New finite-gap solutions for the coupled Burgers equations engendered by the Neumann systems*

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(Received 2 Januray 2010; revised manuscript received 6 March 2010)

On the tangent bundle TS^{N-1} of the unit sphere S^{N-1}, this paper reduces the coupled Burgers equations to two Neumann systems by using the nonlinearization of the Lax pair, whose Liouville integrability is displayed in the scheme of the r-matrix technique. Based on the Lax matrix of the Neumann systems, the Abel–Jacobi coordinates are appropriately chosen to straighten out the restricted Neumann flows on the complex torus, from which the new finite-gap solutions expressed by Riemann theta functions for the coupled Burgers equations are given in view of the Jacobi inversion.

Keywords: coupled Burgers equations, Lax matrix, Jacobi inversion, finite-gap solutions
PACC: 0420J, 0290

1. Introduction

The finite dimensional integrable systems (FDISs) are a class of remarkable nonlinear dynamical systems in the realm of mathematical physics; and a series of important examples have arisen in the development of classical mechanics, such as the Kovakevski top, the geodesic flow, the constrained harmonic oscillator and so on.[1−5] After the Moser's work[6] and the Flaschka's statement that FDISs could be generated by constraining infinite dimensional integrable systems on finite dimensional invariant submanifolds,[7] many new FDISs were found through a certain constraint between spectral potentials and eigenfunctions by the nonlinearization technique.[8,9] Interestingly, those resultant FDISs not only enrich the theory of integrable systems itself, but also pave an effective way to solve soliton equations.

The finite-gap solution is a kind of exact solution of integrable nonlinear evolution equations (NLEEs) and is very interesting since it can also be reduced to the multi-soliton solution and the elliptic function solution. Recently the finite dimensional integrable Hamiltonian systems have been already used to obtain finite-gap solutions of integrable NLEEs in both (1+1)- and (2+1)-dimensions under the Bargmann constraint through employing a Lax matrix, Abel–Jacobi coordinates and the theory of algebraic curves.[10−16] Now a natural question arises whether the Neumann systems (i.e. FDISs with an inherent geometric condition) could be used to obtain exact solutions of integrable NLEEs. In 1983 Moser studied the relation between the FDIS and the Dirac–Poisson bracket under the Neumann constraint.[6] Later on, Sklyanin gave a scheme to seek the separation of variables from the r-matrix structure.[17] A general r-matrix approach to deal with both Neumann (type) and Bargmann (type) systems was proposed by Qiao and Zhou.[11,13,18−21] As a practical application for Neumann type systems, the Camassa–Holm equation was the first example shown to deduce algebro-geometric (or finite-gap) solutions on a symplectic submanifold by Qiao[21] in view of the Neumann constraint.

In the present paper, we want to combine the nonlinearization theory[8,9] with the procedure for Neumann type systems shown in the paper[21] to derive new finite-gap solutions of the coupled Burgers equations (CBEs)

$$\begin{aligned} u_t &= u_{xx} + 2uu_x + 2v_x, \\ v_t &= -v_{xx} + 2u_xv + 2uv_x. \end{aligned} \tag{1}$$

*Project supported by the Scientific Foundation of the Southeast University of China (Grant No. KJ2009359), the National Natural Science Foundation of China (Grant No. 10871182) and the U. S. Army Research Office (contract/grant number W911NF-08-1-0511) and Texas grant NHARP 2010.
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This system is able, with the deformation condition $v = 0$, to be reduced to the well-known Burgers equation

$$u_t = u_{xx} + 2uu_x, \tag{2}$$

which describes wave processes in acoustics and hydrodynamics. Actually, this paper is a continuation of the work of Ref. [22]. Compared with the FDISs found from the Lax representations of an integrable hierarchy,[22] here we more emphasize the approach to obtain the finite-gap solutions of the CBEs by using the Neumann systems on TS^{N-1}. For this purpose, we give a new integrable decomposition of the Neumann systems associated with the CBEs and discuss how to handle the Neumann systems based on the r-matrix structure and the theory of algebraic curves for finite dimensional integrable Hamiltonian systems.[10–16] First of all, under the Neumann constraint, the CBEs are conditionally reduced to a pair of Neumann systems, whose involutive solutions give the special solutions of the CBEs. To prove the Liouville integrability of the Neumann systems in the framework of r-matrix theory, we create their Lax matrix and r-matrix formulation under the Dirac–Poisson bracket on TS^{N-1}, which corresponds the Neumann constraint, instead of the standard Poisson bracket. Secondly, the Lax matrix is used to select two sets of elliptic coordinates, and the special solutions of the CBEs are expressed as symmetric functions with respect to these elliptic coordinates. A Riemann surface of hyperelliptic curve and Abel–Jacobi coordinates are employed to linearize the restricted Neumann flows so that they can be integrated by quadratures and then the explicit solutions of the CBEs are given under the Abel–Jacobi coordinates. Finally, the Jacobi inversion problem is investigated to generate new finite-gap solutions of the CBEs in terms of Riemann theta functions.

2. Integrable reduction of the coupled Burgers equations

In this section, we decompose the CBEs into two Neumann systems, and then prove their Liouville integrability in the framework of r-matrix theory. We begin with the spectral problem[22,23]

$$\varphi_x = \boldsymbol{U}\varphi,$$
$$\boldsymbol{U} = \begin{pmatrix} -\frac{1}{2}\lambda + \frac{1}{2}u & -v \\ 1 & \frac{1}{2}\lambda - \frac{1}{2}u \end{pmatrix},$$
$$\boldsymbol{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{3}$$

where λ is a spectral parameter, and u and v are two spectral potentials. To derive an integrable hierarchy associated with Eq. (3), we compute the Lenard sequence $\{g_j\}$ $(-1 \leq j \in \mathbb{Z})$ by

$$\boldsymbol{K}\boldsymbol{g}_{j-1} = \boldsymbol{J}\boldsymbol{g}_j, \quad \boldsymbol{J}\boldsymbol{g}_{-1} = 0, \quad j \geq 0, \tag{4}$$

where

$$\boldsymbol{K} = \begin{pmatrix} 2\partial & \partial^2 + \partial u \\ -\partial^2 + u\partial & v\partial + \partial v \end{pmatrix},$$
$$\boldsymbol{J} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \quad \partial = \partial/\partial x. \tag{5}$$

Let us take $\boldsymbol{g}_{-1} = (0,1)^{\mathrm{T}}$ and $\boldsymbol{g}_{-2} = (1/2,0)^{\mathrm{T}}$ as two special kernel elements from the kernel of the Lenard operator $\boldsymbol{J}$. Then, it is easy to see that $\ker \boldsymbol{J} = \{\varrho_1 g_{-1} + \varrho_2 g_{-2} | \forall \varrho_1, \varrho_2 \in \mathbb{R}\}$, and each g_j can be determined by the recursion formula Eq. (4), for example

$$\boldsymbol{g}_0 = \begin{pmatrix} v \\ u \end{pmatrix}, \quad \boldsymbol{g}_1 = \begin{pmatrix} -v_x + 2uv \\ u_x + u^2 + 2v \end{pmatrix}.$$

