Acta Applicandae Mathematicae **83:** 199–220, 2004.

Integrable Hierarchy, 3 × 3 Constrained Systems, and Parametric Solutions

ZHIJUN QIAO
Department of Mathematics, University of Texas-Pan American, Edinburg, TX 78539, U.S.A. and T-CNLS, Los Alamos National Laboratory, Los Alamos, NM 87545, U.S.A. and Institute of Mathematics, Fudan University, Shanghai 200433, P.R. China

(Received: 20 March 2003; in final form: 16 December 2003)

Abstract. This paper provides a new integrable hierarchy. The DP equation: $m_t + um_x + 3mu_x = 0$, $m = u - u_{xx}$, proposed recently by Degasperis and Procesi, is the first member in the negative order hierarchy while the first equation in the positive order hierarchy is: $m_t = 4(m^{-2/3})_x - 5(m^{-2/3})_{xxx} + (m^{-2/3})_{xxxxx}$. The whole hierarchy is shown Lax-integrable through solving a key matrix equation. To obtain the parametric solutions for the whole hierarchy, we separately discuss the negative order and the positive order hierarchies. For the negative order hierarchy, its 3 × 3 Lax pairs and corresponding adjoint representations are cast in Liouville-integrable Hamiltonian canonical systems under the Dirac–Poisson bracket defined on a symplectic submanifold of $\mathbb{R}^{6N}$. Based on the integrability of those finite-dimensional canonical Hamiltonian systems we give the parametric solutions of all equations in the negative order hierarchy. In particular, we obtain the parametric solution of the DP equation. Moreover, for the positive order hierarchy, we consider a different constraint and process a procedure similar to the negative case to obtain the parametric solutions of the positive order hierarchy. In a special case, we give the parametric solution of the 5th-order PDE $m_t = 4(m^{-2/3})_x - 5(m^{-2/3})_{xxx} + (m^{-2/3})_{xxxxx}$. Finally, we discuss the stationary solutions of the 5th-order PDE, which may be included in the parametric solution.

Mathematics Subject Classifications (2000): 35Q53, 58F07, 35Q35.

Key words: Hamiltonian system, matrix equation, zero curvature representation, integrable equation, parametric solution.

1. Introduction

The inverse scattering transformation (IST) method is a powerful tool to solve integrable nonlinear evolution equations (NLEEs) (Gardner *et al.*, 1967). This method has been successfully applied to get soliton solutions of the integrable NLEEs. These examples include the well-known KdV equation (Korteweg and de Vries, 1895), which is related to a 2nd-order operator (i.e. Schrödinger operator) spectral problem (Levitan and Gasymov, 1964; Marchenko, 1950), the remarkable Ablowitz–Kaup–Newell–Segur (AKNS) equations (Ablowitz *et al.*, 1973, 1974), which is associated with the Zakharov–Shabat (ZS) spectral problem (Zakharov and Shabat, 1972), and other higher-dimensional integrable equations.

To seek for new integrable systems has been an important topic in the theory of integrable system. Kaup (1980) studied the inverse scattering problem for cubic eigenvalue equations of the form $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi$, and showed a 5th-order partial differential equation (PDE) $Q_t + Q_{xxxxx} + 30(Q_{xxx}Q + \frac{5}{2}Q_{xx}Q_x) + 180Q_xQ^2 = 0$ (called the KK equation) integrable. Afterwards, Kupershmidt (1984) constructed a super-KdV equation and presented the integrability of the equation through giving bi-Hamiltonian property and Lax form. Konopelchenko and Dubrovsky (1984) presented a 5th-order equation $u_t = (u^{-2/3})_{xxxxx}$ and pointed out that this equation is a reduction of some $(2 + 1)$-dimensional equation. We have already found the parametric solution and singular traveling wave solutions of this equation (Qiao, 2002c).

Recently, Degasperis and Procesi (1999) proposed a new integrable equation: $m_t + um_x + 3mu_x = 0$, $m = u - u_{xx}$, called the DP equation, which has the peaked soliton solution. The DP equation is an extention of the shallow water Camassa–Holm (CH) equation (Camassa and Holm, 1993), and is proven to be associated with a 3rd-order spectral problem (Degasperis *et al.*, 2002): $\psi_{xxx} = \psi_x - \lambda m\psi$ and to have a close relationship to a canonical Hamiltonian system under a new nonlinear Poisson bracket (called Peakon Bracket) (Holm and Hone, 2002). The shallow water equation (Camassa and Holm, 1993) was shown to be bi-Hamiltonian and integrable and to have the peaked soliton solution. Families of integrable equations inexplicitly implying the shallow water equation were known to be derivable in the general context of hereditary symmetries in Fuchssteiner and Fokas (1981). The shallow water equation's billiard solutions, piece-wise smooth solutions and algebro-geoemtric solutions were successively treated in Alber *et al.* (1994, 1995, 1999, 2001), Constantin and McKean (1999) and in Qiao (2003).

The purpose of the present paper is twofold:

- we extend the DP equation to a new integrable hierarchy through studying the functional gradient of the spectral problem and a pair of Lenard's operators;
- we connect the DP equation to finite-dimensional integrable systems, give its parametric solution from the point of constraint view, and discuss the stationary solutions.

The whole paper is organized as follows. Next section is saying how to connect a spectral problem to the DP equation and how to cast it into a new hierarchy of NLEEs, and is also giving the pair of Lenards operators for the whole hierarchy. In Section 3, we construct the zero curvature representations for this new hierarchy through solving a key 3×3 matrix equation, and therefore this hierarchy is integrable. In particular, we give the matrix Lax pair of the DP equation, which is equivalent to the form in (Degasperis *et al.*, 2002), as well as the Lax pair for a 5th-order equation $m_t = 4(m^{-2/3})_x - 5(m^{-2/3})_{xxx} + (m^{-2/3})_{xxxxx}$. We will see that the DP equation is included in the negative order hierarchy while the 5th-order equation in the positive order hierarchy. To obtain the parametric solutions for the whole hierarchy, we separately discuss the negative order and the positive order hierarchies. In Section 4, we deal with the negative order hierarchy. Its 3×3 Lax pairs and corresponding adjoint representations are cast in Liouville-integrable canonical Hamiltonian systems under the Dirac–Poisson bracket defined on a symplectic submanifold of $\mathbb{R}^{6N}$. Based on the integrability of those finite-dimensional canonical Hamiltonian systems we give the parametric solutions of all equations in the negative order hierarchy. In particular, we obtain the parametric solution of the DP equation. Section 5 deals with the positive order hierarchy. We consider a different constraint between the potential and the eigenfunctions. Under the constraint the 3×3 Lax pairs and corresponding adjoint representations of the positive hierarchy are changed to Liouville-integrable canonical Hamiltonian systems in the whole $\mathbb{R}^{6N}$. Then we obtain the parametric solutions of the positive order hierarchy. In particular, we give the parametric solution of the 5th-order PDE $m_t = 4(m^{-2/3})_x - 5(m^{-2/3})_{xxx} + (m^{-2/3})_{xxxxx}$. Finally, in Section 6 we discuss the stationary solutions of the 5th-order PDE, which may be included in the parametric solution.

2. Spectral Problems and Lenards Operators

Let us consider the following spectral problem

$$\psi_{xxx} = \frac{1}{\alpha^2}\psi_x - \lambda m\psi \tag{1}$$

and its adjoint problem

$$\psi^*_{xxx} = \frac{1}{\alpha^2}\psi^*_x + \lambda m\psi^*, \tag{2}$$

where λ is a spectral parameter, m is a scalar periodic potential function, ψ and ψ^* are the spectral wave functions corresponding to the same λ, $\alpha = \text{constant}$, and the domain of x is one period: $\Omega = (x_0, x_0 + T)$. Then, we have

$$\nabla\lambda = \frac{\delta\lambda}{\delta m} = \frac{\lambda\psi\psi^*}{E}, \tag{3}$$

where

$$E = -\int_\Omega m\psi\psi^*\,\mathrm{d}x = \text{const.}$$

Here during our computation about the functional gradient $\delta\lambda/\delta m$ of the spectral parameter λ with respect to the potential m, we need the boundary conditions of decaying at infinities or periodical condition for the potential function m. A general calculated method can be found in (Fuchssteiner and Fokas, 1981; Cao, 1989; Tu, 1990).

