds proportional to their maximal heights. This consideration suggests that the appropriate notion of stability for the solitary waves is orbital stability: a wave starting close to a solitary wave should stay close, as long as it exists, to some translate of the solitary wave. The orbit of a solitary wave is the set of all its translates.

Let us now discuss the appropriate notion of stability for the solitary waves of (2.1).

Definition 3.1. The solitary wave $\vec{\varphi}_c$ of (2.1) is stable in X if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for any $\vec{u}_0 \in U_\delta$, if $\vec{u} \in C([0, T); X)$ for some $0 < T \leq \infty$ is a solution to (2.1) with $\vec{u}(0) = \vec{u}_0$, then $\vec{u}(t) \in U_\epsilon$ for all $t \in [0, T)$. Otherwise the solitary wave $\vec{\varphi}_c$ is said to be unstable in X.

As is discussed in [4], some solutions of (2.1) are defined globally in time (e.g. for $0 < \sigma < 2$ and $\inf_{x \in \mathbb{R}} \eta_0 > -1$, or the solitary waves constructed in Section 2) while other waves break in finite time. Note that by stability we mean that even if

a solution which is initially close to a solitary wave blows up in a finite time, it will stay close to some translate of the solitary wave up to the breaking time.

Our main theorem in this section is the following.

Theorem 3.2. *Let $\sigma \leq 1$. All smooth solitary waves of* (2.1) *are stable.*

First from Theorem 2.7 we know that smooth solitary waves exist only when $c > A_1$ or $c < A_2$. For convenience we assume $c > A_1$.

The special case $\sigma = 1$ is settled in [50]. We will show that for general $\sigma < 1$, the problem can be analyzed using the method provided by Grillakis et al. [26].

Let us now make some functional analysis setup. Recalling the functionals E and F are well-defined on X, we may compute their Fréchet derivatives as follows

$$\begin{cases} E'_u = -u_{xx} + u \\ E'_\eta = \eta, \end{cases} \qquad \begin{cases} F'_u = \dfrac{3}{2}u^2 - \dfrac{\sigma}{2}u_x^2 - \sigma u u_{xx} + \eta + \dfrac{1}{2}\eta^2 - Au \\ F'_\eta = u + u\eta. \end{cases}$$

Using these notation we see that a solitary wave $\vec{\varphi}_c$ of (2.1) satisfies

$$cE'(\vec{\varphi}_c) - F'(\vec{\varphi}_c) = 0. \tag{3.2}$$

Denote

$$L_c = -\partial_x\left((c - \sigma\varphi)\partial_x\right) - 3\varphi + \sigma\varphi_{xx} + c + A.$$

Then the linearized operator $H_c : X \to X^*$ of $cE' - F'$ at $\vec{\varphi}_c$ can be computed as

$$H_c = cE''(\vec{\varphi}_c) - F''(\vec{\varphi}_c) = \begin{pmatrix} L_c & -(1+\eta) \\ -(1+\eta) & c - \varphi \end{pmatrix}.$$

Using (2.10) we have

$$H_c = \begin{pmatrix} L_c & -\dfrac{c}{c-\varphi} \\ -\dfrac{c}{c-\varphi} & c - \varphi \end{pmatrix}. \tag{3.3}$$

We see easily that H_c is self-adjoint and bounded from below, i.e., $H_c \geq aI$ for some constant a and I is the identity operator.

The next lemma states some spectral properties about H_c.

Lemma 3.3. *Let $c > A_1$ and $\vec{\varphi}_c$ be a smooth solitary wave of* (2.1). *The spectrum of H_c satisfies the following properties.*

(1) *The essential spectrum of H_c is positive and bounded away from zero.*
(2) *The kernel of H_c is spanned by $\partial_x\vec{\varphi}_c$.*
(3) *H_c has exactly one negative simple eigenvalue λ_1 corresponding to eigenfunction $\vec{\chi} = (\chi, \mu)$.*

Proof. The proof is inspired by Lemma 3.2 in [50]. The details are as follows.

