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Cusp solitons and cusp-like singular solutions for nonlinear equations

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Abstract

This paper gives two new families of nonlinear partial differential equations (PDEs). One has cusp soliton solution while the other possesses the cusp-like singular traveling wave solution. A typical integrable system: Harry–Dym (HD) equation is able to be contained in both families and has cusp soliton solution as well as cusp-like singular traveling wave solution. We prove that the cusp solution of the HD equation is not stable and the cusp-like solution is not included in the parametric solutions of the HD equation.

1. Introduction

The Harry–Dym (HD) equation is an important integrable model in soliton theory [2]. This equation is related to the classical string problem [10] and has many applications in theoretical and experimental physics [3]. It has the bi-Hamiltonian structures, Lax pair, and cusp soliton solution [11] by the inverse scattering transformation [1,12,13]. The HD equation is able to work out from the Wadati–Konno–Ichikawa (WKI) hierarchy [13,9] through some reductions. Besides the HD equation, other nonlinear partial differential equations (PDEs), such as normalized Boussinesq equation and Ito-type wave equation, were also found to possess the cusp solutions [5–7]. However, the discussion about the cusp and cusp-like solutions is very few in the literature. In this paper, we will present two new families of nonlinear PDEs. One possesses the cusp soliton solution and includes the HD equation as its special case, while the other has the cusp-like singular traveling wave solution which also involves the HD equation. That is, the typical integrable system: the HD equation has cusp soliton solution (see Section 2.3) as well as cusp-like singular traveling wave solution (see Section 3). We prove that the cusp solution of the HD equation is not stable (see Section 2.3) and the cusp-like solution is not included in the parametric solutions of the HD equation (see Section 4).

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Throughout this paper, we give the following conventions: $u_{nx} = \frac{\partial^n u}{\partial x^n}$, $n = 0, 1, 2, \ldots$, $u_t = \frac{\partial u}{\partial t}$, $\partial = \frac{\partial}{\partial x}$, $\partial^n = \frac{\partial^n}{\partial x^n}$. Denote the set of all real numbers and integers by $\mathbb{R}$ and $\mathbb{Z}$, respectively. a, A represent two arbitrary constants in $\mathbb{R}$.

2. A family of nonlinear PDEs and cusp solutions

2.1. 1st order PDE

Let us first start from the simplest PDE

$$u_t + u_x = 0. \tag{1}$$

Apparently, this equation has the general solution

$$u = f(x - t), \quad \forall f. \tag{2}$$

Thus, the following cusp solution

$$u = \cosh^{-2}\xi, \quad \xi = a(x - t) + \tanh\xi + A, \tag{3}$$

is a special form of (2).

2.2. 2nd order PDE

Next, we consider the second order PDE

$$(2 - u)u_t - (1 - u)^{3/2}u_{2x} = 0. \tag{4}$$

This equation possesses the following cusp solution

$$u = \cosh^{-2}\xi, \quad \xi = a(x - at) + \tanh\xi + A. \tag{5}$$

In fact, computing

$$\xi_x = a\coth^2\xi, \quad \xi_t = -a^2\coth^2\xi,$$
$$u_x = \frac{-4a}{\sinh 2\xi}, \qquad u_t = \frac{4a^2}{\sinh 2\xi},$$
$$u_{2x} = 2a^2\frac{\cosh 2\xi}{\sinh^4\xi},$$

implies $(2 - u)u_t - (1 - u)^{3/2}u_{2x} = 0$.

2.3. 3rd order PDE i.e. Harry–Dym (HD) equation

In the case of the 3rd order PDE, we take the Harry–Dym (HD) equation

$$u_t + (1 - u)^3 u_{3x} = 0, \tag{6}$$

as a typical example. It possesses the following cusp solution

$$u = \cosh^{-2}\xi, \quad \xi = a(x - 4a^2t) + \tanh\xi + A. \tag{7}$$

This coincides with the result of Wadati, Ichikawa and Shimizu [11] obtained by using the inverse scattering transformation method [12,13,1].

In fact, making a transformation $U^{-1/2} = 1 - u$, $T = 2t$, $X = x$ casts Eq. (6) to the standard Harry–Dym (HD) equation:

$$U_T = \left(\frac{1}{\sqrt{U}}\right)_{3X} = \partial_X^3 U^{-1/2}, \tag{8}$$

where $\partial_X^3 = \frac{\partial^3}{\partial X^3}$. Therefore, the HD equation (8) has the following cusp soliton solution

$$U = \coth^4\xi, \quad \xi = a(X - 2a^2T) + \tanh\xi + A. \tag{9}$$

In Section 3, we will study a family of higher-order Harry–Dym type equations having the co-called cusp-like singular solutions other than cusp solutions.

Some calculations yield

$$u_{3x} = -8a^3 \frac{\cosh^5 \xi}{\sinh^7 \xi}, \quad u_t = \frac{16a^3}{\sinh 2\xi}.$$

Thus $u = \cosh^{-2}\xi$ satisfies the HD equation $u_t + (1 - u)^3 u_{3x} = 0$. (See Figs. 1 and 2).