Now, we denote the function $\nabla_1\lambda$ by

$$\nabla_1\lambda = \frac{\lambda(\psi\psi^*_x - \psi^*\psi_x)}{E}. \tag{4}$$

Then, we have the following equality:

$$\overline{K}(\nabla\lambda, \nabla_1\lambda)^{\mathrm{T}} = \lambda \bar{J}(\nabla\lambda, \nabla_1\lambda)^{\mathrm{T}},$$

where $\overline{K}$ and $\bar{J}$ are two matrix operators

$$\overline{K} = \begin{pmatrix} -4\partial + 5\alpha^2\partial^3 - \alpha^4\partial^5 & 0 \\ 0 & \partial^3 - \frac{1}{\alpha^2}\partial \end{pmatrix},$$
$$\bar{J} = \begin{pmatrix} 0 & 3\alpha^4(2m\partial + \partial m) \\ m\partial + 2\partial m & 0 \end{pmatrix},$$
$$\partial = \frac{\partial}{\partial x},$$

or we rewrite Equation (5) as the following Lenard spectral problem form

$$K\nabla\lambda = \lambda^2 J\nabla\lambda, \tag{5}$$

where

$$K = 4\partial - 5\alpha^2\partial^3 + \alpha^4\partial^5,$$
$$J = 3\alpha^6(2m\partial + \partial m)(-\alpha^2\partial^3 + \partial)^{-1}(m\partial + 2\partial m).$$

Without loss of generality, we assume $\alpha = 1$ below. Then, K, J read

$$K = 4\partial - 5\partial^3 + \partial^5, \tag{6}$$
$$J = 3(2m\partial + \partial m)(\partial - \partial^3)^{-1}(m\partial + 2\partial m). \tag{7}$$

Remark 1. Here we do not care about the Hamiltonian properties of the operators K, J (although they are a pair of Hamiltonian operators), but need

$$K^{-1} = (\partial - \partial^3)^{-1}(4 - \partial^2)^{-1},$$
$$J^{-1} = \frac{1}{27}m^{-2/3}\partial^{-1}m^{-1/3}(\partial - \partial^3)m^{-1/3}\partial^{-1}m^{-2/3}.$$

They yield

$$\mathcal{L} = J^{-1}K = \frac{1}{27}m^{-2/3}\partial^{-1}m^{-1/3}(\partial - \partial^3)m^{-1/3}\partial^{-1}m^{-2/3}(4\partial - 5\partial^3 + \partial^5),$$
$$\mathcal{L}^{-1} = K^{-1}J = 3(\partial - \partial^3)^{-1}(4 - \partial^2)^{-1}(2m\partial + \partial m)(\partial - \partial^3)^{-1}(m\partial + 2\partial m),$$

which are actually the two recursive operators we need in the next section.

3. Zero Curvature Representations and an Integrable Hierarchy

Letting $\psi = \psi_1$, we change Equation (1) to a 3×3 matrix spectral problem

$$\Psi_x = U(m, \lambda)\Psi, \tag{8}$$

$$U(m, \lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -m\lambda & 1 & 0 \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}. \tag{9}$$

Apparently, the Gateaux derivative matrix $U_*(\xi)$ of the spectral matrix U in the direction $\xi \in C^\infty(\mathbb{R})$ at point m is

$$U_*(\xi) \triangleq \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} U(m + \epsilon\xi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\xi\lambda & 0 & 0 \end{pmatrix} \tag{10}$$

which is obviously an injective homomorphism.

For any given C^∞-function G, we construct the following 3×3 matrix equation with respect to $V = V(G)$

$$V_x - [U, V] = U_*(KG - \lambda^2 JG). \tag{11}$$

THEOREM 1. *For the spectral problem* (8) *and an arbitrary C^∞-function G, the matrix equation* (11) *has the following solution*

$$V = \lambda \begin{pmatrix} \Gamma G + 3\lambda\partial\Theta^{-1}\Upsilon G & 3G_x - 3\lambda\Theta^{-1}\Upsilon G & -6G \\ \Gamma G_x + 3\lambda(\partial^2\Theta^{-1}\Upsilon G + 2mG) & -2(G - G_{xx}) & -3G_x - 3\lambda\Theta^{-1}\Upsilon G \\ \Gamma G_{xx} + 3\lambda(\partial + \lambda m)\Theta^{-1}\Upsilon G & -\Theta G - 3\lambda(\partial^{-1}\Upsilon G - 2mG) & -2G - G_{xx} - 3\lambda\partial\Theta^{-1}\Upsilon G \end{pmatrix}, \tag{12}$$

where $\Theta = \partial - \partial^3$, $\Upsilon = m\partial + 2\partial m$, $\Gamma = 4 - \partial^2$. *Therefore,* $J = 3\Upsilon^*\Theta^{-1}\Upsilon$, $K = \Gamma\Theta$.

Proof. Let

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix},$$

and substitute it into Equation (11). That is a over-determined equation. Using the calculation technique (Qiao, 2002a) which needs to compare all coefficients according to the various powers of parameter λ on the right-hand side of (11) and to analysize all elements, we obtain the following results:

$$V_{11} = \lambda\Gamma G + 3\lambda^2\partial\Theta^{-1}\Upsilon G,$$
$$V_{12} = 3\lambda G_x - 3\lambda^2\Theta^{-1}\Upsilon G,$$
$$V_{13} = -6\lambda G,$$
$$V_{21} = \lambda\Gamma G_x + 3\lambda^2(\partial^2\Theta^{-1}\Upsilon G + 2mG),$$

$$V_{22} = -2\lambda(G - G_{xx}),$$
$$V_{23} = -3\lambda G_x - 3\lambda^2\Theta^{-1}\Upsilon G,$$
$$V_{31} = \lambda\Gamma G_{xx} + 3\lambda^2(\partial + \lambda m)\Theta^{-1}\Upsilon G,$$
$$V_{32} = -\lambda\Theta G - 3\lambda^2(\partial^{-1}\Upsilon G - 2mG),$$
$$V_{33} = -2\lambda G - \lambda G_{xx} - 3\lambda^2\partial\Theta^{-1}\Upsilon G,$$

which completes the proof. □

THEOREM 2. *Let* $G_0 \in \operatorname{Ker} J = \{G \in C^\infty(\mathbb{R}) \mid JG = 0\}$ *and* $G_{-1} \in \operatorname{Ker} K = \{G \in C^\infty(\mathbb{R}) \mid KG = 0\}$. *We define the Lenard's sequence*

$$G_j = \begin{cases} \mathcal{L}^j G_0, & j \in \mathbb{Z}, \\ \mathcal{L}^{j+1} G_{-1}, & j \in \mathbb{Z}. \end{cases} \tag{13}$$

Then,

(1) *the all vector fields* $X_k = JG_k$, $k \in \mathbb{Z}$ *satisfy the following commutator representation*

$$V_{k,x} - [U, V_k] = U_*(X_k), \quad \forall k \in \mathbb{Z}; \tag{14}$$

(2) *the following hierarchy of nonlinear evolution equations*

$$m_{t_k} = X_k = JG_k, \quad \forall k \in \mathbb{Z}, \tag{15}$$

possesses the zero curvature representation

$$U_{t_k} - V_{k,x} + [U, V_k] = 0, \quad \forall k \in \mathbb{Z}, \tag{16}$$

where

$$V_k = \sum V(G_j)\lambda^{2(k-j-1)}, \qquad \sum = \begin{cases} \sum_{j=0}^{k-1}, & k > 0, \\ 0, & k = 0, \\ -\sum_{j=k}^{-1}, & k < 0, \end{cases} \tag{17}$$

and $V(G_j)$ *is given by Equation* (12) *with* $G = G_j$.

Proof. (1) For $k = 0$, it is obvious. For $k < 0$, we have

$$\begin{aligned} V_{k,x} - [U, V_k] &= -\sum_{j=k}^{-1}(V_x(G_j) - [U, V(G_j)])\lambda^{2(k-j-1)} \\ &= -\sum_{j=k}^{-1} U_*(KG_j - \lambda^2 KG_{j-1})\lambda^{2(k-j-1)} \\ &= U_*\left(\sum_{j=k}^{-1} KG_{j-1}\lambda^{2(k-j)} - KG_j\lambda^{2(k-j-1)}\right) \\ &= U_*(KG_{k-1} - KG_{-1}\lambda^{2k}) \\ &= U_*(KG_{k-1}) \\ &= U_*(X_k). \end{aligned}$$

For the case of $k > 0$, it is similar to prove.