(1) Since φ, φ_x and φ_{xx} all decay exponentially at infinity, it follows from Weyl's essential spectrum theorem that the essential spectrum of H_c is the same as that of its asymptotic operator O_∞ as $|x| \to \infty$, where

$$O_\infty = \begin{pmatrix} -c\partial_{xx} + c + A & -1 \\ -1 & c \end{pmatrix}.$$

When $c > A_1$, we have $c^2 + Ac - 1 > 0$. Hence there is some constant $\delta = \delta(c, A) \in (0, 1)$ such that

$$\begin{aligned} 2|\psi\omega| &\le 2(1-\delta)\sqrt{c(c+A)}|\psi\omega| \\ &\le (1-\delta)\left[(c+A)\psi^2 + c\omega^2\right] \end{aligned}$$

for any $\vec{\psi} = (\psi, \omega) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$. Therefore

$$\begin{aligned} (O_\infty \vec{\psi}, \vec{\psi}^T)_{L^2\times L^2} &= \int_{\mathbb{R}} \left[c\psi_x^2 + (c+A)\psi^2 - 2\psi\omega + c\omega^2\right] dx \\ &\ge \int_{\mathbb{R}} \left[\delta(c+A)\psi^2 + \delta c\omega^2\right] dx \ge \delta c \|\vec{\psi}\|^2_{L^2\times L^2}. \end{aligned}$$

Hence O_∞ is positive when $c > A_1$, and then the essential spectrum of H_c is $[a_0, \infty)$ for some $a_0 > 0$ and there are finitely many eigenvalues located to the left of a_0.

(2) If $\vec{\psi} = (\psi, \omega)$ is an eigenfunction of H_c corresponding to the eigenvalue zero, then

$$\begin{aligned} -\partial_x\left((c-\sigma\varphi)\psi_x\right) + (-3\varphi + \sigma\varphi_{xx} + c + A)\psi - \frac{c}{c-\varphi}\omega = 0, \\ -\frac{c}{c-\varphi}\psi + (c-\varphi)\omega = 0. \end{aligned}$$

From the second equation we get $\omega = \frac{c\psi}{(c-\varphi)^2}$. Hence the first equation can be expressed as a zero eigenvalue problem for $\mathscr{K}_c : H^1 \to H^{-1}$:

$$\mathscr{K}_c\psi := -\partial_x\left((c-\sigma\varphi)\psi_x\right) + \left(-3\varphi + \sigma\varphi_{xx} + c + A - \frac{c^2}{(c-\psi)^3}\right)\psi = 0. \tag{3.4}$$

We now use the fact that $\varphi(x), \varphi_x(x), \varphi_{xx}(x) \to 0$ exponentially fast as $|x| \to \infty$ while $c - \sigma\varphi$ is positive and bounded away from zero when φ is smooth. Similar to [16], it follows that the spectral equation $\mathscr{K}_c\psi = 0$ can be transformed by the Liouville substitution

$$z = \int_0^x \frac{dy}{\sqrt{c - \sigma\varphi(y)}}, \quad \theta(z) = (c - \sigma\varphi(x))^{1/4}\psi(x),$$

into

$$\mathscr{L}_c\theta(z) := \left(-\partial_z^2 + q_c(z) + c + A - \frac{1}{c}\right)\theta(z) = 0,$$

where

$$q_c(z) = -3\varphi(x) + \frac{3\sigma}{4}\varphi_{xx}(x) - \frac{c^2}{(c-\varphi(x))^3} + \frac{1}{c} - \frac{\sigma^2\varphi_x^2(x)}{8(c-\sigma\varphi(x))}.$$

Since $q_c \to 0$ exponentially as $|z| \to \infty$, we deduce that $\mathscr{L}_c : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$ is self-adjoint with essential spectrum $[c + A - \frac{1}{c}, \infty)$. Because $c > A_1$, we know $c + A - \frac{1}{c} > 0$. We may have finitely many eigenvalues of $\mathscr{L}_c$ located to the left of $c + A - \frac{1}{c}$. The nth eigenvalue (in increasing order) has, up to a constant multiple, a unique eigenfunction with precisely $(n-1)$ zeros (see for example, [23] for details). Thus the operator $\mathscr{K}_c$ has the same spectral properties.

