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COMMUTATOR REPRESENTATIONS OF THREE ISOSPECTRAL EQUATION HIERARCHIES

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Based on Cao's theory, this paper presents the commutator representation of three new isospectral equation hierarchies and explains that two different nonlinear isospectral evolution equation hierarchies can be associated with the same spectral problem.

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§1. INTRODUCTION AND GENERAL THEORY

It is a very interesting topic to seek the commutator representation of isospectral equations or the Lax representation. Its key lies in finding out a differenti al operator W_m such that $L_t = [W_m, L]$. Professor Cao Cewen presented the correct frame of this problem in [1] and elaborated the significance and the function of commutator representations. We have know that the commutator representation lays a foundation for the discussion of Lax system nonlinearization and Liouville's integrable system[2,3]. So far, some results[4-7] on the commutator representation have been obtained. In this paper by using Cao's thought[1], we supply the commutator representation of three new nonlinear isospectral evolution equations[8], but the method here is a little different from [4-7]. As for the integrable systems produced by nonlinearization of three spectral problem in [8], we have had some conclusion which are presented in another paper.

Consider the spectral problem

$$y_x = U_y = U(\lambda, w)y, \tag{1.1}$$

where λ is the spectral parameter, and $w = w(x)$ is a potential functional vector. The underlying interval Ω is $(-\infty, \infty)$ or $(0, T)$ and the potential $w(x)$ decays to zero at infinity or is periodic. $y_x = \frac{\partial y}{\partial x}$; $x \in \Omega$. Each element in $U(\lambda, w)$ is related to λ, $w(x)$ and its derivatives.

Following the methid offered in [9,10], we consider the auxiliary problem

$$y_t = V(^{(n)}y = V^{(n)}(\lambda, w)y, \quad n \geq 0. \tag{1.2}$$

Then the zero curvature equation determined by the compatibility condition of (1.1) and (1.2)

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0, \quad n \geq 0$$

can give nonlinear integrable evolution equation

$$w_t = J\mathcal{L}^n f_0, \quad n \geq 0, \tag{1.3}$$

Where J and $\mathcal{L}$ are integro-differential operators which depend on the potential $w(x)$, and J is generally a Hamitonian operator. $\Phi = \mathcal{L}^*$ is the public recursive operator of equation hierarchies (1.3).

Definition 1.1. Integro-differential operators J and $K = J\mathcal{L}$ are called the pair of Lenard's operators corresponding to the spectral (1.1).

Choose $G_{-1} \in \operatorname{Ker} J = \{G|JG = 0\}$ and order $KG_{-1} = JG_j$, $j = 0, 1, 2, \cdots$, the sequence $\{G_j\}_{j=-1}^{\infty}$ recursively obtained as above is called the Lenard's recursive sequence of (1.1). Generally, G_j is the poiynomial of w and its derivatives, and is unique if its constant term is required to be zero. $X_j = JG_j = KG_{j-1}$ is said to be the vector field of (1.1). The hierachy $w_t = X_j(w)$ produced by X_j is called the nonlinear evolution equation hierarchies of (1.1).

Rewrite (1.1) as

$$L(w)y = \lambda y. \tag{1.4}$$

Definition 1.2. $L_{*w}(\xi) = \frac{d}{d\varepsilon}|_{\varepsilon=0} L(w + \varepsilon\xi), \quad \forall \xi.$
Simply write L_{*w} as L_* below.

Consider the operator equation generated through the pair of Lenard's operator K and J

$$[V, L] = L_*(KG) - L_*(JG)L, \tag{1.5}$$

where G is a certain vector function; $V = V(G)$ is an undetermined differential operator; and $[\cdot, \cdot]$ is the Lie bracket.

Theorem 1.1. Let $\{G_j\}$ be the Lenard's sequence. For each G_j ($j = -1, 0, 1, \cdots$) there exists an operator solution $V_j = V(G_j)$ of (1.5). Then

$$[W_m, L] = L_*(X_m), \quad m = 0, 1, 2, \cdots, \tag{1.6}$$

where $W_m = \sum_{j=0}^{m} V_{j-1} L^{m-j}$.