Assume that the time dependence of φ for the spectral problem Eq. (3) obeys the differential equation

$$\varphi_{t_n} = \boldsymbol{V}^{(n)}\varphi, \ \boldsymbol{V}^{(n)} = \begin{pmatrix} V_{11}^{(n)} & V_{12}^{(n)} \\ V_{21}^{(n)} & -V_{11}^{(n)} \end{pmatrix}, \ n \geq 0, \tag{6}$$

where

$$V_{11}^{(n)} = \frac{1}{2}(\partial g^{(2)} - (\lambda - u)g^{(2)}),$$
$$V_{12}^{(n)} = \partial g^{(1)} - v g^{(2)},$$
$$V_{21}^{(n)} = g^{(2)}, \quad \boldsymbol{g} = (g^{(1)}, g^{(2)})^{\mathrm{T}} = \sum_{j=0}^{n} \boldsymbol{g}_{j-2}\lambda^{n-j}.$$

Then the compatibility condition of Eqs. (3) and (6) yields the zero curvature equation $\boldsymbol{U}_{t_n} - \boldsymbol{V}_x + [\boldsymbol{U}, \boldsymbol{V}] = 0$, which leads to the following integrable hierarchy:

$$(u,v)^{\mathrm{T}}_{t_n} = \boldsymbol{J}\boldsymbol{g}_{n-1}, \qquad n \geq 1. \tag{7}$$

Apparently, the first nontrivial member $(t = t_2)$ of the integrable hierarchy Eq. (7) is the CBEs Eq. (1) admitting the Lax pair Eq. (3) and

$$\boldsymbol{\varphi}_t=\boldsymbol{V}^{(2)}\varphi,\boldsymbol{V}^{(2)}=\begin{pmatrix}-\frac{1}{2}\lambda^2+\frac{1}{2}u_x+\frac{1}{2}u^2 & -\lambda v+v_x-uv\\ \lambda+u & \frac{1}{2}\lambda^2-\frac{1}{2}u_x-\frac{1}{2}u^2\end{pmatrix}. \tag{8}$$

Let $\lambda_1,\lambda_2,\ldots,\lambda_N$ be N distinct eigenvalues, and $\boldsymbol{\varphi}=(p_j,q_j)^{\mathrm{T}}$ be the eigenfunction corresponding to λ_j. Then from the spectral problem Eq. (3), we have

$$\begin{pmatrix}p_j\\ q_j\end{pmatrix}_x=\begin{pmatrix}-\frac{1}{2}\lambda_j+\frac{1}{2}u & -v\\ 1 & \frac{1}{2}\lambda_j-\frac{1}{2}u\end{pmatrix}\begin{pmatrix}p_j\\ q_j\end{pmatrix},\quad 1\leq j\leq N. \tag{9}$$

The functional gradient of the eigenvalue λ_j with respect to the spectral potentials u and v is given by[22]

$$\boldsymbol{\nabla}\lambda_j=\begin{pmatrix}\dfrac{\delta\lambda_j}{\delta u}\\ \dfrac{\delta\lambda_j}{\delta v}\end{pmatrix}=\begin{pmatrix}\mathrm{Tr}\left(\boldsymbol{\Xi}\dfrac{\partial\boldsymbol{U}}{\partial u}\right)\\ \mathrm{Tr}\left(\boldsymbol{\Xi}\dfrac{\partial\boldsymbol{U}}{\partial v}\right)\end{pmatrix}=\begin{pmatrix}-p_jq_j\\ q_j^2\end{pmatrix},\quad \boldsymbol{\Xi}=\begin{pmatrix}-p_jq_j & p_j^2\\ -q_j^2 & p_jq_j\end{pmatrix}, \tag{10}$$

where Tr means the trace of a matrix. Consider the Neumann constraint[9,22]

$$\boldsymbol{g}_{-1}=\sum_{j=1}^{N}\boldsymbol{\nabla}\lambda_j,\quad \text{i.e. } \langle\boldsymbol{p},\boldsymbol{q}\rangle=0,\quad \langle\boldsymbol{q},\boldsymbol{q}\rangle=1, \tag{11}$$

where $\boldsymbol{p}=(p_1,\ldots,p_N)^{\mathrm{T}}$, $\boldsymbol{q}=(q_1,\ldots,q_N)^{\mathrm{T}}$, and $\langle\cdot,\cdot\rangle$ stands for the standard inner product in $\mathbb{R}^N$. Differentiating both sides of Eq. (11) with respect to x and using Eq. (9) we obtain

$$u=\langle\boldsymbol{\Lambda q},\boldsymbol{q}\rangle,\quad v=\langle\boldsymbol{p},\boldsymbol{p}\rangle, \tag{12}$$

where $\boldsymbol{\Lambda}=\mathrm{diag}(\lambda_1,\ldots,\lambda_N)$. Substituting Eq. (12) into Eq. (9) gives the known Neumann system,[22]

$$\begin{aligned}&\boldsymbol{p}_x=-\frac{1}{2}\boldsymbol{\Lambda p}+\frac{1}{2}\langle\boldsymbol{\Lambda q},\boldsymbol{q}\rangle\boldsymbol{p}-\langle\boldsymbol{p},\boldsymbol{p}\rangle\boldsymbol{q},\\ &\boldsymbol{q}_x=\boldsymbol{p}+\frac{1}{2}\boldsymbol{\Lambda q}-\frac{1}{2}\langle\boldsymbol{\Lambda q},\boldsymbol{q}\rangle\boldsymbol{q},\\ &\langle\boldsymbol{p},\boldsymbol{q}\rangle=0,\quad \langle\boldsymbol{q},\boldsymbol{q}\rangle=1.\end{aligned} \tag{13}$$

Remark 1 Equations $(13)_1$ and $(13)_2$ can be regarded as a canonical Hamiltonian flow on the tangent bundle TS^{N-1} defined by equation $(13)_3$ (see Eq. (21) below). Namely, in vector fields of $\mathbb{R}^{2N}$ with coordinates $(\boldsymbol{p},\boldsymbol{q})$, the flow is locally tangent to TS^{N-1}.