(2) Noticing $U_{t_k} = U_*(m_{t_k})$, we obtain

$$U_{t_k} - V_{k,x} + [U, V_k] = U_*(m_{t_k} - X_k).$$

The injectiveness of U_* implies that result (2) is correct. □

So, the hierarchy (15) has Lax pair and is therefore integrable. In particular, through choosing $G_{-1} = -\frac{1}{6} \in \operatorname{Ker} K$, (15) reads

$$m_{t_k} = -J\mathcal{L}^{k+1} \cdot \frac{1}{6}, \quad k = -1, -2, \ldots, \tag{18}$$

where $\mathcal{L} = J^{-1}K$. Set $m = u - u_{xx}$, then it is easy to see the first equation in the hierarchy is exactly the DP equation (Degasperis and Procesi, 1999)

$$m_t + um_x + 3mu_x = 0, \quad t = t_{-1}. \tag{19}$$

This equation has the following Lax pair:

$$\begin{aligned} \Psi_x &= U(m, \lambda)\Psi, \\ \Psi_t &= V(m, \lambda)\Psi, \end{aligned} \tag{20}$$

where $U(m, \lambda)$ is defined by Equation (9), and $V(m, \lambda)$ is given by

$$V(m, \lambda) = \begin{pmatrix} u_x + \frac{2}{3}\lambda^{-1} & -u & -\lambda^{-1} \\ u & -\frac{1}{3}\lambda^{-1} & -u \\ u_x + um\lambda & 0 & -u_x - \frac{1}{3}\lambda^{-1} \end{pmatrix}, \tag{21}$$

which can be changed to the form in (Degasperis *et al.*, 2002)

$$\psi_t + \lambda^{-1}\psi_{xx} + u\psi_x - \left(u_x + \frac{2}{3}\lambda^{-1}\right)\psi = 0, \quad \psi = \psi_1. \tag{22}$$

Let us choose a kernel element G_0 from $\operatorname{Ker} J$: $G_0 = m^{-2/3}$. Then Equation (15) reads the following integrable hierarchy

$$m_{t_k} = J\mathcal{L}^k m^{-2/3}, \quad k = 0, 1, 2, \ldots. \tag{23}$$

In particular, the equation

$$m_t = 4(m^{-2/3})_x - 5(m^{-2/3})_{xxx} + (m^{-2/3})_{xxxxx}, \tag{24}$$

has the Lax pair:

$$\begin{aligned} \Psi_x &= U(m, \lambda)\Psi, \\ \Psi_t &= V_1(m, \lambda)\Psi, \end{aligned} \tag{25}$$

where $U(m, \lambda)$ is defined by Equation (9), and $V_1(m, \lambda)$ is given by

$$V_1(m,\lambda)=\lambda\begin{pmatrix} \Gamma m^{-2/3} & 3(m^{-2/3})_x & -6m^{-2/3} \\ \Gamma(m^{-2/3})_x+6\lambda m^{1/3} & 2(m^{-2/3})_{xx}-2m^{-2/3} & -3(m^{-2/3})_x \\ \Gamma(m^{-2/3})_{xx} & (m^{-2/3})_{xxx}-(m^{-2/3})_x+6\lambda m^{1/3} & -2m^{-2/3}-(m^{-2/3})_{xx} \end{pmatrix} \quad (26)$$

with the operator $\Gamma = 4 - \partial^2$. Equation (24) is therefore a new integrable equation.

In the next two sections we will give parametric solutions for the negative order hierarchy (18) and the positive order hierarchy (23).

4. Parametric Solution of the Negative Order Hierarchy

To get the parametric solution, we use the constrained method which leads finite-dimensional integrable systems to the PDEs. Becasue Equation (8) is a 3rd-order eigenvalue problem, we have to investigate itself together with its adjoint problem when we adopt the nonlinearized procedure (Cao, 1989). Ma and Strampp (1994) already studied the AKNS and its its adjoint problem, a 2×2 case, by using the so-called symmetry constraint method. Now, we are discussing a 3×3 problem related to the hierarchy (15). Let us return to the spectral problem (8) and consider its adjoint problem

$$\Psi^*_x = \begin{pmatrix} 0 & 0 & m\lambda \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}\Psi^*, \quad \Psi^* = \begin{pmatrix} \psi_1^* \\ \psi_2^* \\ \psi_3^* \end{pmatrix}, \quad (27)$$

where $\psi^* = \psi_3^*$.

4.1. NONLINEARIZED SPECTRAL PROBLEMS ON A SYMPLECTIC SUBMANIFOLD

Let λ_j $(j = 1, \ldots, N)$ be N distinct spectral values of (8) and (27), and q_{1j}, q_{2j}, q_{3j} and p_{1j}, p_{2j}, p_{3j} be the corresponding spectral functions, respectively. Then we have

$$\begin{aligned} q_{1x} &= q_2, \\ q_{2x} &= q_3, \\ q_{3x} &= -m\Lambda q_1 + q_2, \end{aligned} \quad (28)$$

and

$$\begin{aligned} p_{1x} &= m\Lambda p_3, \\ p_{2x} &= -p_1 - p_3, \\ p_{3x} &= -p_2, \end{aligned} \quad (29)$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, $q_k = (q_{k1}, q_{k2}, \ldots, q_{kN})^{\mathrm{T}}$, $p_k = (p_{k1}, p_{k2}, \ldots, p_{kN})^{\mathrm{T}}$, $k = 1, 2, 3$.

Now, we consider the following $(6N-2)$-dimensional symplectic submanifold in $\mathbb{R}^{6N}$:

$$M = \{(p, q)^{\mathrm{T}} \in \mathbb{R}^{6N} \mid F = 0, G = 0\}, \tag{30}$$

where $p = (p_1, p_2, p_3)^{\mathrm{T}}$, $q = (q_1, q_2, q_3)^{\mathrm{T}}$, $F = \langle \Lambda q_1, p_3 \rangle - 1$, $G = \langle \Lambda q_2, p_3 \rangle - \langle \Lambda q_3, p_2 \rangle + \langle \Lambda q_2, p_1 \rangle$, and $\langle \cdot, \cdot \rangle$ stands for the standard inner product in $\mathbb{R}^N$.

Remark 2. The constraint $F = 0$ comes from the spectral functional gradient formula (3), and the other constraint $G = 0$ has the relationship with $F = 0$ through $\mathrm{d}^3 F(x)/\mathrm{d}x = 3G(x)$.

When we restrict the above two systems in $\mathbb{R}^{6N}$ to the submanifold M, we obtain a constraint of m relationship to the spectral function p, q:

$$m = 2\frac{\langle \Lambda q_2, p_2 \rangle - \langle \Lambda q_3, p_1 + p_3 \rangle}{\langle \Lambda^2 q_2, p_3 \rangle + \langle \Lambda^2 q_1, p_2 \rangle}. \tag{31}$$

Remark 3. $\langle \Lambda^2 q_2, p_3 \rangle + \langle \Lambda^2 q_1, p_2 \rangle \neq 0$ is necessary because it assures M is a symplectic submanifold in $\mathbb{R}^{6N}$.

Under the constraint (31) the two systems (28)and (29) are nonlinearized as follows:

$$\begin{aligned} q_{1x} &= q_2, \\ q_{2x} &= q_3, \\ q_{3x} &= -2\frac{\langle \Lambda q_2, p_2 \rangle - \langle \Lambda q_3, p_3 \rangle - \langle \Lambda q_3, p_1 \rangle}{\langle \Lambda^2 q_2, p_3 \rangle + \langle \Lambda^2 q_1, p_2 \rangle}\Lambda q_1 + q_2; \end{aligned} \tag{32}$$

and

$$\begin{aligned} p_{1x} &= 2\frac{\langle \Lambda q_2, p_2 \rangle - \langle \Lambda q_3, p_3 \rangle - \langle \Lambda q_3, p_1 \rangle}{\langle \Lambda^2 q_2, p_3 \rangle + \langle \Lambda^2 q_1, p_2 \rangle}\Lambda p_3, \\ p_{2x} &= -p_1 - p_3, \\ p_{3x} &= -p_2. \end{aligned} \tag{33}$$

They are forming a $(6N-2)$-dimensional nonlinear system on M with respect to p, q. Is it integrable? To see this, in $\mathbb{R}^{6N}$ we modify the usual standard Poisson bracket (Arnol'd, 1978) of two functions F_1, F_2 as follows:

$$\{F_1, F_2\} = \sum_{i=1}^{3}\left(\left\langle \frac{\partial F_1}{\partial q_i}, \frac{\partial F_2}{\partial p_i} \right\rangle - \left\langle \frac{\partial F_1}{\partial p_i}, \frac{\partial F_2}{\partial q_i} \right\rangle\right) \tag{34}$$

which is still antisymmetric, bilinear and satisfies the Jacobi identity.