Note that $\eta_x = \frac{c\varphi_x}{(c-\varphi)^2}$ and (2.7) imply that $\mathscr{K}_c(\varphi_x) = 0$. Since φ_x has exactly one zero. Therefore the zero eigenvalue of $\mathscr{K}_c$ is simple, and there is exactly one negative eigenvalue while the rest of the spectrum is positive and bounded away from zero. Hence the zero eigenvalue of H_c is simple and the kernel is spanned by $\vec{\varphi}_x$.

(3) The operator H_c is related to a quadratic form $Q_c(\vec{\psi})$ with $\vec{\psi} = (\psi, \omega) \in X$, which is defined as the coefficient of ϵ^2 in the Taylor's expansion of $cE(\vec{\varphi}_c + \epsilon\vec{\psi}) - F(\vec{\varphi}_c + \epsilon\vec{\psi})$ and is given by

$$\begin{aligned} Q_c(\vec{\psi}) &= \frac{1}{2}\int_{\mathbb{R}} \left[(c-\sigma\varphi)\psi_x^2 + (-3\varphi + \sigma\varphi_{xx} + c + A)\psi^2 - \frac{2c}{c-\varphi}\psi\omega + (c-\varphi)\omega^2 \right] dx \\ &= \frac{1}{2}\int_{\mathbb{R}} \left[(c-\sigma\varphi)\psi_x^2 + \left(-3\varphi + \sigma\varphi_{xx} + c + A - \frac{c^2}{(c-\varphi)^3} \right)\psi^2 \right] dx \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} \left[(c-\varphi)\left(\frac{c}{(c-\varphi)^2}\psi - \omega \right)^2 \right] dx \\ &:= Q_c^{(1)}(\psi) + G(\vec{\psi}). \end{aligned}$$

Note that the quadratic form $Q_c^{(1)}(\psi)$ is related to the operator $\mathscr{K}_c$ and $G(\vec{\psi})$ is nonnegative.

Let f be a nontrivial eigenfunction corresponding to the unique negative eigenvalue of $\mathscr{K}_c$. Then

$$Q_c\left(f, \frac{c}{(c-\varphi)^2} f \right) = Q_c^{(1)}(f) < 0.$$

So H_c has a negative eigenvalue, say, $\lambda_1 < 0$. Applying the min-max characterization of eigenvalues to H_c yields

$$\lambda_2 = \max_{\vec{\psi} \in X} \min_{\substack{\vec{\omega} \in X \setminus \{0\} \\ (\vec{\psi}, \vec{\omega}) = 0}} \frac{Q_c(\vec{\omega})}{\|\omega\|_X^2}.$$

Choosing $\vec{\psi} = (f, 0)$ leads to

$$\lambda_2 \geq \min_{\substack{\vec{\omega} \in X\setminus\{0\} \\ ((f,0),\vec{\omega})=0}} \frac{Q_c(\vec{\omega})}{\|\omega\|_X^2} = \min_{\substack{(g,h) \in X\setminus\{0\} \\ ((f,0),(g,h))=0}} \frac{Q_c^{(1)}(g) + G(g,h)}{\|(g,h)\|_X^2} \geq 0.$$

The last inequality is due to $Q_c^{(1)}(g) \geq 0$ for all g such that $(g, f)_{H^1} = 0$ and that $G(g, h) \geq 0$. Therefore λ_1 is simple. Denote the corresponding eigenfunction by $\vec{\chi} = (\chi, \mu)$. From the result in (2) we see that $\lambda_2 = 0$ is also simple. This completes the proof of the lemma. □

Remark 3.4. (i) Notice that the above lemma applies to all smooth solitary waves of (2.1) without the restriction that $\sigma \leq 1$.

Under the assumption $c < A_2$, we can consider the operator

$$H_c = -\begin{pmatrix} L_c & -\dfrac{c}{c-\varphi} \\ -\dfrac{c}{c-\varphi} & c-\varphi \end{pmatrix} = \begin{pmatrix} -L_c & \dfrac{c}{c-\varphi} \\ \dfrac{c}{c-\varphi} & -c+\varphi \end{pmatrix}.$$

In this setting we have for smooth solitary wave that $c < c - A_2 \leq \varphi(x) < 0$ and $c - \sigma\varphi$ is bounded away from zero. By a similar argument, all properties of H_c in Lemma 3.3 are still valid.