The initial problem

$$u_t + (1-u)^3 u_{3x} = 0, \quad u(x, t_0) = \cosh^{-2}\xi, \quad \xi = a(x - 4a^2 t_0) + \tanh \xi + A,$$

has the solution

$$u(x, t) = \cosh^{-2}\xi, \quad \xi = a(x - 4a^2 t) + \tanh \xi + A.$$

Let us now see why the cusp solution is not stable. Let $\epsilon(x, t_0)$ be a perturbation term, namely,

$$v(x, t_0) = u(x, t_0) + \epsilon(x, t_0),$$

then corresponding to this initial value the equation $u_t + (1 - u)^3 u_{3x} = 0$ is assumed to have the solution

$$v(x, t) = u(x, t) + \epsilon(x, t).$$

Then we have

$$\epsilon_t + (1 - u - \epsilon)^3 \epsilon_{3x} + [3(1-u)\epsilon - 3(1-u)^2 - \epsilon^2]\epsilon u_{3x} = 0,$$

which can be linearized as

$$\epsilon_t - 3(1-u)^2 u_{3x}\epsilon + (1-u)^3 \epsilon_{3x} - 3(1-u)^2 \epsilon\epsilon_{3x} = 0. \tag{10}$$

We can choose good enough ϵ such that ϵ_{3x} is also very small. Thus the above equation is approximate to

$$\epsilon_t - 3(1-u)^2 u_{3x}\epsilon = 0. \tag{11}$$

Therefore, $\epsilon(x, t)$ has the following approximate formula

$$\epsilon(x, t) \simeq c e^{3(1-u)^2 u_{3x}}, \quad c = \text{constant}. \tag{12}$$

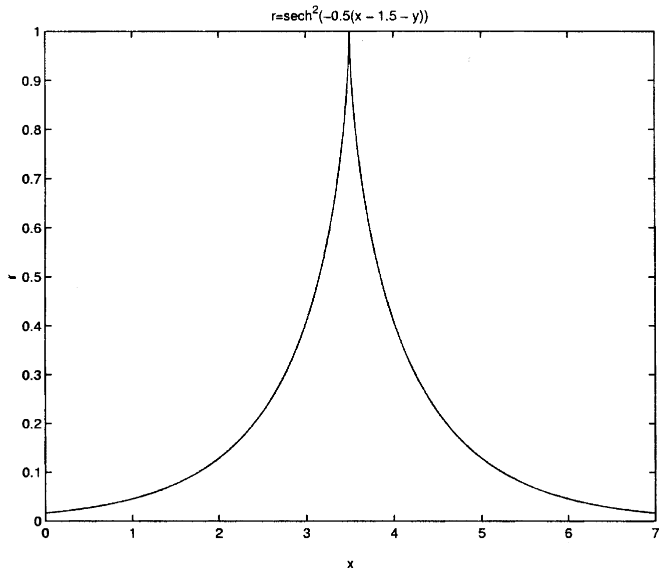

Fig. 1. Plane cusp solution for the well-known HD equation at $t = 1.5$, $a = -0.5$.

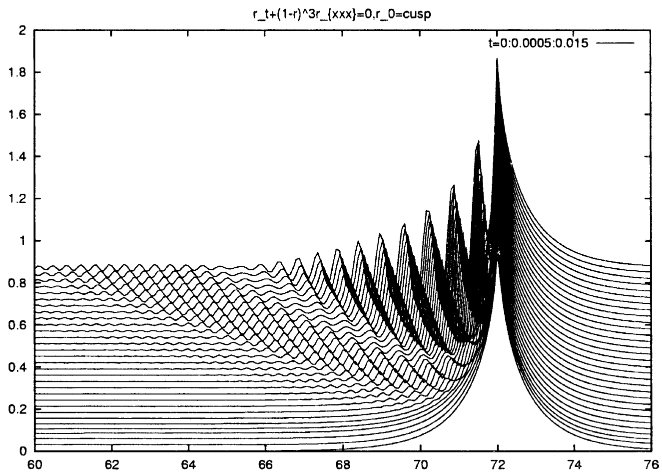

Fig. 2. Cusp traveling wave solution for the well-known HD equation under the cusp initial condition. Apparently, it is not stable. Mathematical reason is displayed in Section 2.3

Note

$$u_{3x} > 0, \quad x < 0,$$
$$u_{3x}\ \text{non-existence}, \quad x = 0,$$
$$u_{3x} < 0, \quad x > 0.$$

Thus, when $x < 0$, Eq. (12) changes more, but when $x > 0$, Eq. (12) changes less. That is why the cusp traveling wave solution takes on like Fig. 2. So, the cusp solution is not stable.

But, in the case of Gaussian initial condition, the Gaussian solution is stable, see the Figure.