Simultaneously, Eq. (8) (the time part of the Lax pair) provides the second new Neumann system

$$\begin{aligned}\boldsymbol{p}_t\ =\ &-\frac{1}{2}\boldsymbol{\Lambda}^2\boldsymbol{p}+\langle\boldsymbol{\Lambda p},\boldsymbol{q}\rangle\boldsymbol{p}+\frac{1}{2}\langle\boldsymbol{\Lambda}^2\boldsymbol{q},\boldsymbol{q}\rangle\boldsymbol{p}-\langle\boldsymbol{p},\boldsymbol{p}\rangle\boldsymbol{\Lambda q}\\ &-\langle\boldsymbol{\Lambda p},\boldsymbol{p}\rangle\boldsymbol{q},\\ \boldsymbol{q}_t\ =\ &\boldsymbol{\Lambda p}+\langle\boldsymbol{\Lambda q},\boldsymbol{q}\rangle\mathbf{p}+\frac{1}{2}\boldsymbol{\Lambda}^2\boldsymbol{q}\\ &-\langle\boldsymbol{\Lambda p},\boldsymbol{q}\rangle\boldsymbol{q}-\frac{1}{2}\langle\boldsymbol{\Lambda}^2\boldsymbol{q},\boldsymbol{q}\rangle\boldsymbol{q},\\ \langle\boldsymbol{p},\boldsymbol{q}\rangle\ =\ &0,\quad \langle\boldsymbol{q},\boldsymbol{q}\rangle=1,\end{aligned} \tag{14}$$

which can also be interpreted as a vector field tangent to TS^{N-1}. And a direct computation shows that the following proposition is true.

Proposition 1 Let $(\boldsymbol{p}(x,t),\boldsymbol{q}(x,t))^{\mathrm{T}}$ be a compatible solution of the Neumann systems (13) and (14). Then

$$u(x,t)=\langle\boldsymbol{\Lambda q},\boldsymbol{q}\rangle,\quad v(x,t)=\langle\boldsymbol{p},\boldsymbol{p}\rangle \tag{15}$$

are special solutions of the CBEs Eq. (1).

In what follows, we discuss the Liouville integrability of the Neumann systems in the framework of r-matrix theory, instead of the Moser constraint method.[6,22] A known fact is that the Lax matrix and r-matrix formulation contain almost all the necessary information for FDISs. For instance, the Lax matrix gives all possible integrals of motion of Neumann systems on TS^{N-1}, and the r-matrix formulation guarantees the involutivity of integrals of motion (see Refs. [20], [21] and [24] and references therein). For this reason, a lengthy calculation yields the following Lax equations.

Proposition 2 The Neumann systems (13) and (14) possess a Lax matrix $\boldsymbol{L}(\lambda)$ that satisfies the Lax equations

$$\boldsymbol{L}_x(\lambda)=[\bar{\boldsymbol{U}},\boldsymbol{L}(\lambda)],\quad \boldsymbol{L}_t(\lambda)=[\bar{\boldsymbol{V}}^{(2)},\boldsymbol{L}(\lambda)], \tag{16}$$

where

$$\boldsymbol{L}(\lambda)=\begin{pmatrix}-\dfrac{1}{2} & 0\\ 0 & \dfrac{1}{2}\end{pmatrix}+\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\begin{pmatrix}p_jq_j & -p_j^2\\ q_j^2 & -p_jq_j\end{pmatrix}\triangleq\begin{pmatrix}A(\lambda) & B(\lambda)\\ C(\lambda) & -A(\lambda)\end{pmatrix}, \tag{17}$$

$$\bar{\boldsymbol{U}}=\begin{pmatrix}-\dfrac{1}{2}\lambda+\dfrac{1}{2}\langle\boldsymbol{\Lambda q},\boldsymbol{q}\rangle & -\langle\boldsymbol{p},\boldsymbol{p}\rangle\\ 1 & \dfrac{1}{2}\lambda-\dfrac{1}{2}\langle\boldsymbol{\Lambda q},\boldsymbol{q}\rangle\end{pmatrix}, \tag{18}$$

and

$$\bar{\boldsymbol{V}}^{(2)}=\begin{pmatrix}-\dfrac{1}{2}\lambda^2+\langle\boldsymbol{\Lambda p},\boldsymbol{q}\rangle+\dfrac{1}{2}\langle\boldsymbol{\Lambda}^2\boldsymbol{q},\boldsymbol{q}\rangle & -\lambda\langle\boldsymbol{p},\boldsymbol{p}\rangle-\langle\boldsymbol{\Lambda p},\boldsymbol{p}\rangle\\ \lambda+\langle\boldsymbol{\Lambda q},\boldsymbol{q}\rangle & \dfrac{1}{2}\lambda^2-\langle\boldsymbol{\Lambda p},\boldsymbol{q}\rangle-\dfrac{1}{2}\langle\boldsymbol{\Lambda}^2\boldsymbol{q},\boldsymbol{q}\rangle\end{pmatrix}. \tag{19}$$

Actually, the Lax matrix Eq. (17) and the auxiliary matrix Eq. (18) were discussed in Ref. [25] to classify the FDISs. The standard Poisson bracket of two functions f and g in the symplectic space $(\mathbb{R}^{2N}, \mathrm{d}p\wedge\mathrm{d}q)$ is defined by[26]

$$\{f,g\}=\sum_{j=1}^{N}\left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j}-\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}\right)=\left\langle\frac{\partial f}{\partial\boldsymbol{q}},\frac{\partial g}{\partial\boldsymbol{p}}\right\rangle-\left\langle\frac{\partial f}{\partial\boldsymbol{p}},\frac{\partial g}{\partial\boldsymbol{q}}\right\rangle.$$

Noticing that the current Neumann systems (13) and (14) possess the same constraints

$$F\triangleq\langle\boldsymbol{p},\boldsymbol{q}\rangle=0,\quad \boldsymbol{G}\triangleq\langle\boldsymbol{q},\boldsymbol{q}\rangle-1=0, \tag{20}$$

we introduce the Dirac–Poisson bracket

$$\{f,g\}_{\mathrm{D}}=\{f,g\}+\frac{1}{\{F,G\}}(\{f,F\}\{G,g\}-\{f,G\}\{F,g\}),$$

on the tangent bundle

$$TS^{N-1}=\{(\boldsymbol{p},\boldsymbol{q})\in\mathbb{R}^{2N}|F=0,\ G=0\}, \tag{21}$$

of the $(N-1)$-dimensional unit sphere $S^{N-1}=\{\boldsymbol{q}\in\mathbb{R}^N,\langle\boldsymbol{q},\boldsymbol{q}\rangle=1\}$. Under the Dirac–Poisson bracket, the Neumann systems (13) and (14) can be rewritten in the canonical Hamiltonian form

$$\boldsymbol{p}_x=\{\boldsymbol{p},H_0\}_{\mathrm{D}},\quad \boldsymbol{q}_x=\{\boldsymbol{q},H_0\}_{\mathrm{D}},$$

and

$$\boldsymbol{p}_t=\{\boldsymbol{p},H_1\}_{\mathrm{D}},\quad \boldsymbol{q}_t=\{\boldsymbol{q},H_1\}_{\mathrm{D}},$$

with the Hamiltonians

$$H_0=\frac{1}{2}(\langle\boldsymbol{\Lambda p},\boldsymbol{q}\rangle+\langle\boldsymbol{p},\boldsymbol{p}\rangle), \tag{22}$$

and

$$H_1=\frac{1}{2}(\langle\boldsymbol{\Lambda}^2\boldsymbol{p},\boldsymbol{q}\rangle+\langle\boldsymbol{p},\boldsymbol{p}\rangle\langle\boldsymbol{\Lambda q},\boldsymbol{q}\rangle+\langle\boldsymbol{\Lambda p},\boldsymbol{p}\rangle). \tag{23}$$

Through a direct but tedious calculation, we can prove the following lemma.