Obviously,

$$\{F, G\} = \langle \Lambda^2 q_2, p_3 \rangle + \langle \Lambda^2 q_1, p_2 \rangle \neq 0. \tag{35}$$

Because we are discussing the system on the submanifold M, we need to introduce the so-called Dirac–Poisson bracket of two functions f, g on M:

$$\{f, g\}_D = \{f, g\} + \frac{1}{\{F, G\}}(\{f, F\}\{G, g\} - \{f, G\}\{F, g\}) \tag{36}$$

which is satisfying the Jacobi identity.

Now, we choose a simple Hamiltonian

$$H = \langle q_2, p_1 + p_3\rangle + \langle q_3, p_2\rangle, \tag{37}$$

then, the two systems (32) and (33) have the canonical Hamiltonian form on M:

$$\begin{aligned} q_{1j,x} &= \{q_{1j}, H\}_D, \\ q_{2j,x} &= \{q_{2j}, H\}_D, \\ q_{3j,x} &= \{q_{3j}, H\}_D; \\ p_{1j,x} &= \{p_{1j}, H\}_D, \\ p_{2j,x} &= \{p_{2j}, H\}_D, \\ p_{3j,x} &= \{p_{3j}, H\}_D, \end{aligned} \tag{38}$$

which can be in a brief form rewritten as:

$$\begin{aligned} q_x &= \{q, H\}_D, \\ p_x &= \{p, H\}_D. \end{aligned} \tag{39}$$

In this calculation procedure, we have used

$$\begin{aligned} &\{H, G\} = 2(\langle \Lambda q_2, p_2\rangle - \langle \Lambda q_3, p_1 + p_3\rangle), \\ &\{H, F\} = 0. \end{aligned}$$

It is easy to check that $H_x = 0$, i.e. H is invariant along the flow (39). Assume $H = C_D$ (C_D is constant) along this flow and

$$u = \frac{1}{2}(\langle q_1, p_2\rangle + \langle q_2, p_3\rangle) - \frac{1}{2}H, \tag{40}$$

then we have

$$u - u_{xx} = m, \tag{41}$$

which is exactly related to the DP equation (19).

To show the integrability of canonical system (39), we need to consider the nonlinearization of the time part of the Lax representations.

4.2. NONLINEARIZED TIME PART ON THE SYMPLECTIC SUBMANIFOLD

Let us turn to the time part (20) of the Lax pair for the DP equation (19). Then the corresponding adjoint problem reads

$$\Psi_t^* = \begin{pmatrix} -u_x - \frac{2}{3}\lambda^{-1} & -u & -u_x - um\lambda \\ u & \frac{1}{3}\lambda^{-1} & 0 \\ \lambda^{-1} & u & u_x + \frac{1}{3}\lambda^{-1} \end{pmatrix} \Psi^*, \quad \Psi^* = \begin{pmatrix} \psi_1^* \\ \psi_2^* \\ \psi_3^* \end{pmatrix}. \tag{42}$$

We also consider the constrained system of the time part on M. Thus, under the constraints (31), (40), and

$$u_x = \frac{1}{2}(\langle q_3 - q_1, p_3\rangle - \langle q_1, p_1\rangle), \tag{43}$$

Equations (20) and (42) are nonlinearized as:

$$\begin{aligned} q_{1t} &= u_x q_1 + \frac{2}{3}\Lambda^{-1} q_1 - u q_2 - \Lambda^{-1} q_3, \\ q_{2t} &= u q_1 - \frac{1}{3}\Lambda^{-1} q_2 - u q_3, \\ q_{3t} &= u_x q_1 + um\Lambda q_1 - \frac{1}{3}\Lambda^{-1} q_3 - u_x q_3; \end{aligned} \tag{44}$$

and

$$\begin{aligned} p_{1t} &= -u_x p_1 - \frac{2}{3}\Lambda^{-1} p_1 - u p_2 - u_x p_3 - um\Lambda p_3, \\ p_{2t} &= u p_1 + \frac{1}{3}\Lambda^{-1} p_2, \\ p_{3t} &= \Lambda^{-1} p_1 + u p_2 + u_x p_3 + \frac{1}{3}\Lambda^{-1} p_3, \end{aligned} \tag{45}$$

respectively, where each q_k, p_k and Λ are the same as in Subsection 4.1, and Λ^{-1} is the inverse of Λ.

Let

$$\begin{aligned} I = {} & \frac{2}{3}\langle \Lambda^{-1} q_1, p_1\rangle - \frac{1}{3}\langle \Lambda^{-1} q_2, p_2\rangle - \frac{1}{3}\langle \Lambda^{-1} q_3, p_3\rangle - \langle \Lambda^{-1} q_3, p_1\rangle + \\ & + \frac{1}{4}\langle q_1, p_2\rangle^2 - \frac{1}{4}\langle q_2, p_3\rangle^2 - \frac{1}{2} H\langle q_1, p_2\rangle - \\ & - u(\langle q_2, p_1\rangle + \langle q_3, p_2\rangle - H) - u_x^2, \end{aligned} \tag{46}$$

then the two systems (44) and (45) are expressed in a canonical Hamiltonian form on M:

$$\begin{aligned} q_t &= \{q, I\}_D, \\ p_t &= \{p, I\}_D, \end{aligned} \tag{47}$$

where $p = (p_1, p_2, p_3)^{\mathrm{T}}$, $q = (q_1, q_2, q_3)^{\mathrm{T}}$.

In the above calculations, we use the following equalities:

$$F = \langle \Lambda q_1, p_3 \rangle - 1 = 0,$$
$$G = \langle \Lambda q_2, p_1 + p_3 \rangle - \langle \Lambda q_3, p_2 \rangle = 0,$$
$$F_x = \langle \Lambda q_2, p_3 \rangle - \langle \Lambda q_1, p_2 \rangle = 0,$$
$$F_{xx} = \langle \Lambda q_3, p_3 \rangle - 2\langle \Lambda q_2, p_2 \rangle + \langle \Lambda q_1, p_1 \rangle + 1 = 0,$$
$$\{G, I\} = 2u(\langle \Lambda q_2, p_2 \rangle - \langle \Lambda q_3, p_1 + p_3 \rangle),$$
$$\{F, I\} = 0.$$

THEOREM 3.

$$\{H, I\}_D = 0, \tag{48}$$

that is, two Hamiltonian flows commute on M.

Proof. By the definition,

$$\{H, I\}_D = \{H, I\} + \frac{1}{\{F, G\}}(\{H, F\}\{G, I\} - \{H, G\}\{F, I\}), \tag{49}$$

we need to compute each Poisson bracket in this equality.

$$\{H, I\} = u_x(\langle p_2, q_1 - q_3 \rangle - \langle q_2, p_1 \rangle) + u(\langle p_3, q_1 - q_3 \rangle - \langle q_1, p_1 \rangle) = 0,$$
$$\{H, F\} = \langle \Lambda q_1, p_2 \rangle - \langle \Lambda q_2, p_3 \rangle = (-\langle \Lambda q_1, p_3 \rangle + 1)_x = 0,$$
$$\{F, I\} = 2u_x(\langle \Lambda q_1, p_3 \rangle - 1) + u(-\langle \Lambda q_1, p_3 \rangle + 1)_x = 0,$$

complete the proof. □

By Theorem 2, we know that the hierarchy (18) has the Lax representation:

$$\Psi_x = U(m, \lambda)\Psi, \tag{50}$$

$$\Psi_{t_j} = -\sum_{k=j}^{-1} V(G_k)\lambda^{2(j-k-1)}\Psi, \quad j < 0, \tag{51}$$

where $V(G_k)$ is given by Equation (12) with $G = G_k$.

In last subsection, we have investigated the nonlinearized systems of spectral problem and its adjoint. Now, we study the nonlinearizations of time part (51) and its adjoint problem:

$$\Psi^*_{t_j} = \sum_{k=j}^{-1} V^{\mathrm{T}}(G_k)\lambda^{2(j-k-1)}\Psi^*, \quad j < 0, \tag{52}$$

where $V^{\mathrm{T}}(G_k)$ is the transpose of $V(G_k)$.