Proof.

$$\begin{aligned}
[W_m, L] &= \sum_{j-0}^{m} [V_{j-1}, L] L^{m-j} \\
&= \sum_{j=0}^{m} L_*(KG_{j-1}) L^{m-j} - L_*(JG_{j-1}) L^{m-j+1} \\
&= \sum_{j=0}^{m} L_*(JG_j) L^{m-j} - L_*(JG_{j-1}) L^{m-j+1} \\
&= L_*(X_m).
\end{aligned}$$

Theorem 1.2. Let the conditions of Theorem 1.3 be tenable and L_* be an injective homomorphism. Then the evolution equation hierarchies of (1.1) has the commutator

$$L_t = [W_m, L], \tag{1.7}$$

in (1.7). W_m is the same as in Theorem 1.3.

Proof.

$$L_t = L_*(w_t),$$
$$L_t - [W_m, L] = L_*(w_t) - L_*(X_m) = L_*(w_t - X_m).$$

Note L_* is injective.

Corollary 1.1. $w_t = X_m(w)$ is the compatibility condition of $Ly = \lambda y$ and $y_t = W_m y$.

Corollary 1.2. The potential vector function $w(x)$ satisfies the stationary nonlinear system determined by the vector field X_j

$$X_N + a_1 X_{N-1} + \cdots + a_N X_0 = 0, \tag{1.8}$$

if and only if

$$[W_N + a_1 W_{N-1} + \cdots + a_N W_0, L] = 0, \tag{1.9}$$

where $a_1, a_2, \cdots, a_N$ are constants.

Proof.

$$\begin{aligned}[W_N + a_1 W_{N-1} + \cdots + a_N W_0, L] &= L_*(X_N) + a_1 L_*(X_{N-1}) + \cdots + a_N L_*(X_0) \\ &= L_*(X_N + a_1 X_{N-1} + \cdots + a_N X_0).\end{aligned}$$

L_* is injective, so Corollary 1.2 is right.

§2. COMMUTATOR REPRESENTATION OF THREE EQUATION HIERARCHIES

The three spectral problems studied in this paper whose pattern is $y_x = Uy$ can be all written as

$$Ly = (L_1 + L_2 \partial) y = \lambda y, \tag{2.1}$$

where the 2×2 matrix differential operator $L = L_1 + L_2$, $\partial = \frac{d}{\partial x}$, λ is the spectral parameter, L_2 is reversible, and $y = (y_1, y_2)^T$.

Let a 2×2 matrix differential operator $V = V_1 + V_2 L$ in which V_1, V_2 are 2×2 function matrixes. Through calculation, the Lie bracket $[V, L]$ is (note $\partial = L_2^{-1}(L - L_1)$):

$$\begin{aligned}[V, L] =& [V_1 + V_2 L, L] \\ =& [V_1, L] - L_2 V_{1,x} - [V_1, L_2] L_2^{-1} L_1 \\ &+ ([V_1, L_2] L_2^{-1} + [V_2, L_1] - L_2 V_{2,x} - [V_2, L_2] L_2^{-1} L_1) L + [V_2, L_2] L_2^{-1} L^2. \end{aligned} \tag{*}$$

Using (*), we try to obtain the commutator representation of the isospectral equations assocuated with three spectral problems below

$$y_x = U_1 y, \quad U_1 = \begin{pmatrix} 1 & \lambda u - 1 \\ \lambda u + 1 & -1 \end{pmatrix}; \tag{2.2}$$

$$y_x = U_2 y, \quad U_2 = \begin{pmatrix} 1 & -\lambda + v + 1 \\ -\lambda + v - 1 & -1 \end{pmatrix}; \tag{2.3}$$

$$y_x = U_3 y, \quad U_3 = \begin{pmatrix} 1 & -\lambda^2 + \lambda u + v + 1 \\ -\lambda^2 + \lambda u + v - 1 & -1 \end{pmatrix}. \tag{2.4}$$

2.1. The spectral problem (2.2) is equivalent to

$$Ly = \lambda y, \quad L = \frac{1}{u}\begin{pmatrix} -1 & \partial + 1 \\ \partial - 1 & 1 \end{pmatrix}. \tag{2.5}$$

$$L = L_1 + L_2\partial, \quad L_1 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix},$$

$$L_2 = \frac{1}{u}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad y = (y_1, y_2)^T.$$