2.4. 4th order PDE

The fourth order PDE we propose here is

$$(2 + 5u)u_t - (1 - u)^{9/2}u_{4x} = 0. \tag{13}$$

This equation has the following cusp solution

$$u = \cosh^{-2}\xi, \quad \xi = a(x - 4a^3t) + \tanh\xi + A. \tag{14}$$

Because of

$$u_{4x} = 8a^4(6 + \cosh(2\xi))\frac{\cosh^6\xi}{\sinh^{10}\xi}, \quad u_t = \frac{16a^4}{\sinh 2\xi},$$

a direct check reveals that Eq. (14) satisfies the 4th order PDE (13).

2.5. 5th order PDE

We consider the fifth order PDE

$$(2 + 18u + 15u^2)u_t + (1 - u)^6u_{5x} = 0. \tag{15}$$

It possesses the following cusp solution

$$u = \cosh^{-2}\xi, \quad \xi = a(x - 8a^4t) + \tanh\xi + A. \tag{16}$$

The 5th order Eq. (15) does not belong to the HD hierarchy:

$$u_{t_j} = J(J^{-1}K)^j \cdot u^{-1/2}, \quad j = 1, 2, \ldots, \tag{17}$$

where $K = \partial^3$, $J = \partial u + u\partial$, and the sign $\cdot$ means the operator action on the function $u^{-1/2}$. Apparently, the first one is the HD equation (8), the second one is also 5th order PDE but not Eq. (15) and has no cusp solution.

A direct calculation of

$$u_{5x} = -4a^5(99 + 40\cosh(2\xi) + \cosh(4\xi))\frac{\cosh^7\xi}{\sinh^{13}\xi},$$
$$u_t = \frac{32a^5}{\sinh 2\xi},$$

verifies that Eq. (16) satisfies the 5th order PDE (15).

2.6. 6th order PDE

The sixth order PDE we consider here is

$$(4 + 94u + 252u^2 + 105u^3)u_t - (1-u)^{15/2}u_{6x} = 0. \tag{18}$$

It possesses the following cusp solution

$$u = \cosh^{-2}\xi, \quad \xi = a(x - 8a^5t) + \tanh\xi + A. \tag{19}$$

After computing

$$u_{6x} = 2a^6(2140 + 1399\cosh(2\xi) + 100\cosh(4\xi) + \cosh(6\xi))\frac{\cosh^8\xi}{\sinh^{16}\xi},$$
$$u_t = \frac{32a^6}{\sinh 2\xi},$$

we find that Eq. (19) is a solution of the 6th order PDE (18).

2.7. 7th order PDE

The seventh order PDE we study here is

$$(1 + 54u + 330u^2 + 420u^3 + 105u^4)u_t + (1-u)^9u_{7x} = 0. \tag{20}$$

It possesses the following cusp solution

$$u = \cosh^{-2}\xi, \quad \xi = a(x - 64a^6t) + \tanh\xi + A. \tag{21}$$

A lengthy calculation

$$u_{7x} = -a^7(58355 + 51296\cosh(2\xi) + 6604\cosh(4\xi) + 224\cosh(6\xi) + \cosh(8\xi))\frac{\cosh^9\xi}{\sinh^{19}\xi},$$
$$u_t = \frac{256a^7}{\sinh 2\xi},$$

implies that Eq. (21) satisfies the 7th order PDE (20).

2.8. General case

In general, we consider the following PDE

$$P_{n-2}(u)u_t + (-1)^n(1-u)^{\frac{3}{2}n}u_{(n+1)x} = 0, \quad n = 1, 2, 3, \ldots, \tag{22}$$

where $P_{n-2}(u)$ is a $(n-2)$-th order polynomial defined by:

$$P_{n-2}(u) = \begin{cases} 2 - u, & n = 1, \\ \sum_{k=0}^{n-2} a_k u^{n-2-k}, & n = 2, 3, \ldots, \end{cases} \tag{23}$$

with the constants a_k, $k = 0, 1, 2, \ldots, n-2$, to be determined. Eq. (22) possesses the cusp soliton solution:

$$u = \cosh^{-2}\xi, \quad \xi = ax - ct + \tanh\xi + A, \tag{24}$$

where $c = c(a,n)$ is a constant related to constant a and n. When we substitute Eq. (24) into Eq. (22) and expand every term in terms of $\cosh^2\xi$, then each constant a_k is uniquely determined and constant c is expressed in term of constant a. We detailedly describe this procedure as follows.

Proposition 1. *Assume*

$$u_{nx} = 2^{7-n}(-a)^n \frac{\cosh^{n+2}\xi}{\sinh^{3n-2}\xi} \sum_{k=0}^{n-3} b_k \cosh 2k\xi, \quad n \geqslant 3, \quad b_{n-3} = 1, \tag{25}$$

where the constants $b_0, b_1, \ldots, b_{n-4}$ *may be determined from the solution Ansatz (24). Then*

$$u_{(n+1)x} = 2^{6-n}(-a)^{n+1} \frac{\cosh^{n+3}\xi}{\sinh^{3n+1}\xi} \sum_{k=0}^{n-2} d_k \cosh 2k\xi, \quad d_{n-2} = 1, \tag{26}$$

where constants $d_0, d_1, \ldots, d_{n-3}$ *are determined by constants* $b_0, b_1, \ldots, b_{n-4}$.