Lemma 1 The components of the Lax matrix $\boldsymbol{L}(\lambda)$ satisfy the identities

$$\begin{aligned}
\{A(\lambda),A(\mu)\}_{\mathrm{D}}&=0,\\
\{B(\lambda),B(\mu)\}_{\mathrm{D}}&=2(-B(\lambda)(2A(\mu)+1)+B(\mu)(2A(\lambda)+1)),\\
\{C(\lambda),C(\mu)\}_{\mathrm{D}}&=0,\\
\{A(\lambda),B(\mu)\}_{\mathrm{D}}&=\frac{2}{\mu-\lambda}(-B(\mu)+B(\lambda))-2B(\mu)C(\lambda),\\
\{A(\lambda),C(\mu)\}_{\mathrm{D}}&=\frac{2}{\mu-\lambda}(-C(\lambda)+C(\mu))+2C(\lambda)C(\mu),\\
\{B(\lambda),C(\mu)\}_{\mathrm{D}}&=\frac{4}{\mu-\lambda}(-A(\mu)+A(\lambda))-2C(\mu)(2A(\lambda)+1).
\end{aligned}$$

Let $\boldsymbol{L}_1(\lambda)=\boldsymbol{L}(\lambda)\otimes\boldsymbol{I}$, $\boldsymbol{L}_2(\mu)=\boldsymbol{I}\otimes\boldsymbol{L}(\mu)$ and $\{\boldsymbol{L}_1(\lambda),\boldsymbol{L}_2(\mu)\}_{\mathrm{D}}^{jk,mn}=\{\boldsymbol{L}_1(\lambda)^{jm},\boldsymbol{L}_2(\mu)^{kn}\}_{\mathrm{D}}$, where $\boldsymbol{I}$ is the 2×2 unit matrix, $\otimes$ stands for the tensor product of two matrices, and $\boldsymbol{L}_1(\lambda)^{jm}$ and $\boldsymbol{L}_2(\mu)^{kn}$ are the jm entry of $\boldsymbol{L}_1(\lambda)$ and the kn entry of $\boldsymbol{L}_2(\mu)$, respectively. By Lemma 1, we can readily derive the dynamical r-matrix formula of the Neumann systems (13) and (14).

Proposition3 The Lax matrix $\boldsymbol{L}(\lambda)$ satisfies the following fundamental Dirac–Poisson bracket with a dynamical r-matrix structure

$$\{\boldsymbol{L}_1(\lambda), \boldsymbol{L}_2(\mu)\}_{\mathrm{D}} = [\boldsymbol{r}_{12}, \boldsymbol{L}_1(\lambda)] - [\boldsymbol{r}_{21}, \boldsymbol{L}_2(\mu)], \quad (24)$$

where

$$\begin{aligned} \boldsymbol{r}_{12} &= -\frac{2}{\mu-\lambda}\boldsymbol{P} - \boldsymbol{Q}_{12} - \boldsymbol{R}_{12}, \\ \boldsymbol{r}_{21} &= -\frac{2}{\mu-\lambda}\boldsymbol{P} - \boldsymbol{Q}_{21} + \boldsymbol{R}_{21}, \end{aligned} \quad (25)$$

with

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\boldsymbol{Q}_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{Q}_{21} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\boldsymbol{R}_{12} = \begin{pmatrix} 0 & 0 & 0 & B(\mu) \\ 0 & 0 & -C(\mu) & 0 \\ 0 & 0 & C(\mu) & -2A(\mu) \\ 0 & 0 & 0 & -C(\mu) \end{pmatrix},$$

$$\boldsymbol{R}_{21} = \begin{pmatrix} 0 & 0 & 0 & B(\lambda) \\ 0 & C(\lambda) & 0 & -2A(\lambda) \\ 0 & -C(\lambda) & 0 & 0 \\ 0 & 0 & 0 & -C(\lambda) \end{pmatrix}.$$

Based on the Lax matrix $\boldsymbol{L}(\lambda)$, one can easily obtain the integrals of motion that guarantee the Liouville integrability of Neumann systems (13) and (14). Let us calculate

$$\det \boldsymbol{L}(\lambda) = -\frac{1}{2}\mathrm{tr}\boldsymbol{L}^2(\lambda) = -\frac{1}{4} + \sum_{k=1}^{N} \frac{E_k}{\lambda - \lambda_k}, \quad (26)$$

where

$$E_k = p_k q_k + \sum_{j=1, j\neq k}^{N} \frac{(p_k q_j - p_j q_k)^2}{\lambda_k - \lambda_j}, \quad 1 \le k \le N.$$

From Eqs. (24) and (20), by direct calculations we obtain[20,21]

$$\{E_k, E_l\}_{\mathrm{D}} = 0, \quad k, l = 1, 2, \ldots, N.$$

Furthermore, $E_1, E_2, \ldots, E_N$ are functionally independent in a dense open subset on TS^{N-1}. In fact, by Ref. [26], we just need to prove the linear independence of the differentials $\mathrm{d}E_1, \mathrm{d}E_2, \ldots, \mathrm{d}E_N$. Due to

$$\begin{aligned} &\left.\begin{vmatrix} \frac{\partial E_1}{\partial p_1} & \frac{\partial E_2}{\partial p_1} & \cdots & \frac{\partial E_N}{\partial p_1} \\ \frac{\partial E_1}{\partial p_2} & \frac{\partial E_2}{\partial p_2} & \cdots & \frac{\partial E_N}{\partial p_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial E_1}{\partial p_N} & \frac{\partial E_2}{\partial p_N} & \cdots & \frac{\partial E_N}{\partial p_N} \end{vmatrix}\right|_{p_i=0, q_i\neq 0} \\ &= \begin{vmatrix} q_1 & 0 & 0 & \cdots & 0 \\ 0 & q_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & q_N \end{vmatrix} \\ &= \prod_{i=1}^{N} q_i \neq 0, \end{aligned}$$

$E_1, E_2, \ldots, E_N$ are functionally independent of a dense open subset on TS^{N-1}.[27] Therefore, the Neumann systems (constrained Hamiltonian systems on TS^{N-1}) $(TS^{N-1}, \{,\}_{\mathrm{D}}, F_j)$ with $F_j = \sum_{k=1}^{N} \lambda_k^j E_k$ are completely integrable in the Liouville sense. Noticing that $H_0 = F_1/2$ and $H_1 = F_2/2$, we have a crucial theorem as follows.

Theorem 1 The Neumann systems (13) and (14) are completely integrable in the Liouville sense.