$L_{*u}(\xi) = -\frac{\xi}{u}L$, $\forall\xi$, L_* is injective.
The pair of Lenard's operators K, J are

$$K = \frac{1}{2}\partial u^{-1}\partial^{-1}, \quad J = \partial^{-1}, \quad \partial\partial^{-1} = \partial^{-1}\partial = 1, \; J^{-1}K = \frac{1}{2}\partial^2 u^{-1}\partial^{-1}.$$

Define the Lenard's sequence $\{G_j\}$ recursively: $G_{-1} = 0$, $G_j = J^{-1}KG_{j-1}$ $(j = -1, 0, 1, \cdots)$. $G_j(x)$ is the polynomial of $u(x)$'s inverse u^{-1} and its derivatives. $X_j(u) = JG_j$, $j = 0, 1, 2, \cdots$. Let $\partial^{-1}0 = -2$. Then the first few results of calculations are $X_0 = u^{-2}u_x$, $X_1 = \frac{1}{2}(u^{-3}u_x)_x$, and $X_2 = -\frac{1}{8}(u^{-1}(u^{-2}))xx)_x$.

The representative equations produced by X_j are

$$u_t = u^{-2}u_x, \quad j = 0, \tag{2.6}$$

$$u_t = \frac{1}{2}(u^{-3}u_x)_x, \quad j = 1. \tag{2.7}$$

The former is the semi-classical Mkdv equation ($u_t = u^{-2}u_x$ can be changed into $v_t = v^2 v_x$ through transformation $v = u^{-1}$), the latter is the nonlinear diffusion equation.

Let $G(x)$ be an arbitrary smooth function on Ω, $J = \partial^{-1}$, $K = \frac{1}{2}\partial u^{-1}\partial^{-1}$, and L expressed as (2.5). Then the operator equation

$$[V, L] = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}(L_*(KG) - L_*(JG)L) \tag{2.8}$$

has the operator solution

$$V = V(G) = \frac{1}{2}u^{-2}\partial^{-1}G\begin{pmatrix} -\partial & \partial \\ -\partial & \partial \end{pmatrix}. \tag{2.9}$$

In fact, if let $A = \frac{1}{2}u^{-1}\partial^{-1}G$. Then $V = V(G) = \begin{pmatrix} A & -A \\ A & -A \end{pmatrix}L$. Letting $V_1 = 0$, $V_2 = \begin{pmatrix} A & -A \\ A & -A \end{pmatrix}$ in $(*)$ and substituting the expressions of L and L_2 into $(*)$, by calculation we have

$$\begin{aligned}[V, L] &= \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}(-\frac{A_x}{u}L + 2AL^2) \\ &= \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}(L_*(KG) - L_*(JG)L).\end{aligned}$$

Order $V_j = V(G_j) = \frac{1}{2}u^{-2}\partial^{-1}G\begin{pmatrix} -\partial & \partial \\ -\partial & \partial \end{pmatrix}$ for each G_j ($j = -1, 0, 1, \cdots$). In light of the result narrated in §1, we have

$$[W_m, L] = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} L_*(X_m), \; m = 0, 1, 2, \cdots, \tag{2.10}$$

where $W_m = \sum_{j=0}^{m} V_{j-1}L^{m-j}$. Furthermore the hierarchy $u_t = X_m(u)$ produced by $X_m(u)$ has the commutator representation

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} L_0 = [W_m, L]. \tag{2.11}$$

Under $y_{xt} = 0$ and $\lambda_t = 0$, $u_t = X_m(u)$ is the compatibility condition of (2.5) and

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} y_t = W_m y$$

for

$$\begin{aligned}
&\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} L_t y + \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} L y_t = \lambda W_m y, \\
&\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} L = -L\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \\
&LW_m y = L\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} y_t = \frac{1}{u}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} y_{xt} = 0, \\
&\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} L_t y = [W_m, L]y.
\end{aligned}$$

Corollary 1.2 in §1 is correct for the vector field $X_m(u)$ of (2.2).