Proof. Noticing the following identities

$$\begin{aligned} \xi_x &= a\coth^2\xi, \\ 2\sinh^2\xi &= \cosh 2\xi - 1, \\ 2\cosh^2\xi &= \cosh 2\xi + 1, \\ 2\cosh\xi\cosh\eta &= \cosh(\xi-\eta) + \cosh(\xi+\eta), \\ 2\sinh\xi\sinh\eta &= -\cosh(\xi-\eta) + \cosh(\xi+\eta), \end{aligned}$$

and making derivative in x on both sides of Eq. (25), we obtain

$$\begin{aligned} u_{(n+1)x} &= B_n \frac{\cosh^{n+3}\xi}{\sinh^{3n+1}\xi}\left[\sinh 2\xi \sum_{k=1}^{n-3} b_k \sinh 2k\xi + \left((n+2)\sinh^2\xi - (3n-2)\cosh^2\xi\right)\sum_{k=0}^{n-3} b_k \cosh 2k\xi\right] \\ &= B_{n+1} \frac{\cosh^{n+3}\xi}{\sinh^{3n+1}\xi}\left[2(2n + (n-2)\cosh 2\xi)\sum_{k=0}^{n-3} b_k \cosh 2k\xi - 2\sinh 2\xi \sum_{k=1}^{n-3} k b_k \sinh 2k\xi\right] \\ &= B_{n+1} \frac{\cosh^{n+3}\xi}{\sinh^{3n+1}\xi}(d_0 + d_1\cosh 2\xi + \cdots + d_{n-3}\cosh 2(n-3)\xi + \cosh 2(n-2)\xi), \end{aligned}$$

where $B_n = 2^{7-n}(-a)^n \cdot a$, $d_0, d_1, \ldots, d_{n-3}$ are evidently determined by constants $b_0, b_1, \ldots, b_{n-4}$ and n, and in particular the last term $\cosh 2(n-2)\xi$ is obtained through calculating

$$2(n-2)\cosh 2(n-3)\xi\cosh 2\xi - 2(n-3)\sinh 2(n-3)\xi\sinh 2\xi = \cosh 2(n-2)\xi + (4n-5)\cosh 2(n-4)\xi.$$

For the u defined by Eq. (24), we have

$$u_t = \frac{4c}{\sinh 2\xi} = \frac{2c}{\cosh\xi\sinh\xi}. \tag{27}$$

Substituting Eqs. (24) and (26) into Eq. (22) and choosing the constant $c = -2^{5-n}a^{n+1}$ lead to

$$\sum_{k=0}^{n-2}\left(\frac{a_k}{\cosh^{2(n-2-k)}\xi} + \frac{d_k}{\cosh^{2(n-2)}\xi}\cosh 2k\xi\right) = 0, \quad d_{n-2} = 1, \tag{28}$$

i.e.

$$\sum_{k=0}^{n-2}\left(a_k\cosh^{2k}\xi + d_k\cosh 2k\xi\right) = 0, \quad d_{n-2} = 1. \tag{29}$$

Because $\cosh 2k\xi$ must be a k-th degree polynomial of term $\cosh^2\xi$, all constants a_k are able to be uniquely given through constants $d_0, d_1, \ldots, d_{n-3}$. Thus, we obtain the following theorem.

Theorem 1. *The* $(n+1)$*-th order equation (22) has the cusp solution given by Eq. (24) with the speed* $c = -2^{5-n\ an+1}$ *(*a *is an arbitrary constant). The coefficient constants* a_k, $k = 0, 1, \ldots, n-2$, *in Eq. (22) are determined by Eq. (29).*

By this theorem, all figures of the cusp solutions (24) for Eq. (22) have the same shape as the Harry–Dym case (see Fig. 1), but their velocities are different with the different order $n + 1$ of (22). These cusp solutions (24) are unstable (see Fig. 2).

3. Another family of nonlinear PDEs and cusp-like singular solutions

Based on the discussion of the HD equation (8), we furthermore propose the following more generalized equation:

$$u_t = \partial^l u^m, \quad l \geqslant 1, \quad m \neq 1, \quad l \in \mathbb{Z}, \quad m \in \mathbb{R}. \tag{30}$$

Assume this equation has the traveling wave solution

$$u(x,t) = f(\xi), \quad \xi = x - ct, \tag{31}$$

then

$$cf + \frac{d^{l-1}}{d\xi^{l-1}} f^m + c_0 = 0, \quad c_0 = \text{constant}. \tag{32}$$

In order to solve this equation, we set $f = \xi^\alpha$ where α is a constant to be determined. Substituting this into Eq. (32) and choosing $c_0 = 0$, we obtain

$$\alpha = \frac{l-1}{m-1}, \quad c = -\prod_{k=0}^{l-2} \left(\frac{m(l-1)}{m-1} - k \right). \tag{33}$$

Thus, the general Eq. (30) has the following traveling wave solution

$$u(x,t) = \left[x + t \prod_{k=0}^{l-2} \left(\frac{m(l-1)}{m-1} - k \right) \right]^{\frac{l-1}{m-1}}. \tag{34}$$

Apparently, if $m < 1$ this solution has singularity at $x_0 = ct_0$ (t_0 is some time), and if $m > 1$ this solution is a smooth traveling wave solution.