Remark 2 The involutivity of H_0 and H_1 implies that the Neumann systems (13) and (14) are compatible and their constrained Hamiltonian phase flows in the sense of Dirac–Poisson bracket commute. Therefore, their commonly compatible solution is called the involutive solution of the two consistent Neumann systems (13) and (14).[22,26]

3. Straightening out of the restricted Neumann flows

To straighten out the restricted Neumann flows reduced from the CBEs, we introduce two sets of elliptic coordinates $\mu_1, \mu_2, \ldots, \mu_{N-1}$ and $\nu_1, \nu_2, \ldots, \nu_{N-1}$ through equations

$$B(\lambda) = -\sum_{j=1}^{N} \frac{p_j^2}{\lambda - \lambda_j} = -\langle \boldsymbol{p}, \boldsymbol{p}\rangle \frac{m(\lambda)}{a(\lambda)},$$

$$C(\lambda)=\sum_{j=1}^{N}\frac{q_j^2}{\lambda-\lambda_j}=\langle\boldsymbol{q},\boldsymbol{q}\rangle\frac{n(\lambda)}{a(\lambda)},\tag{27}$$

where

$$\begin{aligned}a(\lambda)&=\prod_{k=1}^{N}(\lambda-\lambda_k),\\ m(\lambda)&=\prod_{k=1}^{N-1}(\lambda-\mu_k),\\ n(\lambda)&=\prod_{k=1}^{N-1}(\lambda-\nu_k).\end{aligned}\tag{28}$$

By employing Eqs. (11), (27) and (28), one can easily calculate

$$\begin{aligned}\frac{\langle\boldsymbol{\Lambda p},\boldsymbol{p}\rangle}{\langle\boldsymbol{p},\boldsymbol{p}\rangle}&=\sum_{j=1}^{N}\lambda_j-\sum_{j=1}^{N-1}\mu_j\triangleq\sigma-\sigma_1,\\ \langle\boldsymbol{\Lambda q},\boldsymbol{q}\rangle&=\sum_{j=1}^{N}\lambda_j-\sum_{j=1}^{N-1}\nu_j\triangleq\sigma-\sigma_2.\end{aligned}\tag{29}$$

Based on equations Eqs. (11) and (29), $(\mu_1,\mu_2,\ldots,\mu_{N-1},\nu_1,\nu_2,\ldots,\nu_{N-1})$ may be viewed as a set of canonical coordinates of the tangent bundle TS^{N-1}, and $(\nu_1,\nu_2,\ldots,\nu_{N-1})$ is a set of elliptic coordinates of the $N-1$ dimensional sphere $\langle\boldsymbol{q},\boldsymbol{q}\rangle=1$. Meanwhile, from Eqs. (12) and (19) a direct calculation gives

$$u=\sigma-\sigma_2,\quad \partial\ln v=\sigma_1-\sigma_2,\tag{30}$$

$$\begin{aligned}\bar{V}_{12}^{(2)}&=-v(\lambda+\sigma-\sigma_1),\\ \bar{V}_{21}^{(2)}&=\lambda+\sigma-\sigma_2.\end{aligned}\tag{31}$$

By Eqs. (15) and (24), $\det\boldsymbol{L}(\lambda)$ can be written as

$$\begin{aligned}\det\boldsymbol{L}(\lambda)&=-A(\lambda)^2-B(\lambda)C(\lambda)=-\frac{b(\lambda)}{4a(\lambda)}\\ &=-\frac{R(\lambda)}{4a^2(\lambda)},\end{aligned}\tag{32}$$

where

$$b(\lambda)=\prod_{k=1}^{N}(\lambda-\lambda_{N+k}),$$

$$R(\lambda)=a(\lambda)b(\lambda)=\prod_{k=1}^{2N}(\lambda-\lambda_k).$$

The combination of Eqs. (27) and (32) results in

$$A(\mu_k)=\frac{\sqrt{R(\mu_k)}}{2a(\mu_k)},\quad A(\nu_k)=\frac{\sqrt{R(\nu_k)}}{2a(\nu_k)},\quad 1\le k\le N-1.\tag{33}$$

By the first equation in Eq. (27), a simple calculation leads to

$$\left.\frac{\mathrm{d}B(\lambda)}{\mathrm{d}x}\right|_{\mu_k}=\langle\boldsymbol{p},\boldsymbol{p}\rangle\frac{\prod\limits_{i=1,i\neq k}^{N-1}(\mu_k-\mu_i)}{a(\mu_k)}\frac{\mathrm{d}\mu_k}{\mathrm{d}x}.$$

Therefore by the first equation in Eq. (16), the evolution of the elliptic coordinate μ_k with respect to x is

$$\frac{\mathrm{d}\mu_k}{\mathrm{d}x}=\frac{\sqrt{R(\mu_k)}}{\prod\limits_{i=1,i\neq k}^{N-1}(\mu_k-\mu_i)},\quad 1\le k\le N-1.\tag{34}$$

Similarly, we can arrive at

$$\frac{\mathrm{d}\nu_k}{\mathrm{d}x}=-\frac{\sqrt{R(\nu_k)}}{\prod\limits_{i=1,i\neq k}^{N-1}(\nu_k-\nu_i)},\quad 1\le k\le N-1,\tag{35}$$

and the evolutions of μ_k and ν_k with respect to the time variable t are

$$\begin{aligned}\frac{\mathrm{d}\mu_k}{\mathrm{d}t}&=\frac{(\mu_k-\sigma_1+\sigma)\sqrt{R(\mu_k)}}{\prod\limits_{i=1,i\neq k}^{N-1}(\mu_k-\mu_i)},\\ \frac{\mathrm{d}\nu_k}{\mathrm{d}t}&=\frac{(-\nu_k+\sigma_2-\sigma)\sqrt{R(\nu_k)}}{\prod\limits_{i=1,i\neq k}^{N-1}(\nu_k-\nu_i)},\\ 1\le k&\le N-1.\end{aligned}\tag{36}$$

These formulae naturally bring us to think about the Riemann surface Γ of the hyperelliptic curve given by the affine equation $\xi^2=R(\lambda)$, whose genus is $N-1$. For the same value of λ, there exist two points $(\lambda,\sqrt{R(\lambda)})$ and $(\lambda,-\sqrt{R(\lambda)})$ on the upper and lower sheets of Γ. However, two infinity points are not the branch points because of $\deg R(\lambda)=2N$. Under an alternative local coordinate $z=\lambda^{-1}$, they are regarded as $\infty_1=(0,1)$ and $\infty_2=(0,-1)$ respectively. Let $a_1,a_2,\ldots,a_{N-1},\ b_1,b_2,\ldots,b_{N-1}$ be a set of regular cycle paths on Γ, which are independent if they have the intersection numbers:

$$\begin{aligned}&a_i\circ a_j=b_i\circ b_j=0,\quad a_i\circ b_j=\delta_{ij},\\ &i,j=1,2,\ldots,N-1.\end{aligned}$$