Let $\Psi_k = (q_{1k}, q_{2k}, q_{3k})^{\mathrm{T}}$, $\Psi^*_k = (p_{1k}, p_{2k}, p_{3k})^{\mathrm{T}}$ be the eigenfunctions corresponding to N the eigenvalues λ_k ($k = 1, \ldots, N$) of (8) and (27). Let us start from the constraint $G_{-1} = -\frac{1}{6}\sum_{j=1}^{N}\nabla\lambda_j$. This constraint is giving the symplectic submanifold M we need. Let the two antisymmetric operators act on this constraint, we have:

$$G_j = -\frac{1}{6}\langle\Lambda^{2j+3}q_1, p_3\rangle, \quad j < 0. \tag{53}$$

Therefore, a complicated calculation yields the following formulations:

$$\begin{aligned}
&G_j - G_{j,xx} = \frac{1}{6}(\langle\Lambda^{2j+3}q_1, p_1\rangle + \langle\Lambda^{2j+3}q_3, p_3\rangle - 2\langle\Lambda^{2j+3}q_2, p_2\rangle),\\
&\Gamma G_j = \frac{1}{6}(\langle\Lambda^{2j+3}q_1, p_1\rangle + \langle\Lambda^{2j+3}q_3, p_3\rangle - 2\langle\Lambda^{2j+3}q_2, p_2\rangle - 3\langle\Lambda^{2j+3}q_1, p_3\rangle),\\
&\Gamma G_{j,x} = \frac{1}{2}(\langle\Lambda^{2j+3}q_1, p_2\rangle + \langle\Lambda^{2j+3}q_2, p_1\rangle - \langle\Lambda^{2j+3}q_3, p_2\rangle),\\
&\Gamma G_{j,xx} = \frac{1}{2}\big[m(\langle\Lambda^{2j+4}q_1, p_2\rangle + \langle\Lambda^{2j+4}q_2, p_3\rangle) + 2\langle\Lambda^{2j+3}q_3, p_1\rangle +\\
&\qquad + \langle\Lambda^{2j+3}q_3, p_3\rangle - \langle\Lambda^{2j+3}q_1, p_1 + p_3\rangle\big],\\
&\Theta G_j = \frac{1}{2}(\langle\Lambda^{2j+3}q_2, p_1 + p_3\rangle - \langle\Lambda^{2j+3}q_3, p_2\rangle),\\
&\partial^{-1}\Upsilon G_j - 2mG_j = -\frac{1}{6}(\langle\Lambda^{2j+2}q_2, p_1 + p_3\rangle + \langle\Lambda^{2j+2}q_3, p_2\rangle),\\
&\partial^2\Theta^{-1}\Upsilon G_j + 2mG_j = \frac{1}{6}(\langle\Lambda^{2j+2}q_2, p_1\rangle + \langle\Lambda^{2j+2}q_3, p_2\rangle - \langle\Lambda^{2j+2}q_1, p_2\rangle),\\
&\Theta^{-1}\Upsilon G_j = -\frac{1}{6}(\langle\Lambda^{2j+2}q_1, p_2\rangle + \langle\Lambda^{2j+2}q_2, p_3\rangle).
\end{aligned}$$

Substituting these equalities to Equations (51) and (52), with a similar computational method to Subsection 4.1 we find the nonlinearized systems of the time part (51) and the adjoint time part (52) are cast in a canonical Hamiltonian system on the $(6N-2)$-dimensional symplectic submanifold M:

$$\begin{aligned} q_{t_k} &= \{q, F_k\}_D,\\ p_{t_k} &= \{p, F_k\}_D, \end{aligned} \qquad k = -1, -2, -3, \ldots, \tag{54}$$

where

$$\begin{aligned}
F_k = {}& \frac{2}{3}\langle\Lambda^{2k+1}q_1, p_1\rangle - \frac{1}{3}\langle\Lambda^{2k+1}q_2, p_2\rangle - \frac{1}{3}\langle\Lambda^{2k+1}q_3, p_3\rangle - \langle\Lambda^{2k+1}q_3, p_1\rangle +\\
&+ \sum_{j=k}^{-2}\Big[-\frac{1}{12}\langle\Lambda^{2j+3}q_1, p_1\rangle\langle\Lambda^{2(k-j)-1}q_1, p_1\rangle -\\
&- \frac{1}{3}\langle\Lambda^{2j+3}q_2, p_2\rangle\langle\Lambda^{2(k-j)-1}q_2, p_2\rangle - \frac{1}{4}\langle\Lambda^{2j+3}q_3, p_3\rangle\langle\Lambda^{2(k-j)-1}q_3, p_3\rangle +
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{3}\langle\Lambda^{2j+3}q_2, p_2\rangle\langle\Lambda^{2(k-j)-1}q_1, p_1\rangle - \frac{1}{6}\langle\Lambda^{2j+3}q_3, p_3\rangle\langle\Lambda^{2(k-j)-1}q_1, p_1\rangle +\\
&+\frac{1}{3}\langle\Lambda^{2j+3}q_3, p_3\rangle\langle\Lambda^{2(k-j)-1}q_2, p_2\rangle - \frac{1}{4}\langle\Lambda^{2j+3}q_1, p_2\rangle\langle\Lambda^{2(k-j)-1}q_1, p_2\rangle +\\
&+\frac{1}{4}\langle\Lambda^{2j+3}q_1, p_3\rangle\langle\Lambda^{2(k-j)-1}q_1, p_3\rangle + \frac{1}{4}\langle\Lambda^{2j+3}q_2, p_3\rangle\langle\Lambda^{2(k-j)-1}q_2, p_3\rangle +\\
&+\frac{1}{2}(\langle\Lambda^{2j+3}q_1, p_2\rangle - \langle\Lambda^{2j+3}q_2, p_3\rangle)\times\\
&\times(\langle\Lambda^{2(k-j)-1}q_3, p_2\rangle - \langle\Lambda^{2(k-j)-1}q_2, p_1\rangle) + \frac{1}{2}\langle\Lambda^{2j+3}q_1, p_3\rangle\times\\
&\times\left.(\langle\Lambda^{2(k-j)-1}q_1, p_1\rangle - 2\langle\Lambda^{2(k-j)-1}q_3, p_1\rangle - \langle\Lambda^{2(k-j)-1}q_3, p_3\rangle)\right] +\\
&+\sum_{j=k}^{-1}\left[-\frac{1}{4}\langle\Lambda^{2j+2}q_1, p_1\rangle\langle\Lambda^{2(k-j)}q_1, p_1\rangle - \frac{1}{4}\langle\Lambda^{2j+2}q_3, p_3\rangle\langle\Lambda^{2(k-j)}q_3, p_3\rangle +\right.\\
&+\frac{1}{4}\langle\Lambda^{2j+2}q_1, p_2\rangle\langle\Lambda^{2(k-j)}q_1, p_2\rangle - \frac{1}{4}\langle\Lambda^{2j+2}q_1, p_3\rangle\langle\Lambda^{2(k-j)}q_1, p_3\rangle -\\
&-\frac{1}{4}\langle\Lambda^{2j+2}q_2, p_3\rangle\langle\Lambda^{2(k-j)}q_2, p_3\rangle - \frac{1}{2}(\langle\Lambda^{2j+2}q_1, p_2\rangle + \langle\Lambda^{2j+2}q_2, p_3\rangle)\times\\
&\times(\langle\Lambda^{2(k-j)}q_3, p_2\rangle + \langle\Lambda^{2(k-j)}q_2, p_1\rangle) + \frac{1}{2}\langle\Lambda^{2j+2}q_1, p_3\rangle\langle\Lambda^{2(k-j)}q_3, p_3\rangle +\\
&\left.+\frac{1}{2}\langle\Lambda^{2j+2}q_3, p_3\rangle\langle\Lambda^{2(k-j)}q_1, p_1\rangle - \frac{1}{2}\langle\Lambda^{2j+2}q_1, p_3\rangle\langle\Lambda^{2(k-j)}q_1, p_1\rangle\right].
\end{aligned}
$$

Remark 4. (1) In the above calculation procedure we used the following facts:

$$
\begin{aligned}
&\{q_1, F\} = \{q_2, F\} = 0,\\
&\{q_3, F\} = \Lambda q_1,\\
&\{p_1, F\} = \{p_2, F\} = 0,\\
&\{p_3, F\} = -\Lambda p_3,\\
&\{F, F_k\} = 0,\\
&\{G, F_k\} = (\langle\Lambda q_2, p_2\rangle - \langle\Lambda q_3, p_1 + p_3\rangle)(\langle\Lambda^{2k+2}q_1, p_2\rangle + \langle\Lambda^{2k+2}q_2, p_3\rangle),
\end{aligned}
$$

and on the symplectic submanifold M the equalities $F = G = F_x = F_{xx} = 0$ hold.

(2) It should be possible to give a solution for the Hamiltonian systems (54) by Maple or Mathematica. But that needs to figure out a special code program with integral–differential operator action, particularly for the inverse operator action. This is also for the case of the matrix equation (11).

