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M-Shape peakons, dehisced solitons, cuspons and new 1-peak solitons for the Degasperis–Procesi equation

Zhijun Qiao

Department of Mathematics, The University of Texas-Pan American, 1201W University Drive, Edinburg, TX 78541, United States

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Abstract

In this paper, we investigate all possible single traveling solitary wave solutions of the Degasperis–Procesi (DP) equation under the boundary condition $u \to A$ (A is a constant) as $x \to \pm\infty$. Regular peakons of the DP equation correspond to the case of $A = 0$. In the case of $A \neq 0$, we find new exact soliton solutions including cuspon, peakon, M-shape peakon, dehisced soliton, and double dehisced 1-peak soliton. In particular, we propose three new types of soliton solutions – M-shape peakon, dehisced soliton, and double dehisced 1-peak soliton, which are given in an explicit form. The most interesting is: for the DP equation the cuspon is a limit of those new peaked solutions solutions. We show some graphs to explain our new solutions.

1. Introduction

The balanced wave equation

$$m_t + m_x u + bmu_x = 0, \quad m = u - u_{xx} \tag{1}$$

was first proposed by Holm and Staley [10] in 2003, and recently has arisen a lot of attractive attentions. This family is integrable only when $b = 2,3$ [14]. $b = 2$ corresponds to the Camassa–Holm equation [4], while $b = 3$ matches the Degasperis–Procesi (DP) equation [7]. The DP equation has Lax pair [8] (therefore is integrable), and can be also extended to a whole integrable hierarchy of equations with parametric solutions under some constraints [16].

An important issue to study both the CH equation and the DP equation is to find their new solutions through investigating their intrinsic mathematical structures. Their peaked solitons, shock peakons, traveling solitary wave solutions both cusped and smooth, multi-peakon solutions, global weak solutions, piece-wise smooth solutions, and algebro–geoemtric solutions were successively dealt with in the articles [1–6,9–13,15–20].

In an earlier paper, we dealt with the traveling wave solutions for the CH equation under inhomogeneous boundary condition and found new soliton solutions both smooth and peaked [19]. In [11], the author studied the traveling wave

E-mail address: qiao@utpa.edu

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solutions of the DP equation, but did not give explicit solutions. In the present paper, we study all possible single exact/explicit soliton solutions of the DP equation

$$m_t + m_x u + 3mu_x = 0, \quad m = u - u_{xx} \tag{2}$$

through setting the traveling wave solution under the only boundary condition $u \to A$ (A is a constant) as $x \to \pm\infty$. We focus on the case of nonzero constant. A because $A = 0$ corresponds to the regular peakon solution [10]. In the case of $A \neq 0$, we give new exact soliton solutions including cuspon, peakon, M-shape peakon, dehisced soliton, and double dehisced 1-peak soliton. In particular, we propose three new types of soliton solutions – M-shape peakon, dehisced soliton, and double dehisced 1-peak soliton, which are presented in an explicit form. The most interesting is: for the DP equation the cuspon is a limit of those new peaked soliton solutions. The cusp soliton was first proposed by Wadati et al. [21] in 1980. Here in our paper, we will give new cuspon solution and show some graphs to see what our all new solutions look like.

2. New solitons

Let us consider the traveling wave solution of the DP equation (2) through a generic setting $u(x,t) = U(x - ct) = U(\xi)$, where c is the wave speed, and $\xi = x - ct$. Substituting it into the DP equation (2) gives

$$(U - c)(U - U'')' + 3U'(U - U'') = 0, \tag{3}$$

where $U' = U_\xi$, $U'' = U_{\xi\xi}$, $U''' = U_{\xi\xi\xi}$.