It is known that

$$\tilde{\omega}_l=\frac{\lambda^{l-1}\mathrm{d}\lambda}{\sqrt{R(\lambda)}},\quad 1\le l\le N-1,$$

are $N-1$ linearly independent holomorphic differentials of Γ. Let

$$A_{ij}=\int_{a_j}\tilde{\omega}_i,\ \mathbf{C}=(A_{ij})^{-1},\ 1\le i,j\le N-1.\tag{37}$$

Then, $\tilde{\omega}_l$ can be normalized into a new basis

$$\omega_j = \sum_{l=1}^{N-1} C_{jl}\tilde{\omega}_l,$$

with the properties

$$\int_{a_i} \omega_j = \sum_{l=1}^{N-1} C_{jl} \int_{a_i} \tilde{\omega}_l = \sum_{l=1}^{N-1} C_{jl}A_{li} = \delta_{ji},$$

and

$$B_{ij} = \int_{b_j} \omega_i, \qquad 1 \le i, j \le N-1,$$

where the matrix $\boldsymbol{B} = (B_{ij})_{(N-1)\times(N-1)}$ is symmetric with the positively definite imaginary part, and it is further used to construct the Riemann theta function on the Riemann surface $\varGamma$. The Abel–Jacobi coordinates are given by

$$\begin{aligned} \rho_j^{(1)}(x,t) &= \sum_{k=1}^{N-1} \int_{\tilde{p}_0}^{\mu_k(x,t)} \omega_j = \sum_{k=1}^{N-1}\sum_{l=1}^{N-1} C_{jl} \int_{\tilde{p}_0}^{\mu_k} \frac{\lambda^{l-1}\mathrm{d}\lambda}{\sqrt{R(\lambda)}}, \\ \rho_j^{(2)}(x,t) &= \sum_{k=1}^{N-1} \int_{\tilde{p}_0}^{\nu_k(x,t)} \omega_j = \sum_{k=1}^{N-1}\sum_{l=1}^{N-1} C_{jl} \int_{\tilde{p}_0}^{\nu_k} \frac{\lambda^{l-1}\mathrm{d}\lambda}{\sqrt{R(\lambda)}}, \end{aligned} \qquad 1 \le j \le N-1, \tag{38}$$

where $\tilde{p}_0$ is a fixed point on $\varGamma$. It is not difficult for one to calculate from the first equation in Eq. (38) that

$$\partial_x \rho_j^{(1)} = \sum_{l=1}^{N-1}\sum_{k=1}^{N-1} C_{jl} \frac{\mu_k^{l-1}\mu_{k,x}}{\sqrt{R(\mu_k)}} = \sum_{l=1}^{N-1}\sum_{k=1}^{N-1} C_{jl} \frac{\mu_k^{l-1}}{\prod\limits_{i=1, i\ne k}^{N-1}(\mu_k-\mu_i)}. \tag{39}$$

On the other hand, we have the algebraic relation

$$I_s = \sum_{k=1}^{N-1} \frac{\mu_k^s}{\prod\limits_{i=1,i\ne k}^{N-1}(\mu_k-\mu_i)} = \delta_{s,N-2}, \quad I_{N-1} = \sigma_1 I_{N-2}, \quad 1 \le s \le N-2. \tag{40}$$

So, we obtain

$$\partial_x \rho_j^{(1)} = \varOmega_j^{(0)}, \quad \varOmega_j^{(0)} = C_{jN-1}, \quad 1 \le j \le N-1. \tag{41}$$

A similar procedure applied for Eq. (41) produce

$$\partial_t \rho_j^{(1)} = \varOmega_j^{(1)}, \quad \partial_x \rho_j^{(2)} = -\varOmega_j^{(0)}, \quad \partial_t \rho_j^{(2)} = -\varOmega_j^{(1)}, \tag{42}$$

where $\varOmega_j^{(1)} = C_{jN-2} + \sigma C_{jN-1}$. Based on the above results, the special solutions of the CBEs under the Abel–Jacobi coordinates can be obtained through a linear superposition

$$\rho_j^{(1)} = \varOmega_j^{(0)}x + \varOmega_j^{(1)}t + \gamma_j^{(1)}, \quad \rho_j^{(2)} = -\varOmega_j^{(0)}x - \varOmega_j^{(1)}t + \gamma_j^{(2)}, \quad 1 \le j \le N-1, \tag{43}$$

with constants of integration

$$\gamma_j^{(1)} = \sum_{k=1}^{N-1} \int_{\tilde{p}_0}^{\mu_k(0,0)} \omega_j, \quad \gamma_j^{(2)} = \sum_{k=1}^{N-1} \int_{\tilde{p}_0}^{\nu_k(0,0)} \omega_j.$$

4. Finite-gap solutions

In Section 3, the special solutions Eq. (43) of the CBEs are given in the Abel–Jacobi coordinates $(\rho^{(1)}, \rho^{(2)})$. To derive explicit solutions of the CBEs in the original coordinates (u, v), the following steps $(\rho^{(1)}, \rho^{(2)}) \Longrightarrow (\mu_k, \nu_k)$ should be completed with the help of the Jacobi inversion.

Let T be the lattice in $\mathbb{C}^{N-1}$ generated by $2(N-1)$ periodic vectors $\{\delta_i, \boldsymbol{B}_j\}$ with components $(\delta_{ij}), (B_{ij})$ respectively. The complex torus $J(\varGamma) = \mathbb{C}^{N-1}/T$ is called the Jacobian of the Riemann surface $\varGamma$. The Abel map $\mathcal{A}:\ \mathrm{Div}(\varGamma) \longrightarrow J(\varGamma)$ is defined by $\mathcal{A}(\tilde{p}) = \int_{\tilde{p}_0}^{\tilde{p}} \omega$ with a natural linear extension to the factor group $\mathrm{Div}(\varGamma)$:

$\mathcal{A}(\sum n_k\tilde{p}_k) = \sum n_k\mathcal{A}(\tilde{p}_k)$, where $\tilde{p}$ is an arbitrary point on $\mathit{\Gamma}$. As one knows, the Riemann theta function on $\mathit{\Gamma}$ is defined by[27]

$$\begin{aligned}\theta(\zeta) &= \sum_{z\in\mathbb{Z}^{N-1}} \exp\left(\pi i\langle \boldsymbol{Bz},\boldsymbol{z}\rangle + 2\pi i\langle \boldsymbol{\zeta},\boldsymbol{z}\rangle\right), \quad \boldsymbol{\zeta}\in\mathbb{C}^{N-1},\\ \langle \boldsymbol{Bz},\boldsymbol{z}\rangle &= \sum_{i,j=1}^{N-1} B_{ij}z_iz_j, \quad \langle \boldsymbol{\zeta},\boldsymbol{z}\rangle = \sum_{i=1}^{N-1} z_i\zeta_i,\end{aligned}$$

with the properties

$$\theta(-\zeta)=\theta(\zeta),\quad \theta(\zeta+\delta_j)=\theta(\zeta),\quad \theta(\zeta+\boldsymbol{B}_j)=\theta(\zeta)\exp\{-\pi i(B_{jj}+2\zeta_j)\}.$$