(ii) We will apply the method of Grillakis et al. [26] to establish the stability of smooth solitary waves. However our problem does not exactly fall into the framework there since the operator J is not onto. But in fact the invertibility of J is only needed to get instability and is not required for stability (see Sections 3 and 4 in [26] for more details). Hence the argument in Section 3 of [26] can still be used here.

Let $\vec{\varphi}_c = (\varphi, \eta)$ be a solitary wave of (2.1). Consider the following scalar function

$$d(c) = \begin{cases} cE(\vec{\varphi}_c) - F(\vec{\varphi}_c) & \text{if } c > A_1, \\ F(\vec{\varphi}_c) - cE(\vec{\varphi}_c) & \text{if } c < A_2. \end{cases} \tag{3.5}$$

The next lemma shows that for $\sigma \leq 1$, $c > A_1$ or $c < A_2$ and $\vec{\varphi}_c = (\varphi, \eta) \in X$ being a smooth solitary wave of (2.1), $d(c)$ is convex in c.

Lemma 3.5. *Assume $\sigma \leq 1$, $c > A_1$ or $c < A_2$ and $\vec{\varphi}_c = (\varphi, \eta)$ is a smooth solitary wave of* (2.1). *Then $d''(c) > 0$.*

Proof. Consider first $c > A_1$. Differentiating $d(c)$ with respect to c and then applying (3.2) we obtain

$$d'(c) = \langle cE'(\vec{\varphi}_c) - F'(\vec{\varphi}_c), \partial_c \vec{\varphi}_c \rangle + E(\vec{\varphi}_c) = E(\vec{\varphi}_c). \tag{3.6}$$

In view of the even symmetry of φ, it follows from (2.10) and (2.23) that

$$d'(c) = E(\vec{\varphi}_c) = \int_0^\infty \left[\varphi_x^2 + \varphi^2 + \frac{\varphi^2}{(c-\varphi)^2}\right] dx$$
$$= \int_0^\infty \varphi^2 \left[\frac{(c-\varphi-A_1)(c-\varphi-A_2)}{(c-\varphi)(c-\sigma\varphi)} + 1 + \frac{1}{(c-\varphi)^2}\right] dx.$$

Recall that $0 < \varphi(x) \le c - A_1$ and $\varphi'(x) < 0$ on $(0, \infty)$ when $c > A_1$. We have from (2.23) that

$$d'(c) = -\int_0^\infty \varphi\varphi_x \sqrt{\frac{(c-\varphi)(c-\sigma\varphi)}{(c-\varphi-A_1)(c-\varphi-A_2)}} \cdot$$
$$\times \left[\frac{(c-\varphi-A_1)(c-\varphi-A_2)}{(c-\varphi)(c-\sigma\varphi)} + 1 + \frac{1}{(c-\varphi)^2}\right] dx. \tag{3.7}$$

Introducing a change of variable $y = c - \varphi(x)$ the above becomes

$$d'(c) = \int_{A_1}^{c} (c-y) \sqrt{\frac{y\,[(1-\sigma)c + \sigma y]}{(y-A_1)(y-A_2)}} \left[\frac{(y-A_1)(y-A_2)}{y\,[(1-\sigma)c+\sigma y]} + 1 + \frac{1}{y^2}\right] dx.$$

Differentiating the above with respect to c we have

$$d''(c) = 0 + \int_{A_1}^{c} \partial_c \left\{(c-y)\sqrt{\frac{(y-A_1)(y-A_2)}{y\,[(1-\sigma)c+\sigma y]}}\right\} dy$$
$$+ \int_{A_1}^{c} \partial_c \left\{(c-y)\sqrt{\frac{y\,[(1-\sigma)c+\sigma y]}{(y-A_1)(y-A_2)}}\left(1 + \frac{1}{y^2}\right)\right\} dy$$
$$:= \int_{A_1}^{C} I_1(y)dy + \int_{A_1}^{C} I_2(y)dy.$$