Let us give some special cases of Eq. (30).

- Choosing $m = -1/2$, $l = 3$, i.e. Eq. (30) becomes the HD equation (8), we have $\alpha = -4/3$, $c = 2/9$. Therefore the HD equation (8) possesses a cusp-like singular traveling wave solution

$$u(x,t) = \left(x - \frac{2}{9} t \right)^{-4/3}. \tag{35}$$

Obviously, this is very different from its cusp solution (9).

- Choosing $m = -2/3$, $l = 5$, i.e. Eq. (30) becomes $u_t = (u^{-2/3})_{5x}$, we have $\alpha = -12/5$, $c = -336/625$. Therefore the equation $u_t = (u^{-2/3})_{5x}$ possesses a cusp-like singular traveling wave solution

$$u(x,t) = \left(x + \frac{336}{625} t \right)^{-12/5}. \tag{36}$$

- Choosing $m = 1/2$, $l = 3$, i.e. Eq. (30) becomes $u_t = (u^{1/2})_{3x}$, we have $\alpha = -4$, $c = -6$. Therefore the equation $u_t = (u^{1/2})_{3x}$ possesses a cusp-like singular traveling wave solution

$$u(x,t) = (x + 6t)^{-4}. \tag{37}$$

- Choosing $m = 2/3$, $l = 5$, i.e. Eq. (30) becomes $u_t = (u^{2/3})_{5x}$, we have $\alpha = -12$, $c = -7920$. Therefore the equation $u_t = (u^{2/3})_{5x}$ possesses a cusp-like singular traveling wave solution

$$u(x,t) = (x + 7920t)^{-12}. \tag{38}$$

The figures of Eqs. (37) and (38) are seen in Fig. 4.

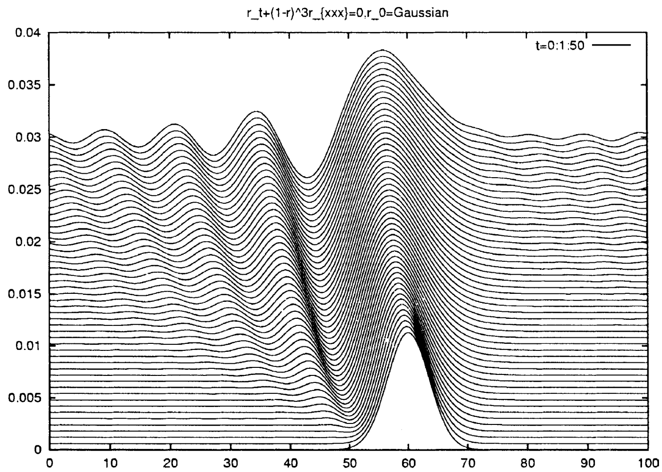

Fig. 3. Initial value problem for the well-known HD equation under the Gaussian initial condition. No cusps appear.

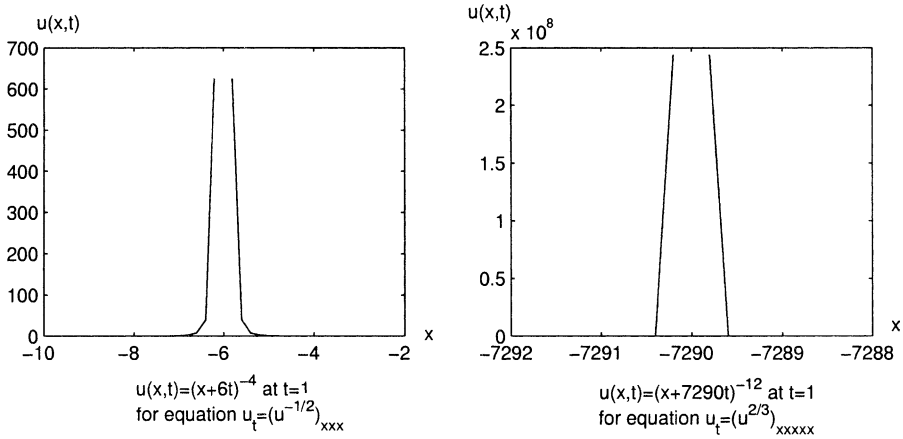

Fig. 4. Two cusp-like singular traveling wave solutions.