Apparently, $F_{-1} = I$. Furthermore, by the Dirac–Poisson bracket (36) on submanifold M we obtain the following theorem.

THEOREM 4. *All Hamiltonian flows* (39) *and* (54) *commute on* M.

Proof. Through a lengthy calculation, we have

$$\{H, F_k\} = 0,$$
$$\{H, F\} = 0,$$
$$\{F, F_k\} = 0,$$
$$\{G, F_k\} = (\langle \Lambda q_2, p_2\rangle - \langle \Lambda q_3, p_1 + p_3\rangle)(\langle \Lambda^{2k+2} q_1, p_2\rangle + \langle \Lambda^{2k+2} q_2, p_3\rangle).$$

Therefore,

$$\{H, F_k\}_D = \{H, F_k\} + \frac{1}{\{F, G\}}(\{H, F\}\{G, F_k\} - \{H, G\}\{F, F_k\}) = 0,$$

which completes the proof. □

So, all Hamiltonian flows (54) are integrable on M. Particularly, the Hamiltonian system (39) is integrable.

4.3. PARAMETRIC SOLUTIONS

THEOREM 5. *Let* $p(x, t_k), q(x, t_k)$ $(p(x, t_k) = (p_1, p_2, p_3)^{\mathrm{T}}, q(x, t_k) = (q_1, q_2, q_3)^{\mathrm{T}}, k = -1, -2, \ldots)$ *be the common solution of the two integrable flows* (39) *and* (54), *then*

$$m = 2\frac{\langle \Lambda q_2, p_2\rangle - \langle \Lambda q_3, p_1 + p_3\rangle}{\langle \Lambda^2 q_2, p_3\rangle + \langle \Lambda^2 q_1, p_2\rangle}, \tag{55}$$

satisfies the negative order hierarchy (18).

Proof. Noticing the following formulas

$$G_k = -\frac{1}{6}\langle \Lambda^{2k+3} q_1, p_3\rangle, \quad k = -1, -2, -3, \ldots,$$
$$\Theta^{-1}\Upsilon G_k = -\frac{1}{6}(\langle \Lambda^{2k+2} q_1, p_2\rangle + \langle \Lambda^{2k+2} q_2, p_3\rangle),$$
$$\Upsilon = m\partial + 2\partial m, \qquad \Upsilon^* = \partial m + 2m\partial, \qquad \Theta = \partial - \partial^3,$$

and Equation (54), we directly compute and find Equation (55) satisfies $m_{t_k} = 3\Upsilon^*\Theta^{-1}\Upsilon G_k$ which completes the proof. □

In particular, we obtain the following theorem.

THEOREM 6. *Let* $p(x, t), q(x, t)$ $(p(x, t) = (p_1, p_2, p_3)^{\mathrm{T}}, \quad q(x, t) = (q_1, q_2, q_3)^{\mathrm{T}})$ *be the common solution of the two integrable flows* (39) *and* (47), *then*

$$m = 2\frac{\langle \Lambda q_2(x,t), p_2(x,t)\rangle - \langle \Lambda q_3(x,t), p_1(x,t) + p_3(x,t)\rangle}{\langle \Lambda^2 q_2(x,t), p_3(x,t)\rangle + \langle \Lambda^2 q_1(x,t), p_2(x,t)\rangle}, \tag{56}$$

$$u = \frac{1}{2}(\langle q_1(x,t), p_2(x,t)\rangle + \langle q_2(x,t), p_3(x,t)\rangle) - \frac{1}{2}H, \tag{57}$$

satisfy the DP equation:

$$m_t + um_x + 3mu_x = 0. \tag{58}$$

Proof. Let

$$A = \langle \Lambda q_2, p_2 \rangle - \langle \Lambda q_3, p_1 + p_3 \rangle,$$
$$B = \langle \Lambda^2 q_2, p_3 \rangle + \langle \Lambda^2 q_1, p_2 \rangle,$$
$$C = \langle \Lambda^2 q_3, p_3 \rangle - \langle \Lambda^2 q_1, p_1 + p_3 \rangle.$$

Then through a lengthy calculation we have

$$A_t = uG + umC - 2u_x A = umC - 2u_x A,$$
$$B_t = G - uC + u_x B = -uC + u_x B,$$
$$A_x = -2G - mC = -mC,$$
$$B_x = C.$$

By the above equalities, we obtain

$$\begin{aligned} m_t + um_x + 3mu_x &= \frac{2}{B^2}[A_t B - AB_t + u(A_x B - AB_x) + 3u_x AB] \\ &= \frac{2}{B^2}(umBC + uAC + uA_x B - uAB_x) \\ &= \frac{2u}{B^2}(B(mC + A_x) + A(C - B_x)) \\ &= 0. \end{aligned}$$

In the above proof procedures, we have used the following equalities: $F = G = 0$, $F_x = F_{xx} = 0$ on M. □

Similarly, we can discuss the parametric solution of the positive order hierarchy (23). That needs us to consider a new kind of constraint and related integrable system, which we deal with in the next section.

5. Parametric Solution of the Positive Order Hierarchy

Let us directly consider the following constraint:

$$G_0 = \sum_{j=1}^{N} E_j \nabla \lambda_j, \tag{59}$$

where $E_j \nabla \lambda_j = \lambda_j q_{1j} p_{3j}$ is the functional gradient of λ_j for the spectral problems (8) and (27), and q_{kj}, p_{kj} $(k = 1, 2, 3)$ are the related eigenfunctions of λ_j. Then Equation (59) is saying

$$m = \langle \Lambda q_1, p_3 \rangle^{-3/2} \tag{60}$$

which composes a new constraint in the whole space $\mathbb{R}^{6N}$. Under this constraint, the spectral problem (8) and its adjoint problem (27) are able to be cast in a Hamiltonian canonical form in $\mathbb{R}^{6N}$:

$$(H^+)\colon\quad \begin{aligned} q_x &= \{q, H^+\},\\ p_x &= \{p, H^+\}, \end{aligned} \tag{61}$$

with the Hamiltonian

$$H^+ = \langle q_2, p_1 + p_3\rangle + \langle q_3, p_2\rangle + \frac{2}{\sqrt{\langle \Lambda q_1, p_3\rangle}}. \tag{62}$$

To see the integrability of the system (61), we take into account of the time part $\Psi_t = V_1(m, \lambda)\Psi$ and its adjoint $\Psi_t = -V_1^{\mathrm{T}}(m, \lambda)\Psi$, where $V_1(m, \lambda)$ is defined by Equation (26). Under the constraint (60), the time part and its adjoint are also nonlinearized as a canonical Hamiltonian system in $\mathbb{R}^{6N}$

$$(F_1)\colon\quad \begin{aligned} q_t &= \{q, F_1\},\\ p_t &= \{p, F_1\}, \end{aligned} \tag{63}$$

with the Hamiltonian

$$\begin{aligned} F_1 = {} & 6\frac{\langle \Lambda^2 q_1, p_2\rangle + \langle \Lambda^2 q_2, p_3\rangle}{\sqrt{\langle \Lambda q_1, p_3\rangle}} - \\ & - \frac{1}{2}(\langle \Lambda q_1, p_1\rangle^2 + 4\langle \Lambda q_2, p_2\rangle^2 + \langle \Lambda q_3, p_3\rangle^2) + \\ & + \frac{3}{2}(\langle \Lambda q_1, p_3\rangle^2 + \langle \Lambda q_2, p_3\rangle^2 - \langle \Lambda q_1, p_2\rangle^2) + \\ & + 3\langle \Lambda q_1, p_3\rangle(\langle \Lambda q_1, p_1\rangle - 2\langle \Lambda q_3, p_1\rangle - \langle \Lambda q_3, p_3\rangle) + \\ & + 2\langle \Lambda q_1, p_1\rangle\langle \Lambda q_2, p_2\rangle + 2\langle \Lambda q_2, p_2\rangle\langle \Lambda q_3, p_3\rangle - \langle \Lambda q_1, p_1\rangle\langle \Lambda q_3, p_3\rangle + \\ & + 3(\langle \Lambda q_1, p_2\rangle - \langle \Lambda q_2, p_3\rangle)(\langle \Lambda q_3, p_2\rangle - \langle \Lambda q_2, p_1\rangle). \end{aligned} \tag{64}$$

After a calculation of the Poisson bracket of $\{H^+, F_1\}$, we know that the two canonical Hamiltonian flows commute, i.e.