If $U - U'' = 0$, then the DP equation (2) has the following smooth solutions of $U(\xi) = c_1 e^{\xi} + c_2 e^{-\xi}$ with any real constants c_1, c_2, which is not interesting for us. On the other hand, the DP equation has the regular peakon solution [10] $u(x,t) = U(\xi) = ce^{-|x-ct|}$ with the following properties

$$U(0) = c, \quad U(\pm\infty) = 0, \quad U'(0-) = c, \quad U'(0+) = -c, \tag{4}$$

where $U'(0-)$ and $U'(0+)$ represent the left-derivative and the right-derivative at 0, respectively. Actually, the peakon solution is a special case of the following proposition.

Proposition 1. *The DP equation (2) has the following piecewise smooth traveling wave solution*

$$u(x,t) = -a\sinh(|x - ct|) + ce^{-|x-ct|}, \tag{5}$$

where $a \in \mathbb{R}$ is an arbitrary constant and c is the wave speed.

In particular, if we take $a = 0$ in this proposition, then (5) exactly gives the regular peakon solution $u(x,t) = ce^{-|x-ct|}$. If we take $a = -c$, (5) reads $u(x,t) = c\cosh(|x - ct|)$ which is a solution of $u - u_{xx} = 0$. If we take $c = 0$, (5) reads a stationary solution $u(x,t) = -a\sinh(|x|)$. If $a \neq 0$, solution (5) is not a soliton because $u(x,t)$ is divergent as $x \to \pm\infty$. So, in the following we shall seek for new soliton solutions under inhomogeneous boundary condition.

Let us now assume that U is neither a constant function nor satisfies $U - U'' = 0$. Then Eq. (3) can be changed to

$$\frac{(U - U'')'}{U - U''} = \frac{3U'}{c - U}. \tag{6}$$

Under the boundary condition $\lim_{\xi\to\pm\infty} U \to A$ (A is a non-zero constant), which is equivalent to the boundary condition $\lim_{x\to\pm\infty} u \to A$ in the DP equation, we take the integration twice on both sides of Eq. (6), and arrive at

$$U'^2 = \frac{(U - A)^2[(U - c + A)^2 + cA]}{(U - c)^2}, \tag{7}$$

which is able to be converted to

$$\frac{(U - c)\mathrm{d}U}{(U - A)\sqrt{(U - c + A)^2 + cA}} = -\mathrm{sign}(\xi)\mathrm{d}\xi. \tag{8}$$

Taking another integral on both sides of Eq. (8) yields the following implicit solution:

$$(V + \sqrt{V^2 + cA})\left(\frac{cA - (c - 2A)V + \sqrt{[(c - A)^2 + A(3A - c)](V^2 + cA)}}{V + c - 2A}\right)^r = e^{-(|\xi| + r\ln 2)}, \tag{9}$$

where $r = \frac{c-A}{\sqrt{(c-A)^2 + A(3A-c)}}$ and $V = U - c + A$.

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Generally, we cannot have an explicit solution $U(\xi)$ from Eq. (9), but we do obtain explicit solutions in the following two cases.

Case 1. $[r=1]$. Namely, $c=3A$ and $A>0$. Then Eq. (9) becomes

$$\frac{(2A-U-\sqrt{U^2-4UA+7A^2})(5A-U+2\sqrt{U^2-4UA+7A^2})}{A-U}=\frac{\mathrm{e}^{-|\xi|-\ln 2}}{A}.$$

Solving this equation for U leads to the explicit form of the solution of the ODE (7):

$$U=\frac{3}{2}A\left(\cosh X+\frac{1}{3}-\cosh\frac{X}{2}\sqrt{2\cosh X-\frac{10}{3}}\right),\qquad X=|\xi|+\ln(6A^2). \tag{10}$$

Therefore, we get the following theorem.