- For the above special cases, generally choosing $m = -n/(n+1)$, $n > 0$, $n \in \mathbb{R}$, then $m < 1$ and Eq. (30) becomes $u_t = \partial^{2n+1}u^{-n/(n+1)}$, and we have $\alpha = -\frac{2n(n+1)}{2n+1}$, $c = -\prod_{k=0}^{2n-1}\left(\frac{2n^2}{2n+1} - k\right)$. Therefore the equation $u_t = \partial^{2n+1}u^{-n/(n+1)}$ possesses a cusp-like singular traveling wave solution

$$u(x,t) = \left[x + t\prod_{k=0}^{2n-1}\left(\frac{2n^2}{2n+1} - k\right)\right]^{-\frac{2n(n+1)}{2n+1}}. \tag{39}$$

Apparently, Eqs. (35) and (36) are two reductions of Eq. (39) when $n = 1, 2$, respectively.

If choosing $l = 2n + 2$, we obtain the following simpler cusp-like singular traveling wave solution

$$u(x,t) = \left(x + t\prod_{k=0}^{2n}(n-k)\right)^{-(n+1)}, \tag{40}$$

for equation $u_t = \partial^{2n+2}u^{-n/(n+1)}$.

Choosing $m = n/(n + 1)$, $n > 0$, $n \in \mathbb{R}$, then $m < 1$ and Eq. (30) becomes $u_t = \partial^l u^{n/(n+1)}$, we have $\alpha = -(n + 1)(l - 1)$, $c = (-1)^l\prod_{k=0}^{l-2}(n(l-1)+k)$. Therefore the equation $u_t = \partial^l u^{n/(n+1)}$ possesses a cusp-like singular traveling wave solution

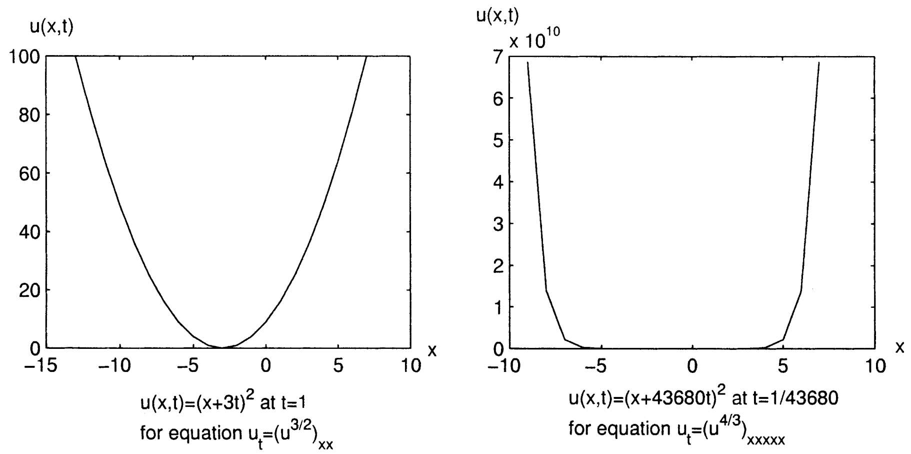

Fig. 5. Two smooth traveling wave solutions.

$$u(x,t) = \left[x + t(-1)^{l+1}\prod_{k=0}^{l-2}(n(l-1)+k)\right]^{-(n+1)(l-1)}. \tag{41}$$

Obviously, Eqs. (37) and (38) are two reductions of Eq. (41) when $n = 1$, $l = 3$; $n = 2$, $l = 5$, respectively.

- Choosing $m = (n + 1)/n$, $n > 0$, $n \in \mathbb{R}$, then $m > 1$ and Eq. (30) becomes $u_t = \partial^l u^{(n+1)/n}$, we have $\alpha = n(l - 1)$, $c = -\prod_{k=0}^{l-2}((n+1)(l-1)-k)$. Therefore this equation possesses a smooth traveling wave solution

$$u(x,t) = \left[x + t\prod_{k=0}^{l-2}((n+1)(l-1)-k)\right]^{n(l-1)}. \tag{42}$$

For example, let $m = 4/3$, $l = 5$, then the equation $u_t = \partial^5 u^{4/3}$ has the following smooth traveling wave solution

$$u(x,t) = (x + 43680t)^{12}, \tag{43}$$

and let $m = 3/2$, $l = 2$, then the equation $u_t = (u^{3/2})_{2x}$ has the smooth traveling wave solution

$$u(x,t) = (x + 3t)^2. \tag{44}$$

The figures of Eqs. (44) and (43) are seen in Fig. 5.