Since φ is smooth, we have that $c - \sigma\varphi > 0$. Hence $(1-\sigma)c + \sigma y = c - \sigma\varphi > 0$. Moreover, we know that $A_2 < 0 < A_1 < y < c$. Further explicit computation shows that

$$I_1(y) = \sqrt{\frac{(y-A_1)(y-A_2)}{y\,[(1-\sigma)c+\sigma y]}} \cdot \left(\frac{[(1-\sigma)c+\sigma y] + y}{2[(1-\sigma)c+\sigma y]}\right) > 0.$$

Here we don't need to assume $\sigma \le 1$. If $\sigma \le 1$, we have

$$I_2(y) = \frac{(y^2+1)\left[(1-\sigma)c + \sigma y + \frac{1}{2}(1-\sigma)(c-y)\right]}{y\sqrt{(y-A_1)(y-A_2)y[(1-\sigma)c+\sigma y]}} > 0.$$

Therefore $d''(c) > 0$.

The other case $c < A_2$ can be handled in a very similar way and hence we omit it. □

The next lemma can be obtained by exactly the same proof as in [26].

Lemma 3.6. *Let $\sigma \leq 1$ and $\vec{\varphi}_c = (\varphi, \eta)$ be a solitary wave of* (2.1). *There exist $\epsilon > 0$ and a unique C^1 map $s : U_\epsilon \to \mathbb{R}$ such that for every $\vec{u} \in U_\epsilon$ and $r \in \mathbb{R}$*

(1)

$$(\vec{u}(\cdot + s(\vec{u})), \vec{\varphi}_x) = 0,$$

(2)

$$s(\vec{u}(\cdot + r)) = s(\vec{u}) - r.$$

By the spectrum analysis in Lemma 3.3 and the convexity of $d(c)$ in Lemma 3.5, we can follow exactly the same idea as in [26] Theorem 3.3 to get

Lemma 3.7. *Let the assumptions of Lemma* 3.5 *hold. There exists a constant $k = k(c) > 0$ such that*

$$\langle H_c(\vec{\psi}), \vec{\psi} \rangle \geq k \|\vec{\psi}\|_X^2, \tag{3.8}$$

for all $\vec{\psi} \in X$ satisfying $(\vec{\varphi}_c, \vec{\psi}) = (\vec{\varphi}_c', \vec{\psi}) = 0$.

The following lemma can be obtained directly from Lemma 3.6 and Lemma 3.7.

Lemma 3.8. *Let the assumptions of Lemma* 3.5 *hold. There exists an $\epsilon > 0$ such that*

$$F(\vec{\varphi}_c) - F(\vec{u}) \geq \frac{k}{4} \|\vec{u}(\cdot + s(\vec{u})) - \vec{\varphi}_c\|_X^2,$$

for $\vec{u} \in U_\epsilon$ satisfying $E(\vec{u}) = E(\vec{\varphi}_c)$.

Proof of Theorem 3.2. In view of Lemma 3.3 and Lemma 3.8, the result of theorem is then a direct consequence of Theorem 3.5 in [26]. □

4. Global Solutions When $\sigma = 0$

In [4], the authors established a blow-up criterion for $\sigma \neq 0$ (cf. Theorem 3.3 in [4]). In fact, the restriction of $\sigma \neq 0$ can be removed using the same argument and hence we get

Theorem 4.1. *Let (u, ρ) be the solution of* (1.2) *with initial data $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > 3/2$, and T the maximal time of existence. Then*

$$T < \infty \Rightarrow \int_0^T \|u_x(\tau)\|_{L^\infty} d\tau = \infty. \tag{4.1}$$

The wave-breaking phenomena for system (1.2) when $\sigma \neq 0$ was discussed in details in [4]. Here we show that when $\sigma = 0$ the solutions constructed in Theorem 2.2 are global-in-time.