2.1′. For the spectral problem (2.2) if we let the pair of Lenard's operators $\widehat{K} = \partial^{-1}$, $\widehat{J} = \frac{1}{2}\partial u^{-1}\partial^{-1}$, we can get the comuutator representation of another nonlinear evolution equation hierarchies of (2.2). Especially we point out $\widehat{J}^{-1}\widehat{K} = 2\partial u \partial^{-1}$. Some results are expressed as below and their proof are the same as 2.1.

Choose $\widehat{G}_{-1} = u_x \in \ker \widehat{J}$. The first few results of the Lenard's recursive sequence $\widehat{G}_{j+1} = \widehat{J}^{-1}\widehat{K}\widehat{G}_j$ ($j = -1, 0, 1, \cdots$) and vector field $\widehat{X}_m = \widehat{J}\widehat{G}_m$ ($m = 0, 1, 2, \cdots$) are

$$\widehat{G}_0 = 2\partial u \partial^{-1} u, \; \widehat{G}_1 = 4\partial u \partial^{-1} u \partial^{-1} u; \; \widehat{X}_0 = u, \; \widehat{X}_1 = 2u\partial^{-1}u.$$

The vector field $\widehat{X}_m = \widehat{J}\widehat{G}_m$ produces another evolution equation hierarchies of (2.2) $u_t = \widehat{X}_m(u)$ with the representative equation as

$$u_t = 2u\partial^{-1}u, \quad m = 1. \tag{2.12}$$

Let $\widehat{G}(x)$ be an arbitrary smooth function on Ω and $\widehat{V} = V(\widehat{G})$ (its expression as (2.9)). Then

$$[\widehat{V}, L] = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} (L_*(\widehat{J}\widehat{G}) - L_*(\widehat{K}\widehat{G})L). \tag{2.13}$$

Order $\widehat{V}_j = V(\widehat{G}_j)$ for each $\widehat{G}_j \in \{\widehat{G}_j\}_{j=-1}^{\infty}$. Then

$$[\widehat{W}_m, L] = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} L_*(\widehat{X}_m), \quad m = 0, 1, 2, \cdots, \tag{2.14}$$

where $\widehat{W}_m = \sum_{j=0}^{m} \widehat{V}_{j-1} L^{m+j-1}$, Hence the evolution equation hierarchy $u_t = \widehat{X}_m(u)$ nominated by another vector field $\widehat{X}_m$ has the commutation representation

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} L_t = [\widehat{W}_m, L]. \tag{2.15}$$

Under $y_{xt} = 0$ and $\lambda_t = 0$, $u_t = \widehat{X}_m(u)$ is the compatibility condition of $Ly = \lambda y$ and $\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} y_t = \widehat{W}_m y$.

Corollary 1.6 in §1 is also right for the other vector field $\widehat{X}_m(u)$ of (2.2).

2.2. The spectral problem (2.3) is equivalent to

$$Ly = \lambda y, \quad L = \begin{pmatrix} v-1 & -\partial - 1 \\ -\partial + 1 & v+1 \end{pmatrix}, \tag{2.16}$$

$$L = L_1 + L_2\partial, \quad L_1 = \begin{pmatrix} v-1 & -1 \\ 1 & v+1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

$$L_*(\xi) = \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix}, \forall \xi, \ L_* \text{ is an injective homomorphism.}$$

$$K = \frac{1}{2}\partial^2 + \partial v, \quad J = \partial, \quad J^{-1}K = \frac{1}{2}\partial + v.$$

Let

$$G_{-1} = 1, \quad G_j = J^{-1}KG_{j-1}, \quad X_j = JG_j = KG_{j-1} \ (j = 0, 1, 2, \cdots).$$

The first few are

$$G_0 = v, \ G_1 = \frac{1}{2}v_x + v^2, \quad G_2 = \frac{1}{4}v_{xx} + \frac{3}{2}vv_x + v^3;$$

$$X_0 = v_x, \ X_1 = \frac{1}{2}v_{xx} + 2vv_x, \quad X_2 = \frac{1}{4}v_{xxx} + \frac{3}{4}(v^2)_{xx} + 3v^2 v_x.$$

The representative equation of $v_t = X_m(v)$ produced by the vector field $X_m(v)$ of (2.16) is

$$v_t = \frac{1}{2}v_{xx} + 2vv_x, \quad m = 1$$

which is the well-known Burgers equation[11].