Theorem 1. *Under the inhomogeneous boundary condition* $\lim_{\xi\to\pm\infty}U\to A(A>0)$ *and* $c=3A$, *the DP equation* (2) *has the following new explicit soliton solution*

$$u(x,t)=U(\xi)=\frac{3}{2}A\left(\cosh X+\frac{1}{3}-\cosh\frac{X}{2}\sqrt{2\cosh X-\frac{10}{3}}\right),\qquad X=|x-3At|+\ln(6A^2). \tag{11}$$

The solution (11) can be classified to the following four categories based on the A's choices:

- if $A>\frac{\sqrt{2}}{2}$, the solution (11) presents new piecewise smooth 1-peak solitons with the following features

$$U(0)=\frac{36A^4+4A^2+1-(6A^2+1)\sqrt{36A^4-20A^2+1}}{8A},$$
$$U'(0+)=\frac{9A^3}{2}-\frac{1}{8A}-\frac{216A^6-60A^4+10A^2-1}{8A\sqrt{36A^4-20A^2+1}},$$
$$U'(0-)=-\frac{9A^3}{2}-\frac{1}{8A}-\frac{216A^6-60A^4+10A^2-1}{8A\sqrt{36A^4-20A^2+1}}.$$

See a representative graph in Fig. 1 for details in the case of $A=1$, $c=3$, where $U(0)=\frac{41-7\sqrt{17}}{8}=1.517$, $U'(0+)=\frac{35}{8}-\frac{165\sqrt{17}}{136}=-0.627$, and $U'(0-)=-\frac{35}{8}+\frac{165\sqrt{17}}{136}=0.627$.
- if $A=\frac{\sqrt{2}}{2}$, the solution (11) presents a new cusp soliton with the following features

$$U(0)=c=3A=\frac{3\sqrt{2}}{2}=2.121,$$
$$U'(0+)=-\infty,$$
$$U'(-0)=\infty.$$

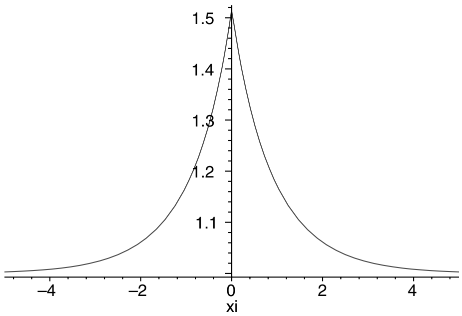

Fig. 1. 2d graph of a new piecewise smooth 1-peak soliton solution for the DP equation under the positive boundary $A=1$ and wave speed $c=3$. The limit of these piecewise smooth 1-peak solitons is a cusp soliton solution (see below) when $A\to\frac{\sqrt{2}}{2}+$.

See the graph in Fig. 2 for details. Actually, this cusp soliton solution is the limit of those piecewise smooth 1-peak solitons when $A \to \frac{\sqrt{2}}{2}+$.

- if $\frac{\sqrt{2}}{6} < A < \frac{\sqrt{2}}{2}$, the solution (11) presents a new type of soliton solution – dehisced soliton with the following features: $U(\xi)$ is real only on interval $[-\ln(2A^2), +\infty)$ or $(-\infty, \ln(2A^2)]$ (noticing: $\frac{1}{9} < 2A^2 < 1$ and $-2\ln 3 < \ln(2A^2) < 0$), and

$$U(-\ln(2A^2)) = U(\ln(2A^2)) = 3A = c,$$
$$U'(-\ln(2A^2)+) = -\infty,$$
$$U'(\ln(2A^2)-) = +\infty.$$

Apparently, $U(0) = \frac{36A^4+4A^2+1-(6A^2+1)\mathrm{i}\sqrt{(18A^2-1)(1-2A^2)}}{8A}$ is a complex number.

- if $0 < A \leqslant \frac{\sqrt{2}}{6}$, the solution (11) presents another new type of soliton solution-double dehisced 1-peak soliton with the following features: $U(\xi)$ is real only on interval $(-\infty, \ln(2A^2)]$ or $[\ln(18A^2), -\ln(18A^2)]$ or $[-\ln(2A^2), +\infty)$ (noticing: $0 < 2A^2 \leqslant \frac{1}{9}$, $0 < 18A^2 \leqslant 1$ and $-\infty < \ln(2A^2) \leqslant -2\ln 3, -\infty < \ln(18A^2) \leqslant 0$), and