Theorem 4.2. *Let $\sigma = 0$. If $(u_0, \rho_0 - 1) \in H^s \times H^{s-1}$, $s > 3/2$, then there exists a unique solution $(u, \rho - 1)$ of* (1.2) *in $C([0, \infty); H^s \times H^{s-1}) \cap C^1([0, \infty); H^{s-1} \times H^{s-2})$*

with $(u, \rho)|_{t=0} = (u_0, \rho_0)$. *Moreover, the solution depends continuously on the initial data and the Hamiltonian* H_1 *is independent of the existence time.*

As discussed in [4], system (1.2) has two associated characteristics q and $\tilde{q}$ given by the following initial-value problems

$$\begin{cases} \dfrac{\partial q}{\partial t} = u(t, q), & 0 < t < T, \\ q(0, x) = x, & x \in \mathbb{R}, \end{cases} \tag{4.2}$$

$$\begin{cases} \dfrac{\partial \tilde{q}}{\partial t} = \sigma u(t, \tilde{q}), & 0 < t < T, \\ \tilde{q}(0, x) = x, & x \in \mathbb{R}, \end{cases} \tag{4.3}$$

where $u \in C^1([0, T), H^{s-1})$ is the first component of the solution (u, ρ) to (1.2) with initial data $(u_0, \rho_0) \in H^s \times H^{s-1}$ with $s > 3/2$ and $T > 0$ is the maximal time of existence. When $\sigma = 0$, the second one $\tilde{q}$ becomes stationary. Thus we will perform the estimates along the first characteristics q.

A direct calculation shows that for $t > 0$, $x \in \mathbb{R}$

$$q_x(t, x) = e^{\int_0^t u_x(\tau, q(\tau, x))d\tau} > 0.$$

Hence $q(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism of the line for each $t \in [0, T)$. Hence the L^∞ norm of any function $v(t, \cdot) \in L^\infty(\mathbb{R})$, $t \in [0, T)$ is preserved under $q(t, \cdot)$ with $t \in [0, T)$, i.e.,

$$\|v(t, \cdot)\|_{L^\infty(\mathbb{R})} = \|v(t, q(t, \cdot))\|_{L^\infty(\mathbb{R})}, \quad t \in [0, T). \tag{4.4}$$

Similarly we have

$$\inf_{x \in \mathbb{R}} v(t, x) = \inf_{x \in \mathbb{R}} v(t, q(t, x)), \quad t \in [0, T), \tag{4.5}$$

$$\sup_{x \in \mathbb{R}} v(t, x) = \sup_{x \in \mathbb{R}} v(t, q(t, x)), \quad t \in [0, T). \tag{4.6}$$

When $\sigma = 0$, we can rewrite system (1.2) as

$$\begin{cases} u_t + \partial_x p * \left(-Au + \dfrac{3}{2}u^2 + \dfrac{1}{2}\rho^2 \right) = 0, \\ \rho_t + (\rho u)_x = 0, \end{cases} \tag{4.7}$$

where $p(x)$ is defined in (2.5)

The following lemma is needed in carrying out the estimates along the "extremal" characteristics.

Lemma 4.3 ([9]). *Let* $T > 0$ *and* $v \in C^1\left([0, T); H^2(\mathbb{R})\right)$. *Then for every* $t \in [0, T)$ *there exists at least one point* $\xi(t) \in \mathbb{R}$ *with*

$$m(t) := \inf_{x \in \mathbb{R}} [v_x(t, x)] = v_x(t, \xi(t)).$$

The function $m(t)$ is absolutely continuous on $(0, T)$ with

$$\frac{dm(t)}{dt} = v_{tx}(t, \xi(t)) \quad \text{a.e. on } (0, T).$$

To prove Theorem 4.2 of global well-posedness of solutions, we need the following estimates for u_x.

Lemma 4.4. *Let $\sigma = 0$ and (u, ρ) be the solution of (4.7) with initial data $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > 3/2$, and T the maximal time of existence. Then*

$$\sup_{x\in\mathbb{R}} u_x(t, x) \le \sup_{x\in\mathbb{R}} u_{0,x}(x) + \frac{1}{2}\left(\sup_{x\in\mathbb{R}} \rho_0^2(x) + C_1^2\right) t, \tag{4.8}$$

$$\inf_{x\in\mathbb{R}} u_x(t, x) \ge \inf_{x\in\mathbb{R}} u_{0,x}(x) + \frac{1}{2}\left(\inf_{x\in\mathbb{R}} \rho_0^2(x) - C_2^2\right) t, \tag{4.9}$$

where the constants above are defined as follows.