Let $G(x)$ be an arbitrary smooth function, and expressed as (2.16). Then the operator equation (1.5) has the solution

$$V = V(G) = \begin{pmatrix} G\partial & \frac{1}{2}G_x \\ \frac{1}{2}G_x & G\partial \end{pmatrix}$$

which is obtained through the calculation of $(*)$. So the evolution equation hierarchy $v_t = X_m(v)$ of (2.3) has the commutator representation

$$L_t = [W_m, L], \tag{2.17}$$

where

$$W_m = \sum_{j=0}^{m} V_{j-1} L^{m-j} = \sum_{j=0}^{m} \begin{pmatrix} G_{j-1}\partial & \frac{1}{2}G_{j-1,x} \\ \frac{1}{2}G_{j-1,x} & G_{j-1}\partial \end{pmatrix} L^{m-j}.$$

Two corollaries in §1 are right here.

2.3. The spectral problem (2.4) is equivalent to

$$Ly = \lambda y, \quad L = \frac{1}{\lambda}\begin{pmatrix} \lambda u + v - 1 & -\partial - 1 \\ 1 - \partial & \lambda u + v + 1 \end{pmatrix}, \tag{2.18}$$

$$L = L_1 + L_2\partial, \quad L_1 = \frac{1}{\lambda}\begin{pmatrix} \lambda u + v - 1 & -1 \\ 1 & \lambda u + v + 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

$$L_{*w}(\xi) = \begin{pmatrix} \xi_1 + \frac{1}{\lambda}\xi_2 & 0 \\ 0 & \xi_1 + \frac{1}{\lambda}\xi_2 \end{pmatrix}, \xi = (\xi_1, \xi_2)^T, \quad w = (u, v)^T,$$

L_* is an injective homomorphism.

$$K = \begin{pmatrix} 0 & \partial \\ \frac{1}{2}\partial^2 + \partial v & 0 \end{pmatrix}, \quad J = \begin{pmatrix} \partial & 0 \\ -\partial u & \partial \end{pmatrix} \quad J^{-1}K = \begin{pmatrix} 0 & 1 \\ \frac{1}{2}\partial + v & u \end{pmatrix}. \tag{2.19}$$

$$G_{-1} = (1, u)^T \in \ker J, \quad G_j = J^{-1}KG_{j-1},$$
$$X_j = JG_j = KG_{j-1} \ (j = 0, 1, 2, \cdots).$$

The first two results are

$$G_0 = \begin{pmatrix} u \\ v_u^2 \end{pmatrix}, \quad G_1 = \begin{pmatrix} v + u^2 \\ \frac{1}{2}u_x + 1uv + v^3 \end{pmatrix}; \quad X_0 = \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \quad X_1 = \begin{pmatrix} 2uu_x + v_x \\ \frac{1}{2}u_{xx} + (uv)_x \end{pmatrix}.$$

The vector field X_m produces the soliton hierarchy $w_t = (u, v)^T = X_m(w)$ of (2.4) with the representative equation as

$$u_t = 2uu_t + v_x, \quad v_t = \frac{1}{2}u_{xx} + (uv)_x, \quad m = 1.$$

Let $A(x)$ be an arbitrary smooth function on Ω. L, K, J are indicated as (2.18) and (2.19). Let $G = (A, uA)^T$. Then the operator solution $V = V(G)$ of the operator equation (1.5) is

$$V = V(G) = \begin{pmatrix} A\partial & \frac{1}{2}A_x \\ \frac{1}{2}A_x & A\partial \end{pmatrix}.$$