Let us consider two special divisors $\sum_{k=1}^{N-1}\tilde{p}_k^{(m)}$ $(m=1,2)$, where

$$\tilde{p}_k^{(1)} = (\mu_k(x,t),\zeta(\mu_k)),\quad \tilde{p}_k^{(2)} = (\nu_k(x,t),\zeta(\nu_k)),$$

with $\zeta^2(\lambda)=R(\lambda)$. Then, we have

$$\mathcal{A}\left(\sum_{k=1}^{N-1}\tilde{p}_k^{(m)}\right) = \sum_{k=1}^{N-1}\mathcal{A}(\tilde{p}_k^{(m)}) = \sum_{k=1}^{N-1}\int_{\tilde{p}_0}^{\tilde{p}_k^{(m)}}\omega = \boldsymbol{\rho}^{(m)},$$

whose component is $\rho_j^{(m)} = \sum_{k=1}^{N-1}\int_{\tilde{p}_0}^{\tilde{p}_k^{(m)}}\omega_j$. According to the Riemann theorem,[28] there exist constant vectors (the Riemann constants) $\boldsymbol{M}^{(1)},\boldsymbol{M}^{(2)}\in\mathbb{C}^{N-1}$ determined by $\mathit{\Gamma}$ such that

- $f^{(1)}(\lambda)=\theta(\mathcal{A}(\zeta(\lambda))-\rho^{(1)}-\boldsymbol{M}^{(1)})$ has exactly $N-1$ zeros at $\mu_1,\ldots,\mu_{N-1}$,
- $f^{(2)}(\lambda)=\theta(\mathcal{A}(\zeta(\lambda))-\rho^{(2)}-\boldsymbol{M}^{(2)})$ has exactly $N-1$ zeros at $\nu_1,\ldots,\nu_{N-1}$.

To make the function $f^{(m)}(\lambda)$ $(m=1,2)$ single valued, $\mathit{\Gamma}$ is cut along all a_k,b_k to form a simply connected region whose boundary is denoted by γ consisting of $4(N-1)$ edges in the order:

$$a_1^+b_1^+a_1^-b_1^-a_2^+b_2^+a_2^-b_2^-\ldots\ldots a_{N-1}^+b_{N-1}^+a_{N-1}^-b_{N-1}^-,$$

where symbols $+$, $-$ denote the orientation. From the residue theorem, we have

$$\sum_{j=1}^{N-1}\mu_j = I(\mathit{\Gamma}) - \sum_{s=1}^{2}\operatorname*{Res}_{\lambda=\infty_s}\lambda d\ln f^{(1)}(\lambda),\quad \sum_{j=1}^{N-1}\nu_j = I(\mathit{\Gamma}) - \sum_{s=1}^{2}\operatorname*{Res}_{\lambda=\infty_s}\lambda d\ln f^{(2)}(\lambda), \tag{44}$$

where

$$I(\mathit{\Gamma}) = \frac{1}{2\pi i}\oint_\gamma \lambda d\ln f^{(m)}(\lambda) = \sum_{j=1}^{N-1}\int_{a_j}\lambda\omega_j,\qquad m=1,2,$$

are two constants independent of $\rho^{(m)}$ (see Refs. [13] and [29] for more details). So, let us calculate residues as follows:

$$\begin{aligned}f^{(m)}(\lambda)|_{\lambda=\infty_s} &= f^{(m)}(z^{-1})|_{z=0}\\ &= \theta\left(\int_{\tilde{p}_0}^{\tilde{p}}\omega - \rho^{(m)} - \boldsymbol{M}^{(m)}\right)\\ &= \theta\left(\int_{\infty_s}^{\tilde{p}}\omega - \pi_s - \rho^{(m)} - \boldsymbol{M}^{(m)}\right)\\ &= \theta\left(\ldots,\int_{\infty_s}^{\tilde{p}}\omega_j - \pi_{sj} - \rho_j^{(m)} - M_j^{(m)},\ldots\right)\\ &= \theta\left(\ldots,(-1)^{s-1}\sum_{l=1}^{N-1}Cjl\int_0^z\frac{z^{-l-1}}{\sqrt{R(z^{-1})}}\mathrm{d}z - \pi_{sj} - \rho_j^{(m)} - M_j^{(m)},\ldots\right)\end{aligned}$$

$$
\begin{aligned}
&= \theta\left(\ldots, \rho_j^{(m)} + M_j^{(m)} + \pi_{sj} + (-1)^s(C_{jN-1}z + \frac{1}{2}(C_{jN-2} + \sigma C_{jN-1})z^2 + \ldots), \ldots\right) \\
&= \theta_s^{(m)}(\rho^{(m)} + M^{(m)} + \pi_s) + (-1)^{s+m-1}\partial_x\theta_s^{(m)} z + \cdots,
\end{aligned}
$$

where we have used the fact that θ is an even function and $\pi_{sj} = \int_{\infty_s}^{\tilde{p}_0} \omega_j$ $(s, m = 1, 2)$. Utilizing the expression of $f^{(m)}(z^{-1})$ with respect to the local coordinate z, we arrive at

$$
\begin{aligned}
z^{-1}\frac{\mathrm{d}\ln f^{(m)}(z^{-1})}{\mathrm{d}z} &= \frac{z^{-1}}{\theta_s^{(m)} + (-1)^{s+m-1}\partial_x\theta_s^{(m)} z + \ldots}\frac{\mathrm{d}f^{(m)}(z^{-1})}{\mathrm{d}z} \\
&= \frac{z^{-1}}{\theta_s^{(m)}}\left(1 - (-1)^{s+m-1}\frac{\partial_x\theta_s^{(m)}}{\theta_s^{(m)}}z + \ldots\right)((-1)^{s+m-1}\partial_x\theta_s^{(m)} + \ldots) \\
&= \frac{(-1)^{s+m-1}\partial_x\theta_s^{(m)}}{\theta_s^{(m)}}z^{-1} - \left(\frac{\partial_x\theta_s^{(m)}}{\theta_s^{(m)}}\right)^2 + \ldots .
\end{aligned}
$$

Therefore, we have

$$
\operatorname*{Res}_{\lambda=\infty_s} \lambda d\ln f^{(m)}(\lambda) = (-1)^{s+m-1}\partial_x \ln\theta_s^{(m)}, \tag{45}
$$

where

$$
\theta_s^{(1)} = \theta(\boldsymbol{\Omega}^{(0)}x + \boldsymbol{\Omega}^{(1)}t + \boldsymbol{\Upsilon}_s), \quad \theta_s^{(2)} = \theta(-\boldsymbol{\Omega}^{(0)}x - \boldsymbol{\Omega}^{(1)}t + \boldsymbol{\Lambda}_s),
$$

with

$$
\Upsilon_{sj} = \gamma_j^{(1)} + M_j^{(1)} + \pi_{sj}, \quad \Lambda_{sj} = \gamma_j^{(2)} + M_j^{(2)} + \pi_{sj}, \quad 1 \le j \le N-1.
$$