$$U(0) = \frac{36A^4 + 4A^2 + 1 - (6A^2+1)\sqrt{(18A^2-1)(2A^2-1)}}{8A},$$
$$U(-\ln(2A^2)) = U(\ln(2A^2)) = 3A,$$
$$U(-\ln(18A^2)) = U(\ln(18A^2)) = 3A,$$
$$U'(0+) = \frac{9A^3}{2} - \frac{1}{8A} - \frac{216A^6 - 60A^4 + 10A^2 - 1}{8A\sqrt{36A^4 - 20A^2 + 1}},$$
$$U'(0-) = -\frac{9A^3}{2} + \frac{1}{8A} + \frac{216A^6 - 60A^4 + 10A^2 - 1}{8A\sqrt{36A^4 - 20A^2 + 1}},$$
$$U'(-\ln(2A^2)+) = -\infty, \quad U'(-\ln(18A^2)-) = +\infty$$
$$U'(-\ln(18A^2)+) = -\infty, \quad U'(-\ln(2A^2)-) = +\infty.$$

In particular, when $A = \frac{\sqrt{2}}{6}$, the corresponding double dehisced soliton solution $U(\xi)$ is defined on $(-\infty, -2\ln 3] \cup [0,0] \cup [2\ln 3, +\infty)$ and discontinuous at $\xi = -2\ln 3$, $\xi = 0$, $\xi = 2\ln 3$ with values of solution $U(0) = U(-2\ln 3) = U(2\ln 3) = \frac{\sqrt{2}}{2}$.

Case 2. $[r = -1]$. Namely, $c = 3A$ and $A < 0$. Then Eq. (9) becomes

$$\frac{5A - U - 2\sqrt{U^2 - 4UA + 7A^2}}{(A-U)(2A - U - \sqrt{U^2 - 4UA + 7A^2})} = \frac{\mathrm{e}^{|\xi| - \ln 2}}{A}.$$

Solving this equation for U gives another explicit solution of the ODE (7):

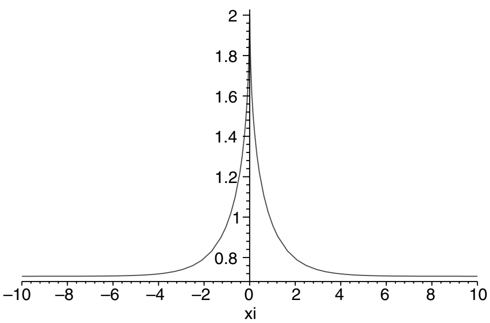

Fig. 2. 2d graph of a cusp soliton solution for the DP equation under the positive boundary $A = \frac{\sqrt{2}}{2}$ and wave speed $c = \frac{3\sqrt{2}}{2}$. This cusp soliton solution is the limit of those piecewise smooth 1-peak solitons when $A \to \frac{\sqrt{2}}{2}+$.

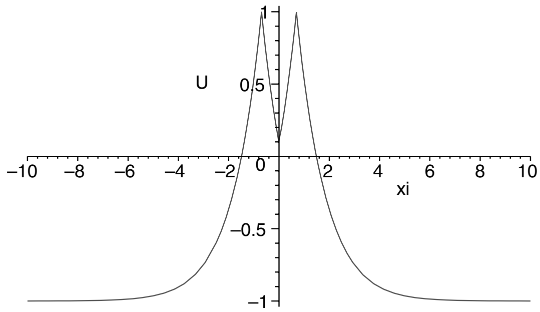

Fig. 3. 2d graph of a piecewise smooth "M"-shape-peaks soliton solution for the DP equation under the negative boundary $A = -1$ and wave speed $c = 3A = -3$.

$$
\begin{aligned}
&U = \frac{1}{2}A(1 - 3\cosh X + |\sinh\frac{X}{2}|\sqrt{18\cosh X + 30}),\\
&X = |\xi| - \ln 2.
\end{aligned} \tag{12}
$$

Therefore, we have the following theorem.