4. Comparison with the parametric solutions for the HD equation

Paper [8] dealt with the parametric solution for the Harry–Dym hierarchy, especially gave the parametric solution of the HD equation (6). Let us briefly recall this result. The spectral problem associated with the HD equation (6) is

$$y_x = My, \tag{45}$$

where

$$y = (y_1, y_2)^{\mathrm{T}}, \quad M = \begin{pmatrix} -i\lambda & \lambda(u-1) \\ -\lambda & i\lambda \end{pmatrix}, \quad i^2 = -1.$$

As shown in Ref. [8], we provided the following constraint

$$u = \frac{1}{\langle \Lambda p, p\rangle^2}, \tag{46}$$

which is located between the potential u and the spectral functions $y_j = (p_j, q_j)^{\mathrm{T}}$ of spectral parameters λ_j ($j = 1, \ldots, N$). Here $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, $p = (p_1, \ldots, p_N)^{\mathrm{T}}$, $q = (q_1, \ldots, q_N)^{\mathrm{T}}$, and $\langle\cdot,\cdot\rangle$represents the standard inner-product in $R^{N.}$

Under this constraint the spectral problem (45) is nonlinearized as a canonical Hamiltonian system in R^{2N}:

$$(H):\quad \begin{cases} q_x = -i\Lambda q + (\langle \Lambda p, p\rangle^{-2} - 1)\Lambda p = \frac{\partial H}{\partial p}, \\ p_x = -\Lambda q + i\Lambda p = -\frac{\partial H}{\partial q}, \end{cases} \tag{47}$$

with

$$H = -i\langle \Lambda p, q\rangle + \frac{1}{2}\langle \Lambda q, q\rangle - \frac{1}{2}\langle \Lambda p, p\rangle - \frac{1}{2}\langle \Lambda p, p\rangle^{-1}. \tag{48}$$

This Hamiltonian system (H) is completely integrable [8], and

$$u = \frac{1}{\langle \Lambda p(x,t), p(x,t)\rangle^2}, \tag{49}$$

is a solution of the HD equation (6), where $q(x,t)$, $p(x,t)$ is a common solution of the compatible Hamiltonian systems (H) and (F_0). Here

$$(F_0):\quad \begin{cases} q_t = \frac{\partial F_0}{\partial p}, \\ p_t = -\frac{\partial F_0}{\partial q}, \end{cases} \tag{50}$$

with

$$F_0 = \langle \Lambda^3 p, p\rangle\langle \Lambda p, p\rangle^{-1} + \langle \Lambda^2 p, p\rangle^2 - \langle \Lambda^3 p, p\rangle\langle \Lambda p, p\rangle + \langle \Lambda^3 q, q\rangle\langle \Lambda p, p\rangle \\ - \langle \Lambda^2 p, q\rangle^2 + 2i(\langle \Lambda^2 p, q\rangle\langle \Lambda^2 p, p\rangle - \langle \Lambda^3 p, q\rangle\langle \Lambda p, p\rangle).$$

The cusp solution (9) of the HD equation (8) cannot be included in its parametric solution (49). In fact, Eq. (47) is equivalent to

$$p_{xx} = -\langle \Lambda p, p\rangle^{-2}\Lambda^2 p. \tag{51}$$

When $N = 1$, this equation becomes

$$p^3 p_{xx} = -1, \tag{52}$$

which has the general solution

$$p(x) = \pm\sqrt{c_1 x^2 - 2c_1 c_2 x + c_1 c_2^2 - \frac{1}{c_1}},\ \forall \text{ constants } c_1 \neq 0,\ c_2 \in \mathbb{R}. \tag{53}$$

Therefore,

$$u = \lambda^{-2} p^{-4} = \lambda^{-2}\left(c_1 x^2 - 2c_1 c_2 x + c_1 c_2^2 - \frac{1}{c_1}\right)^{-2}. \tag{54}$$

Apparently, this is NOT the single cusp solution (9).

Furthermore, the cusp-like singular traveling wave solution (35) of the HD equation (8) cannot be included in its parametric solution (49), either. In fact, let $X = x - \frac{2}{9}t$ and substitute Eq. (35) into the HD spectral problem $\psi_{xx} + \lambda^2 u\psi = 0$. Then we obtain

$$X^{4/3}\psi_{XX} + \lambda^2\psi = 0. \tag{55}$$

Solving this equation gives its general solution

$$\psi = \left(\frac{c_2}{3\lambda} - c_1 X^{1/3}\right)\sin(3\lambda X^{1/3}) - \left(\frac{c_1}{3\lambda} + c_2 X^{1/3}\right)\cos(3\lambda X^{1/3}), \tag{56}$$

where c_1, c_2 are two arbitrary constants, and $\lambda \neq 0$. Let $\lambda_1, \ldots, \lambda_N$ be N spectral parameters of the HD spectral problem $\psi_{xx} + \lambda^2 u\psi = 0$, and each p_j the spectral function corresponding to $\lambda_j \neq 0$. Then

$$p_j = \left(\frac{c_{j2}}{3\lambda_j} - c_{j1} X^{1/3}\right)\sin(3\lambda_j X^{1/3}) - \left(\frac{c_{j1}}{3\lambda_j} + c_{j2} X^{1/3}\right)\cos(3\lambda_j X^{1/3}),\ \forall c_{j1},\ c_{j2} \in \mathbb{R},\ j = 1, \ldots, N.$$