$$C_1 = \sqrt{\frac{3 + A^2}{2}} \|(u_0, \rho_0 - 1)\|_{H^1\times L^2}, \tag{4.10}$$

$$C_2 = \sqrt{2 + C_1^2}. \tag{4.11}$$

Proof. The local well-posedness theorem and a density argument implies that it suffices to prove the desired estimates for $s \ge 3$. Thus we take $s = 3$ in the proof. Also we may assume that

$$u_0 \not\equiv 0. \tag{4.12}$$

Otherwise the results become trivial. Since now $s \ge 3$, we have $u \in C_0^1(\mathbb{R})$. Therefore

$$\inf_{x\in\mathbb{R}} u_x(t, x) \le 0, \quad \sup_{x\in\mathbb{R}} u_x(t, x) \ge 0, \quad t \in [0, T). \tag{4.13}$$

Differentiating the first equation of (4.7) with respect to x and using the identity $-\partial_x^2 p * f = f - p * f$ we obtain

$$u_{tx} = \frac{1}{2}\rho^2 + \frac{3}{2}u^2 + A\partial_x^2 p * u - p * \left(\frac{3}{2}u^2 + \frac{1}{2}\rho^2\right). \tag{4.14}$$

Using Lemma 4.3 and the fact that

$$\sup_{x\in\mathbb{R}} [v_x(t, x)] = -\inf_{x\in\mathbb{R}} [-v_x(t, x)],$$

we can consider $\bar{m}(t)$ and $\bar{\xi}(t)$ as follows

$$\bar{m}(t) := u_x\left(t, \bar{\xi}(t)\right) = \sup_{x\in\mathbb{R}} (u_x(t, x)), \quad t \in [0, T). \tag{4.15}$$

Hence

$$u_{xx}\left(t, \bar{\xi}(t)\right) = 0, \quad \text{a.e.} \quad t \in [0, T). \tag{4.16}$$

Take the trajectory $q(t, x)$ defined in (4.2). Then we know that $q(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism for every $t \in [0, T)$. Therefore there exists $x_1(t) \in \mathbb{R}$ such that

$$q\left(t, x_1(t)\right) = \bar{\xi}(t) \quad t \in [0, T). \tag{4.17}$$

Now let

$$\bar{\zeta}(t) = \rho(t, q(t, x_1)), \quad t \in [0, T). \tag{4.18}$$

Therefore along this trajectory $q(t, x_1)$ equation (4.14) and the second equation of (4.7) become

$$\begin{aligned} \bar{m}'(t) &= \frac{1}{2}\bar{\zeta}^2 + f(t, q(t, x_1)), \\ \bar{\zeta}'(t) &= -\bar{\zeta}\bar{m}, \end{aligned} \tag{4.19}$$

for $t \in [0, T)$, where $'$ denotes the derivative with respect to t and $f(t, q(t, x))$ is given by

$$f = \frac{3}{2}u^2 + A\partial_x^2 p * u - p * \left(\frac{3}{2}u^2 + \frac{1}{2}\rho^2\right). \tag{4.20}$$

We first derive the upper and lower bounds for f for later use in getting the wave-breaking result. Using that $\partial_x^2 p * u = p_x * u_x$, we have

$$\begin{aligned} f &= \frac{3}{2}u^2 + Ap_x * u_x - \frac{3}{2}p * u^2 - \frac{1}{2}p * 1 - p * (\rho - 1) - \frac{1}{2}p * (\rho - 1)^2 \\ &\leq \frac{3}{2}u^2 + A|p_x * u_x| - \frac{1}{2} + |p * (\rho - 1)|. \end{aligned}$$

Since

$$A|p_x * u_x| \leq A\|p_x\|_{L^2}\|u_x\|_{L^2} = \frac{1}{2}A\|u_x\|_{L^2} \leq \frac{1}{4} + \frac{1}{4}A^2\|u_x\|_{L^2}^2, \tag{4.21}$$

$$|p * (\rho - 1)| \leq \|p\|_{L^2}\|\rho - 1\|_{L^2} = \frac{1}{2}\|\rho - 1\|_{L^2} \leq \frac{1}{4} + \frac{1}{4}\|\rho - 1\|_{L^2}^2, \tag{4.22}$$

$$u^2 \leq \frac{1}{2}\int_{\mathbb{R}} (u^2 + u_x^2)dx, \tag{4.23}$$

we obtain the upper bound of f

$$\begin{aligned} f &\leq \frac{1}{4}\|\rho - 1\|_{L^2}^2 + \frac{3}{4}\|u\|_{L^2}^2 + \frac{3 + A^2}{4}\|u_x\|_{L^2} \\ &\leq \frac{3 + A^2}{4}\|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}^2 = \frac{1}{2}C_1^2. \end{aligned} \tag{4.24}$$