It follows from Eqs. (44) and (45) that

$$
\sum_{l=1}^{N-1}\mu_l = I(\Gamma) + \partial_x \ln\frac{\theta_1^{(1)}}{\theta_2^{(1)}}, \quad \sum_{l=1}^{N-1}\nu_l = I(\Gamma) + \partial_x \ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}}. \tag{46}
$$

On the other hand, we have already specified an explicit relation Eq. (30) between the desired solutions $u,\ v$ and the symmetric functions of elliptic coordinates $\mu_k,\ \nu_k$ $(1 \le k \le N-1)$. Substituting Eq. (46) into Eq. (30) we obtain the new finite-gap solutions of the CBEs Eq. (1),

$$
\begin{aligned}
u &= \partial_x \ln\frac{\theta(-\boldsymbol{\Omega}^{(0)}x - \boldsymbol{\Omega}^{(1)}t + \boldsymbol{\Lambda}_1)}{\theta(-\boldsymbol{\Omega}^{(0)}x - \boldsymbol{\Omega}^{(1)}t + \boldsymbol{\Lambda}_2)} - I(\Gamma) + \sigma, \\
v &= \frac{\theta(\boldsymbol{\Omega}^{(0)}x + \boldsymbol{\Omega}^{(1)}t + \boldsymbol{\Upsilon}_1)}{\theta(\boldsymbol{\Omega}^{(0)}x + \boldsymbol{\Omega}^{(1)}t + \boldsymbol{\Upsilon}_2)}\frac{\theta(-\boldsymbol{\Omega}^{(0)}x - \boldsymbol{\Omega}^{(1)}t + \boldsymbol{\Lambda}_1)}{\theta(-\boldsymbol{\Omega}^{(0)}x - \boldsymbol{\Omega}^{(1)}t + \boldsymbol{\Lambda}_2)}\frac{\theta(\boldsymbol{\Omega}^{(1)}t + \boldsymbol{\Upsilon}_2)}{\theta(\boldsymbol{\Omega}^{(1)}t + \boldsymbol{\Upsilon}_1)}\frac{\theta(-\boldsymbol{\Omega}^{(1)}t + \boldsymbol{\Lambda}_2)}{\theta(-\boldsymbol{\Omega}^{(1)}t + \boldsymbol{\Lambda}_1)}v(0,t),
\end{aligned}
$$

which differs from the ones derived in the paper[30] in view of finite dimensional integrable Hamiltonian systems.

5. Conclusions and remarks

In this paper, we obtain the explicit solutions–finite-gap solutions of the CBEs, which are given in terms of Riemann theta functions on the tangent bundle TS^{N-1} of the unit sphere S^{N-1}. This means that a practical procedure is provided to construct explicit solutions of integrable NLEEs from the integrable decomposition under the scheme of Neumann systems. For some integrable NLEEs, it is very difficult to figure out the explicit relationship between spectral potentials and elliptical coordinates in light of the integrable decomposition regarding finite dimensional integrable Hamiltonian systems. Recently, the algebro-geometric solutions of the Camassa–Holm equation was given by using the Neumann type FDIS,[21] where Qiao first time dealt with the Neumann type FDIS to obtain the algebro-geometric solutions of the integrable NLEE. The Neumann systems in this paper are another example for solving integrable NLEEs for finite-gap solutions. We hope to do more integrable NLEEs to obtain algebro-geometric solutions under the scheme of Neumann (type) systems. For this statement, we will provide some other examples in the forthcoming papers.

References

[1] Kovalevski S 1889 *Acta. Math.* **12** 177

[2] Knorrer H 1980 *Invent. Math.* **59** 119

[3] Jacobi C 1984 *Vorlesungen uber Dynamik Gesammelte Werke* (Berlin: Supplementband)

[4] Calogero F 1975 *Lett. Nuovo Cimento* **13** 411

[5] Moser J 1975 *Adv. Math.* **16** 197

[6] Moser J 1986 *Integrable Hamiltonian System and Spectral Theory Proc. Symp. on Differential Geometry and Differential Equations* (Beijing: Science Press) 157

[7] Flascka H 1983 in: *Non-linear Integrable System-Classical Theory and Quantum Theory, 1981 Proceedings of RIMS Symposium* Kyoto, Japan (Singapore: World Scientific)

[8] Cao C W 1990 *Sci. China* A **33** 528

[9] Cao C W and Geng X G 1990 in *Proc. Conf. on Nonlinear Physics* Shanghai 1989, Research Reports in Physics (Berlin: Springer) 68

[10] Cao C W, Wu Y T and Geng X G 1999 *J. Math. Phys.* **40** 3948

[11] Qiao Z J 2001 *Theor. Math. Phys.* **127** 827

[12] Geng X G and Cao C W 2001 *Nonlinearity* **14** 1433

[13] Zhou R G 1997 *J. Math. Phys.* **38** 2535

[14] Chen J B and Geng X G 2006 *Chin. Phys.* **15** 1407

[15] Wu Y Q 2008 *Acta Phys. Sin.* **57** 5390 (in Chinese)

[16] Ji J, Yao Y Q, Yu J and Liu Y Q 2007 *Chin. Phys.* **16** 296

[17] Sklyanin E K 1992 *Commun. Math. Phys.* **150** 181

[18] Qiao Z J and Zhou R G 1997 *Phys. Lett.* A **235** 35

[19] Qiao Z J 2001 *Rev. Math. Phys.* **13** 545

[20] Zhou R G 1998 *J. Math. Phys.* **39** 2848

[21] Qiao Z J 2003 *Commun. Math. Phys.* **239** 309

[22] Cao C W and Geng X G 1990 *J. Phys.* A: *Math. Gen.* **23** 4117

[23] Levi D, Sym A and Wojciechowsk S 1983 *J. Phys.* A: *Math. Gen.* **16** 2423

[24] Babelon O and Viallet C M 1990 *Phys. Lett.* B **237** 411

[25] Qiao Z J 2002 *Finite-dimensional Integrable System and Nonlinear Evolution Equations* (Beijing: Chinese National Higher Education Press)

[26] Arnold V I 1978 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)

[27] Zhou Z X 2002 *J. Phys. Soc. Jpn.* **71** 1857

[28] Griffiths P and Harris J 1994 *Principles of Algebraic Geometry* (New York: Wiley)

[29] Dickey L A 1991 *Soliton Equations and Hamiltonian Systems* (Singapore: World Scientific)

[30] Hon Y C and Fan E G 2005 *J. Math. Phys.* **46** 032701