$$\{H^+, F_1\} = 0.$$

Furthermore, under the constraint (60) the nonlinearized systems of the general time part $\Psi_{t_k} = \sum_{j=0}^{k-1} V(G_j)\lambda^{2(k-j-1)}\Psi$ ($k > 0$, $k \in \mathbb{Z}$) and its adjoint problem produce the following canonical Hamiltonian system in $\mathbb{R}^{6N}$:

$$(F_k)\colon\quad \begin{aligned} q_{t_k} &= \{q, F_k\},\\ p_{t_k} &= \{p, F_k\}, \end{aligned} \qquad k = 1, 2, 3, \ldots, \tag{65}$$

with the Hamiltonian

$$F_k = 6\frac{\langle \Lambda^{2k} q_1, p_2\rangle + \langle \Lambda^{2k} q_2, p_3\rangle}{\sqrt{\langle \Lambda q_1, p_3\rangle}} +$$

$$
\begin{aligned}
&+\sum_{j=0}^{k-1}\Big[-\frac{1}{2}\langle\Lambda^{2j+1}q_1,p_1\rangle\langle\Lambda^{2(k-j)-1}q_1,p_1\rangle-\\
&-2\langle\Lambda^{2j+1}q_2,p_2\rangle\langle\Lambda^{2(k-j)-1}q_2,p_2\rangle-\frac{1}{2}\langle\Lambda^{2j+1}q_3,p_3\rangle\langle\Lambda^{2(k-j)-1}q_3,p_3\rangle+\\
&+2\langle\Lambda^{2j+1}q_2,p_2\rangle\langle\Lambda^{2(k-j)-1}q_1,p_1\rangle-\langle\Lambda^{2j+1}q_3,p_3\rangle\langle\Lambda^{2(k-j)-1}q_1,p_1\rangle+\\
&+2\langle\Lambda^{2j+1}q_3,p_3\rangle\langle\Lambda^{2(k-j)-1}q_2,p_2\rangle-\frac{3}{2}\langle\Lambda^{2j+1}q_1,p_2\rangle\langle\Lambda^{2(k-j)-1}q_1,p_2\rangle+\\
&+\frac{3}{2}\langle\Lambda^{2j+1}q_1,p_3\rangle\langle\Lambda^{2(k-j)-1}q_1,p_3\rangle+\frac{3}{2}\langle\Lambda^{2j+1}q_2,p_3\rangle\langle\Lambda^{2(k-j)-1}q_2,p_3\rangle+\\
&+3(\langle\Lambda^{2j+1}q_1,p_2\rangle-\langle\Lambda^{2j+1}q_2,p_3\rangle)\times\\
&\times(\langle\Lambda^{2(k-j)-1}q_3,p_2\rangle-\langle\Lambda^{2(k-j)-1}q_2,p_1\rangle)+3\langle\Lambda^{2j+1}q_1,p_3\rangle\times\\
&\times(\langle\Lambda^{2(k-j)-1}q_1,p_1\rangle-2\langle\Lambda^{2(k-j)-1}q_3,p_1\rangle-\langle\Lambda^{2(k-j)-1}q_3,p_3\rangle)\Big]+\\
&+\sum_{j=1}^{k-1}\Big[-\frac{3}{2}\langle\Lambda^{2j}q_1,p_1\rangle\langle\Lambda^{2(k-j)}q_1,p_1\rangle-\frac{3}{2}\langle\Lambda^{2j}q_3,p_3\rangle\langle\Lambda^{2(k-j)}q_3,p_3\rangle+\\
&+\frac{3}{2}\langle\Lambda^{2j}q_1,p_2\rangle\langle\Lambda^{2(k-j)}q_1,p_2\rangle-\frac{3}{2}\langle\Lambda^{2j}q_1,p_3\rangle\langle\Lambda^{2(k-j)}q_1,p_3\rangle-\\
&-\frac{3}{2}\langle\Lambda^{2j}q_2,p_3\rangle\langle\Lambda^{2(k-j)}q_2,p_3\rangle-3(\langle\Lambda^{2j}q_1,p_2\rangle+\langle\Lambda^{2j}q_2,p_3\rangle)\times\\
&\times(\langle\Lambda^{2(k-j)}q_3,p_2\rangle+\langle\Lambda^{2(k-j)}q_2,p_1\rangle)+3\langle\Lambda^{2j}q_1,p_3\rangle\langle\Lambda^{2(k-j)}q_3,p_3\rangle+\\
&+3\langle\Lambda^{2j}q_3,p_3\rangle\langle\Lambda^{2(k-j)}q_1,p_1\rangle-3\langle\Lambda^{2j}q_1,p_3\rangle\langle\Lambda^{2(k-j)}q_1,p_1\rangle\Big].
\end{aligned}
$$

Apparently, when $k=1$, F_k is exactly Equation (64). Furthermore, through a lengthy computation, we obtain

$$\{H^+,F_k\}=0,\quad k=1,2,\ldots, \tag{66}$$

which represents each Hamiltonian t-flow (F_k) commutes with Hamiltonian x-flow (H^+). Thus, all Hamiltonian canonical systems (F_k) are integrable in $\mathbb{R}^{6N}$. Particularly, the nonlinearized spectral problems (61) is integrable.

Like last section, we also have a similar theorem

THEOREM 7. *Let* $p(x,t_k),q(x,t_k)$ $(p(x,t_k)=(p_1,p_2,p_3)^{\mathrm{T}}$, $q(x,t_k)=(q_1,q_2,q_3)^{\mathrm{T}}$, $k=1,2,3,\ldots)$ *be the common solution of the two integrable flows* (61) *and* (65)*, then*

$$m=\frac{1}{\sqrt{\langle\Lambda q_1(x,t_k),p_3(x,t_k)\rangle^3}},\quad k=1,2,3,\ldots, \tag{67}$$

satisfy the positive order hierarchy (23).

In particular, we have the following theorem.

THEOREM 8. *Let* $p(x,t), q(x,t)$ $(p(x,t) = (p_1, p_2, p_3)^{\mathrm{T}}, q(x,t) = (q_1, q_2, q_3)^{\mathrm{T}})$ *be the common solution of the two integrable flows* (61) *and* (63), *then*

$$m = \frac{1}{\sqrt{\langle \Lambda q_1(x,t), p_3(x,t)\rangle^3}} \tag{68}$$

is a parametric solution of the 5th-*order equation:*

$$m_t = 4(m^{-2/3})_x - 5(m^{-2/3})_{xxx} + (m^{-2/3})_{xxxxx}. \tag{69}$$

Proof. A direct check is done through substitution of Equations (61) and (63). □

6. Summaries

We have known that the DP equation $m_t + um_x + 3mu_x = 0$, $m = u - m_{xx}$ has peaked soliton solution $u = \mathrm{e}^{-|x+t|}$, and moreover it has multi-peaked-soliton solutions (Holm and Staley, 2002; Holm and Hone, 2002). For more general case, all b-balanced equations $m_t + um_x + bmu_x = 0$, $m = u - m_{xx}$, $b =$ constant are also found to have this kind of solutions (Holm and Staley, 2002). But the finite-dimensional systems, derived from the all b-balanced equations (including the DP equation ($b = 3$) but except the CH equation ($b = 2$)) through multisoliton solutions setting, are not canonical Hamiltonian system (see Holm and Hone, 2002). Here our results show that the DP spectral problem can be developed as a completely integrable canonical Hamiltonian system under a constraint between the potential and the eigenfunctions. Furthermore, we give the parametric solutions for the whole hierarchy, in particular for the DP equation. Our parametric solutions are not given in an explicit form, but from our experience (Qiao, 2001, 2002a, 2003) we believe that our parametric solutions should give perodic/quasi-perodic solutions which are explicit instead of multi-soliton solutions. Our parametric solutions can not include the peaked soliton solutions because the parametric solutions are smooth but the peaked soliton solutions only continous.