Now we turn to the lower bound of f. Similar as before, we get

$$\begin{aligned} -f &\le A|p_x * u_x| + \frac{3}{2}p * u^2 + \frac{1}{2} + |p * (\rho - 1)| + \frac{1}{2}p * (\rho - 1)^2. \\ &\le 1 + \frac{A^2}{4}\|u_x\|_{L^2}^2 + \frac{3}{4}\|u\|_{L^2}^2 + \frac{1}{2}\|\rho - 1\|_{L^2}^2 \\ &\le 1 + \frac{3 + A^2}{4}\|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}^2 = \frac{1}{2}C_2^2, \end{aligned} \tag{4.25}$$

where we have used the inequality

$$p * g^2 \le \frac{1}{2}\|g^2\|_{L^1} = \frac{1}{2}\|g\|_{L^2}^2.$$

Combining (4.24) and (4.25) we obtain

$$|f| \le 1 + \frac{3 + A^2}{4}\|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}^2. \tag{4.26}$$

From (4.13) we know $\bar{m}(t) \ge 0$ for $t \in [0, T)$. From the second equation of (4.19) we obtain that

$$\bar{\zeta}(t) = \bar{\zeta}(0)e^{-\int_0^t \bar{m}(\tau)d\tau}. \tag{4.27}$$

Hence

$$|\rho(t, q(t, x_1))| = |\bar{\zeta}(t)| \le |\bar{\zeta}(0)|.$$

Therefore we have

$$\bar{m}'(t) = \frac{1}{2}\bar{\zeta}^2(t) + f \le \frac{1}{2}\bar{\zeta}^2(0) + \frac{1}{2}C_1^2 \le \frac{1}{2}\left(\sup_{x \in \mathbb{R}} \rho_0^2(x) + C_1^2\right).$$

Integrating the above from over $[0, t]$ we prove (4.8).

To obtain a lower bound for $\inf_{x \in \mathbb{R}} u_x(t, x)$, we use the similar idea. Consider the functions $m(t)$ and $\xi(t)$ as in Lemma 4.3

$$m(t) := u_x(t, \xi(t)) = \inf_{x \in \mathbb{R}} (u_x(t, x)), \quad t \in [0, T). \tag{4.28}$$

Hence

$$u_{xx}(t, \xi(t)) = 0 \quad \text{a.e. } t \in [0, T). \tag{4.29}$$

Again take the characteristics $q(t, x)$ defined in (4.2) and choose $x_2(t) \in \mathbb{R}$ such that

$$q(t, x_2(t)) = \xi(t) \quad t \in [0, T). \tag{4.30}$$

Let

$$\zeta(t) = \rho(t, q(t, x_2)), \quad t \in [0, T). \tag{4.31}$$

Hence along this trajectory $q(t, x_2)$ equation (4.14) and the second equation of (4.7) become

$$\begin{aligned} m'(t) &= \frac{1}{2}\zeta^2 + f(t, q(t, x_2)), \\ \zeta'(t) &= -\zeta m. \end{aligned} \tag{4.32}$$

Since $m(t) \leq 0$, we have from the second equation of the above that

$$|\rho(t, q(t, x_2))| = |\zeta(t)| \geq |\zeta(0)|.$$

Then

$$m'(t) \geq \frac{1}{2}\zeta^2(0) - \frac{1}{2}C_2^2 \geq \frac{1}{2}\left(\inf_{x\in\mathbb{R}} \rho_0^2(x) - C_2^2\right).$$

Integrating the above from over $[0, t]$ we obtain (4.9). This completes the proof of Lemma 4.4. □

Proof of Theorem 4.2. Combining Lemma 4.4 and Theorem 4.1 we easily see that the local solution obtained in Theorem 2.2 can be extended to all of the interval $[0, \infty)$. □

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