Let $G_{j-1} = \binom{B_{j-1}}{B_j}$. Then

$$\begin{aligned} G_j = J^{-1}KG_{j-1} &= \begin{pmatrix} 0 & 1 \\ \frac{1}{2}\partial + v & u \end{pmatrix}\begin{pmatrix} B_{j-1} \\ B_j \end{pmatrix} \\ &= \begin{pmatrix} B_j \\ \frac{1}{2}B_{j-1,x} + vB_{j-1} + uB_j \end{pmatrix} \\ &\triangleq \widetilde{G}_j + \widehat{G}_{j-1}, \end{aligned}$$

where

$$\widetilde{G}_j = \begin{pmatrix} B_j \\ uB_j \end{pmatrix}, \quad \widehat{G}_{j-1} = \begin{pmatrix} 0 \\ \frac{1}{2}B_{j-1,x} + vB_{j-1} \end{pmatrix}.$$

If $y(x)$ is the eigenfunction of (2.18), then

$$L_8(K\widehat{G}_{j-1})y = L_*(J\widehat{G}_{j-1})Ly. \tag{2.20}$$

In fact,

$$J\widehat{G}_{j-1} = \begin{pmatrix} 0 \\ (\frac{1}{2}\partial^2 + v)B_{j-1} \end{pmatrix}, \quad K\widehat{G}_{j-1} = \begin{pmatrix} (\frac{1}{2}\partial^2 + v)B_{j-1} \\ 0 \end{pmatrix}.$$

$$\begin{aligned} L_*(J\widehat{G}_{j-1}) \cdot Ly =& \frac{1}{\lambda}(\frac{1}{2}\partial^2 + \partial v)B_{j-1} \cdot Ly \\ =& (\frac{1}{2}\partial^2 + \partial v)B_{j-1} \cdot y = L_*(K\widehat{G}_{j-1})y, \quad y = (y_1, y_2)^T. \end{aligned}$$

So, if we consider the commutator representation of the evolution equation associated with (2.4) or (2.18), we are sure to have

$$L_*(K\widehat{G}_{j-1}) = L_*(J\widehat{G}_{j-1})L. \tag{2.21}$$

Let $G_j = \binom{B_j}{B_{j+1}} = \binom{B_j}{\frac{1}{2}B_{j-1,x} + vB_{j-1} + uB_j}$ be the Lenard's sequence of (2.18). Then the operator equation

$$[V_j, L] = L_*(KG_j) - L_*(JG_j)L \tag{2.22}$$

has the solution

$$V_j = V(G_j) = \begin{pmatrix} B_j\partial & \frac{1}{2}B_{j,x} \\ \frac{1}{2}B_{j,x} & B_j\partial \end{pmatrix}. \tag{2.23}$$

In fact, $\widetilde{G}_j = G_j - \widehat{G}_{j-1}$. Note $\widetilde{G}_j = (B_j, uB_j)^T$. Hence

$$\begin{aligned} [V_j, L] =& L_*(K\widetilde{G}_j) - L_*(J\widetilde{G}_j)L \\ =& L_*(KG_j) - L_*(K\widehat{G}_{j-1}) - L_*(JG_j)L + L_*(J\widehat{G}_{j-1})L \\ =& L_*(KG_j) - L_*(JG_j)L. \end{aligned}$$

Moreover, the evolution equation $w_t = (u, v)^T = X_m(w)$ of (2.4) has the commutator representation

$$L_t = \sum_{j=0}^{m}\Big[\begin{pmatrix} B_{j-1}\partial & \frac{1}{2}B_{j-1,x} \\ \frac{1}{2}B_{j-1,x} & B_{j-1}\partial \end{pmatrix} L^{m-j}, L\Big], \tag{2.24}$$

where L is expressed as (2.18), each $B_{j-1}\,(j = 0, 1, 2, \cdots, m)$ is determined by the Lenard's sequence G_{j-1}.

Two corollaries in §1 in are also correct here.

Remark 2.1. In 2.1 and 2.1′ of §2 we present the commutator representation of two different nonlinear evolution equation hierarchies associated with the same spectral problem (2.2), on which we have some results[12].

Remark 2.2. As for the commutator representation of the isospetral equation, the spectral operator L which is related to the eigenparameter λ, potented $w(x)$ and $\partial = \frac{\partial}{\partial x}$ is the first time involved in this paper for the spectral problem (2.4) or (2.18).

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