Nevertheless, we would like to directly derive possible traveling wave solutions for the 5th-order PDE

$$m_t = 4(m^{-2/3})_x - 5(m^{-2/3})_{xxx} + (m^{-2/3})_{xxxxx}$$

which is the second equation in the positive order hierarchy (23). Let us set $m^{-2/3} = v$, then this equation becomes

$$-\frac{3}{2}v^{-2/3}v_t = 4v_x - 5v_{xxx} + v_{xxxxx}. \tag{70}$$

Assume this equation has the solution $v = f(\xi)$, $\xi = x - ct$, $c =$ constant, then we have

$$2cf^{-1/2} = 2f^2 - \frac{5}{2}f'^2 + f'f''' - \frac{1}{2}f''^2. \tag{71}$$

The right-hand side of this equation is quadratically homogeneous and the left hand side not. Therefore, we set $f = \mathrm{e}^{a\xi}$, $a =$ constant and substitute it into Equation (71), and obtain

$$c = 0, \tag{72}$$
$$a^4 - 5a^2 + 4 = 0, \tag{73}$$

which implies

$$a = \pm 1, \qquad a = \pm 2. \tag{74}$$

So, the 5th-order equation (70) has the following stationary solutions:

$$1,\ \mathrm{e}^{-x},\ \mathrm{e}^{x},\ \mathrm{e}^{-2x},\ \mathrm{e}^{2x}, \tag{75}$$

which exactly composes the basis of the solution space of the stationary equation $4v_x - 5v_{xxx} + v_{xxxxx} = 0$. Therefore, the 5th-order PDE $m_t = 4(m^{-2/3})_x - 5(m^{-2/3})_{xxx} + (m^{-2/3})_{xxxxx}$ possesses the stationary solutions

$$m(x) = (c_0 + c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{x} + c_3\mathrm{e}^{-2x} + c_4\mathrm{e}^{2x})^{-3/2},$$
$$\forall c_k \in \mathbb{R},\ k = 0, 1, 2, 3, 4. \tag{76}$$

Apparently, $\mathrm{e}^{-\frac{3}{2}x}$, $\mathrm{e}^{\frac{3}{2}x}$, e^{-3x}, e^{3x} satisfy the 5th-order PDE $m_t = 4(m^{-2/3})_x - 5(m^{-2/3})_{xxx} + (m^{-2/3})_{xxxxx}$.

COMPARISON WITH THE PARAMETRIC SOLUTION

All the stationary solutions (76) may be included in the parametric solution (68). For example, the function $m(x) = \mathrm{e}^{-\frac{3}{2}x}$ is cast in Equation (68) when we choose dynamical variables q_1, p_3 such that the following constraint

$$\langle \Lambda q_1(x), p_3(x)\rangle = \mathrm{e}^{x} \tag{77}$$

holds, where q_1, p_3 are the solutions of the integrable Hamiltonian system (61).

The 5th-order PDE (70) is not located in the Harry–Dym hierarchy, and therefore it has no cusp (Wadati *et al.*, 1980) solution. As for what kind of PDEs have the cusp solution, see (Qiao, 2002b).

Acknowledgements

The author would like to express his sincere thanks to D. Holm, M. Mineev and K. Vixie for their fruitful discussions, and also to B. Konopelchenko and F. Magri for their helpful suggestions during their visit at Los Alamos National Laboratory.

This work has been supported by the LANL LDRD Project 20020006ER "Unstable Fluid/Fluid Interfaces", the Data Driven Modeling and Analysis Group, the Foundation for the Author of National Excellent Doctoral Dissertation (FANEDD) of P.R. China, and also the research grant of University of Texas-Pan American, USA.

References

Ablowitz, M. J., Kaup, D. J., Newell, A. C. and Segur, H. (1973) Nolinear evolution equations of physical significance, *Phys. Rev. Lett.* **31**, 125–127.

Ablowitz, M. J., Kaup, D. J., Newell, A. C. and Segur, H. (1974) *Stud. Appl. Math.* **53**, 249–315.

Alber, M. S., Camassa, R., Holm, D. D. and Marsden, J. E. (1994) The geometry of peaked solitons and billiard solutions of a class of integrable PDE's, *Lett. Math. Phys.* **32**, 137–151.

Alber, M. S., Camassa, R., Holm, D. D. and Marsden, J. E. (1995) On the link between umbilic geodesics and soliton solutions of nonlinear PDE's, *Proc. Roy. Soc.* **450**, 677–692.

Alber, M. S., Camassa, R., Fedorov, Y. N., Holm, D. D. and Marsden, J. E. (1999) On billiard solutions of nonlinear PDE's, *Phys. Lett. A* **264**, 171–178.

Alber, M. S., Camassa, R., Fedorov, Y. N., Holm, D. D. and Marsden, J. E. (2001) The complex geometry of weak piecewise smooth solutions of integrable nonlinear PDE's of shallow water and Dym type, *Comm. Math. Phys.* **221**, 197–227.

Arnol'd, V. I. (1978) *Mathematical Methods of Classical Mechanics*, Springer-Verlag, Berlin.

Camassa, R. and Holm, D. D. (1993) An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* **71**, 1661–1664.

Cao, C. W. (1989) Nonlinearization of Lax system for the AKNS hierarchy, *Sci. China A* **32**, 701–707 (in Chinese); also see English Edition: Nonlinearization of Lax system for the AKNS hierarchy, *Sci. Sin. A* **33** (1990), 528–536.

Constantin, A. and McKean, H. P. (1999) A shallow water equation on the circle, *Comm. Pure Appl. Math.* **52**, 949–982.

Degasperis, A., Holm, D. D. and Hone, A. N. W. (2002) A new integrable equation with peakon solutions, *Theoret. and Math. Phys.* **133**, 1463–1474.

Degasperis, A. and Procesi, M. (1999) Asymptotic integrability, in A. Degasperis and G. Gaeta (eds), *Symmetry and Perturbation Theory*, World Scientific, pp. 23–37.

Fokas, A. S. and Anderson, R. L. (1982) On the use of isospectral eigenvalue problems for obtaining hereditary symmetries for Hamiltonian systems, *J. Math. Phys.* **23**, 1066–1073.

Fuchssteiner, B. and Fokas, A. S. (1981) Symplectic structures, their Baecklund transformations and hereditaries, *Physica D* **4**, 47–66.

Gardner, C. S., Greene, J. M., Kruskal, M. D. and Miura, R. M. (1967) Method for solving the Korteweg–de Vries equation, *Phys. Rev. Lett.* **19**, 1095–1097.

Holm, D. D. and Hone, A. N. W. (2002) Note on Peakon bracket, Private communication.

Holm, D. D. and Staley, M. (2002) Private communication.

Kaup, D. J. (1980) On the inverse scattering problem for cubic eigenvalue problems of the class $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi$, *Stud. Appl. Math.* **62**, 189–216.

Konopelchenko, B. G. and Dubrovsky, V. G. (1984) Some new integrable nonlinear evolution equations in $2+1$ dimensions, *Phys. Lett. A* **102**, 15–17.

Kupershmidt, B. A. (1984) A super Korteweg–de Vries equation: An integrable system, *Phys. Lett. A* **102**, 213–215.

Korteweg, D. J. and de Vries, G. (1895) On the change of form long waves advancing in a rectangular canal, and on a new type of long stationary waves, *Phil. Mag.* **39**, 422–443.

Levitan, B. M. and Gasymov, M. G. (1964) Determination of a differential equation by two of its spectra, *Russian Math. Surveys* **19**(2), 1–63.

Ma, W. X. and Strampp, W. (1994) An explicit symmetry constraint for the Lax pairs and the adjoint Lax pairs of AKNS systems, *Phys. Lett. A* **185**, 277–286.

Marchenko, V. A. (1950) Certain problems in the theory of second-order differential operators, *Dokl. Akad. Nauk SSSR* **72**, 457–460 (Russian).

Qiao, Z. J. (2001) Generalized r-matrix structure and algebro-geometric solution for integrable system, *Rev. Math. Phys.* **13**, 545–586.

Qiao, Z. J. (2002a) *Finite-Dimensional Integrable System and Nonlinear Evolution Equations*, Higher Education Press, P.R. China.

Qiao, Z. J. (2002b) Equations possessing cusp solitons and cusp-like singular traveling wave solutions, Preprint.

Qiao, Z. J. (2002c) A new integrable hierarchy, parametric solutions, and traveling wave solutions, Preprint, to appear in *Math. Phys. Anal. Geom.*

Qiao, Z. J. (2003) The Camassa–Holm hierarchy, N-dimensional integrable systems, and algebro-geometric solution on a symplectic submanifold, *Comm. Math. Phys.* **239**, 309–341.

Tu, G. Z. (1990) An extension of a theorem on gradients of conserved densities of integrable systems, *Northeast. Math. J.* **6**, 26–32.

Wadati, M., Ichikawa, Y. H. and Shimizu, T. (1980) Cusp soliton of a new integrable nonlinear evolution equation, *Prog. Theor. Phys.* **64**, 1959–1967.

Zakharov, V. E. and Shabat, A. B. (1972) Exact theory of two dimensional self focusing and one dimensional self modulation of waves in nonlinear media, *Sov. Phys. JETP* **34**, 62–69.