Thus, under choosing $c_{j1} = c_{j2}$, we have

$$\langle \Lambda p, p\rangle = X^{2/3}\langle \Lambda \mathbf{c}_1, \mathbf{c}_1\rangle + \frac{1}{9}\langle \Lambda^{-1}\mathbf{c}_1, \mathbf{c}_1\rangle - \sum_{j=1}^{N} c_{j1}^2\left[\frac{2}{3}X^{1/3}\cos(6\lambda_j X^{1/3}) + \left(\lambda_j X^{2/3} - \frac{1}{9\lambda_j}\right)\sin(6\lambda_j X^{1/3})\right],$$

where $\mathbf{c}_1 = (c_{11}, \ldots, c_{N1})^T$. Apparently, this is NOT equal to $X^{2/3}$ regardless of how to select the constant vector $\mathbf{c}_1$. Therefore

$$\frac{1}{\langle \Lambda p(x,t), p(x,t)\rangle^2} \neq \left(x - \frac{2}{9}t\right)^{-4/3}, \tag{57}$$

i.e. the cusp-like singular traveling wave solution (35) cannot be included in its parametric solution (49).

5. Conclusions

In this paper, we give an approach how to get the cusp soliton solution for the family of PDE (22). All cusp solutions for this family have the same shape figure as the HD equation has, but their velocities are different with the different order $n + 1$ of PDE.

We also propose another new family of PDE (30), which has the so-called cusp-like singular traveling wave solution (34) instead of cusp solution (24). As we discussed in Section 3, if $m < 1$ in Eq. (30), then the solution (34) looks like a cusp, but not a cusp because it has singularity at $x_0 = ct_0$ (here $c = -\prod_{k=0}^{l-2}(\frac{m(l-1)}{m-1} - k)$ and t_0 is some time), and its all plane figures are same in shape as Fig. 4 except different velocities c. If $m > 1$ in Eq. (30), then the solution (34) is a smooth traveling wave solution, which has the same shape as Fig. 5 except different velocities c.

In Ref. [4], we obtained a parametric solution of the 5th-order equation $u_t = (u^{-2/3})_{5x}$. The parametric solution cannot include its traveling wave solution $u = (x + \frac{336}{625}t)^{-\frac{12}{5}}$ because the parametric solution is smooth, but the traveling wave solution is singular.

Traveling wave solution $u = (x + \frac{336}{625}t)^{-\frac{12}{5}}$ for equation $u_t = \partial^5 u^{-2/3}$ is singular at some certain point x with the different time t. That is, this singularity travels with the time t (also see Ref. [4]). Actually, when $m < 1$ the traveling wave solution (34) for general Eq. (30) is also matching this case. A natural question arises here: is Eq. (30) integrable for all $l \geqslant 1$, $m \in \mathbb{R}$ or for what kind of $l \geqslant 1$, $m \in \mathbb{R}$ it is integrable? We will in detail discuss this elsewhere.

The Harry–Dym equation has the cusp-like singular traveling wave solution $u(x,t) = (x - \frac{2}{9}t)^{-4/3}$, but this is not the cusp soliton which Wadati, Ichikawa and Shimizu described this in Ref. [11], because the traveling wave solution is singular, but the cusp is continuous.

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References

[1] Ablowitz MJ, Kaup DJ, Newell AC, Segur H. Stud Appl Math 1974;53:249–315.
[2] Ablowitz MJ, Segur H. Soliton and the inverse scattering transform. Philadelphia: SIAM; 1981.
[3] Calogero F, Degasperis A. Spectral transform and solitons I: Studies in mathematics and applications. Amsterdam: North-Holland; 1982.
[4] Qiao Z. A new integrable hierarchy, and parametric solutions and traveling wave solutions. Math Phys Anal Geom, in press [preprint 2002].
[5] Kawamoto S. Cusp and usual solitons of the normalized Boussinesq equation. J Phys Soc Jpn 1984;53:2465–71.
[6] Kawamoto S. Cusp soliton solutions of the Ito-type coupled nonlinear wave equation. J Phys Soc Jpn 1984;53:1203–5.
[7] Nejoh Y. Cusp solitons, shock waves and envelope solitons in a new nonlinear transmission line. J Phys A 1987;20:1733–41.
[8] Qiao Z. A completely integrable system associated with the Harry–Dym hierarchy. J Nonlin Math Phys 1994;1:65–74.
[9] Qiao Z. A completely integrable system and parametric representation of solutions of the Wadati–Konno–Ichikawa hierarchy. J Math Phys 1995;36:3535–40.
[10] Sabatier PC. On some spectral problems and isospectral evolutions connected with the classical string problem II: Evolution equation. Lett Nuovo Cimento 1979;26:483–6.
[11] Wadati M, Ichikawa YH, Shimizu T. Cusp soliton of a new integrable nonlinear evolution equation. Prog Theor Phys 1980;64:1959–67.
[12] Wadati M, Konno K, Ichikawa YH. A generalization of inverse scattering method. J Phys Soc Jpn 1979;46:1965–6.
[13] Wadati M, Konno K, Ichikawa YH. New integrable nonlinear evolution equations. J Phys Soc Jpn 1979;47:1698–700.