Theorem 2. *Under the inhomogeneous boundary condition* $\lim_{\xi\to\pm\infty} U \to A (A < 0)$ *and* $c = 3A$, *the DP equation* (2) *has the following new explicit single soliton solution*

$$
\begin{aligned}
&u(x,t) = U(\xi) = \frac{1}{2}A\left(1 - 3\cosh X + \left|\sinh\frac{X}{2}\right|\sqrt{18\cosh X + 30}\right),\\
&X = |\xi| - \ln 2 = |x - 3At| - \ln 2.
\end{aligned} \tag{13}
$$

A direct calculation yields

$$
\begin{aligned}
&U(0) = \frac{(\sqrt{105} - 11)A}{8},\\
&U'(0+) = \frac{(315 - 57\sqrt{105})A}{280},\\
&U'(0-) = -\frac{(315 - 57\sqrt{105})A}{280},\\
&U(\ln 2) = U(-\ln 2) = -A,\\
&U'(\ln 2+) = \sqrt{3}A, \quad U'(\ln 2-) = -\sqrt{3}A,\\
&U'(-\ln 2+) = \sqrt{3}A, \quad U'(-\ln 2-) = -\sqrt{3}A.
\end{aligned}
$$

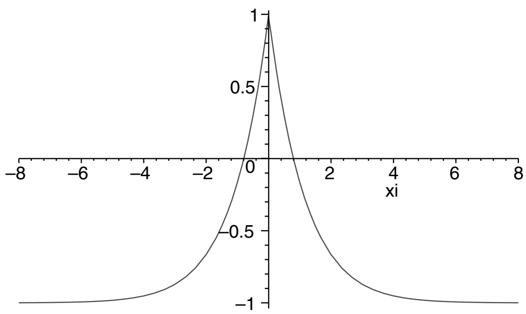

Fig. 4. 2d graph of a new piecewise smooth 1-peak soliton solution for the DP equation under the negative boundary $A = -1$.

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So this solution of the DP equation has three peaks (continuous and piecewise smooth) and its shape is like "M". Therefore, we call this new type of single soliton solution "M"-shape-peaks soliton. See a representative graph in Fig. 3 for more details in the case of $A = -1$, $c = -3$.

Furthermore, if we consider the solution (13) under no absolute value of ξ, then we obtain the following theorem.

Theorem 3. *Under the inhomogeneous boundary condition* $\lim_{\xi\to\pm\infty} U \to A(A<0)$ *and* $c = 3A$ *, the DP equation* (2) *has the following new single peaked soliton solution*

$$\begin{aligned} u(x,t) = U(\xi) &= \frac{1}{2}A\left(1 - 3\cosh\xi + \left|\sinh\frac{\xi}{2}\right|\sqrt{18\cosh\xi + 30}\right), \\ \xi &= x - 3At, \end{aligned} \tag{14}$$

with a piecewise smooth peak at $\xi = 0$ and $U(0) = -A,\ U'(0+) = \sqrt{3}A,\ U'(0-) = -\sqrt{3}A$. See the graph in Fig. 4 for more details.

3. Conclusions

In this paper, we propose three new types of solitons: M-shape peakons, dehisced solitons, and double dehisced 1-peak solitons. The most interesting discovery for the DP equation is that the cuspon is a limit of our new peaked soliton solutions (see Figs. 1 and 2). Through traveling wave setting, the DP equation is converted to the ODE (7) which we solve for all possible single soliton solutions of the DP equation. Actually, the ODE (7) has a physical meaning and can be cast into the Newton equation $U'^2 = V(U) - V(A)$ of a particle with a new potential $V(U)$

$$V(U) = U^2 + \frac{4cA(c-A)}{U-c} + \frac{A(c+A)(c-A)^2}{(U-c)^2}$$

In the paper, we successfully solve the Newton equation $U'^2 = V(U) - V(A)$ and give various soliton solutions both continuous and discontinuous.

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