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New hierarchies of isospectral and non-isospectral integrable NLEEs derived from the Harry–Dym spectral problem [1]

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Abstract

By making use of our previous framework, we display the hierarchies of generalized nonlinear evolution equations (GNLEEs) associated with the Harry–Dym spectral problem, and construct the corresponding generalized Lax representations (GLR) in this paper. It will be clearly seen that the generalized hierarchies can include not only the well-known Harry–Dym hierarchy of isospectral NLEEs but also other new integrable hierarchies of isospectral and non-isospectral NLEEs. Through choosing the so-called 'seed' function these new hierarchies of NLEEs give some new integrable evolution equations, which are very likely to have potential applications in theoretical and experimental physics. All of these hierarchies of NLEEs possess the GLR.

It is well known that, beginning with a proper linear spectral problem $Ly = \lambda y$ (λ is a spectral parameter), we cannot only generate an isospectral ($\lambda_t = 0$) hierarchy of nonlinear evolution equations (NLEEs) integrable by the IST (see Refs. [1–7]), but also produce a corresponding non-isospectral (for example, $\lambda_t = \lambda^n$, $n \geqslant 0$) hierarchy of NLEEs [8] which are often solved still by the IST [9]. Generally, the NLEEs (whether isospectral or non-isospectral hierarchy) by the famous IST [1] possess the Lax representation. Among these are the well-known KdV equation, AKNS equation in 1+1 dimensions and KP equation, Davey–Stewarton equation in 1+2 dimensions [2,10]. In the past two decades the Lax representation of the hierarchy of NLEEs has

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played a very important role in the research and discussion of NLEEs and symmetry. In particular, the Lax representation has been successively used in bi-Hamiltonian structures of finite-dimensional dynamical systems [11,12], the nonlinearization theory of soliton system to produce many new completely integrable systems in the Liouville sense [13,14], and the finite-dimensional restricted flows of the underlying infinite systems [15]. Hence, for a given hierarchy of NLEEs, to find the Lax representation is of great importance.

In a previous paper [16], we mainly discussed the algebraic structure of the operator related to stationary systems. And before doing that, we first presented a so-called operator pattern for generating hierarchies of generalized nonlinear evolution equations (GNLEEs) and their generalized Lax representations (GLR) (see Ref. [16], also see Ref. [17]). In recent years, the time-discrete versions [18] of integrable systems have already attracted a lot of attention. Capel, Nijhoff and their collaborators have obtained many developments [19–21] in this fields, and their methods have been considered for establishing a new Lax pair from the time part of the old Lax pair for the continuous integrable evolution equations [22].

The present paper is inspired on the idea of combining our previous framework [16] with the line of thought of Capel, Nijhoff and their collaborators [20,21,23]. We shall display the hierarchies of GNLEEs associated with the Harry–Dym spectral problem by the Lenard recursive scheme, and construct the corresponding GLR by solving a key operator equation. We shall clearly see that the generalized hierarchies can include not only the well-known Harry–Dym hierarchy of isospectral NLEEs but also other new integrable hierarchies of isospectral and non-isospectral NLEEs. These new hierarchies contain some new integrable evolution equations which may be possibly applied in the study of theoretical and experimental physics. All of these hierarchies of integrable NLEEs have the GLR.

For the convenient use, let us first shortly recall the productive procedure of GNLEEs and GLR. Consider an ordinary $N \times N$ spectral problem

$$Ly = L(u)y = \lambda y\,, \tag{1}$$

where $L = L(u)$ is a spectral operator, $u = (u_1, \ldots, u_l)^T$ is a potential vector function, λ is a spectral parameter, $y = (y_1, \ldots, y_N)^T$. Let $M = (m_{ij})_{N\times N}$, $\tilde{M} = (\tilde{m}_{ij})_{N\times N}$ be two arbitary given $N \times N$ matrix operators. We construct the following two operator equations with respect to the vector functions $G_0 = (G_0^{(1)}, \ldots, G_0^{(l)})^T, G_{-1} = (G_{-1}^{(1)}, \ldots, G_{-1}^{(l)})^T$:

$$L_*(JG_0) = M\,, \tag{2}$$

$$L_*(KG_{-1}) = \tilde{M}\,, \tag{3}$$

where $L_*(\xi)$ stands for the Gateaux derivative operator of $L(u)$ in the direction ξ, and K, J are a pair of Lenard's operators which is usually searched for by the spectral gradient method (SGM) [24,25]. Suppose the solution set $\aleph_J(M) \neq \emptyset$, $\aleph_K(\tilde{M}) \neq \emptyset$ of

Eqs. (2) and (3), then the following NLEEs:

$$u_{t_m} = X_m(u) = \begin{cases} KG_m = K\mathscr{L}^m G_0, & m \geqslant 0,\ G_0 \in \aleph_J(M)\,, \\ JG_m = J\mathscr{L}^{m+1}G_{-1}, & m < 0,\ G_{-1} \in \aleph_K(\tilde{M})\,, \end{cases} \tag{4}$$

are called the hierarchies of GNLEEs associated with Eq. (1). In Eq. (4), $\mathscr{L} = J^{-1}K$.

Let $G = (G^{(1)}, \ldots, G^{(l)})^T$ be an arbitary given vector function. We establish a key operator equation with regard to the operator $V = V(G)$:

$$[V, L] = L_*(KG)L^\beta - L_*(JG)L^\alpha\,, \tag{5}$$

where $[\cdot,\cdot]$ is the commutator, $\alpha, \beta \in$R are two fixed constants, and $\alpha > \beta$.

Suppose $\aleph_J(M) \neq \emptyset$, $\aleph_K(\tilde{M}) \neq \emptyset$, L_* is injective, and for any $G_j = \mathscr{L}^j G_0$, $j \geqslant 0$ or $G_j = \mathscr{L}^{j+1}G_{-1}$, $j < 0$, the operator equation (5) possesses the operator solution $V = V(G_j)$. Then the hierarchies of GNLEEs (4) have the following form of Lax representations ($\eta = \alpha - \beta > 0$):

$$\begin{aligned} L_{t_m} &= [W_m, L] + ML^{(m+1)\eta},\ \ m = 0, 1, 2, \ldots, \\ L_{t_m} &= [W_m, L] + \tilde{M}L^{m\eta},\ \ m = -1, -2, -3, \ldots, \end{aligned} \tag{6}$$

where the operator W_m is defined by

$$W_m = \begin{cases} \sum_{j=0}^m V(G_j)L^{(m-j)\eta-\beta}, & m \geqslant 0\,, \\ -\sum_{j=m}^{-1} V(G_j)L^{(m-j)\eta-\alpha}, & m < 0\,. \end{cases} \tag{7}$$

Eq. (6) is called the generalized Lax representations (GLR) of Eq. (4).

From Eq. (6), we immediately know that the hierarchies of positive order GNLEEs $u_{t_m} = X_m(u)$ $(m = 0, 1, 2, \ldots)$ and the hierarchies of negative order GNLEEs $u_{t_m} = X_m(u)$ $(m = -1, -2, -3, \ldots)$ are the integrability condition of the following two linear problem:

$$\begin{aligned} &Ly = \lambda y, \quad \lambda_{t_m} y = \lambda^{(m+1)\eta} My\,, \\ &y_{t_m} = W_m y = \sum_{j=0}^m V(G_j)L^{(m-j)\eta-\beta}y\,, \quad m \geqslant 0 \end{aligned} \tag{8}$$

and

$$\begin{aligned} Ly &= \lambda y, \quad \lambda_{t_m} y = \lambda^{m\eta}\tilde{M}y\,, \\ y_{t_m} &= W_m y = -\sum_{j=m}^{-1} V(G_j)L^{(m-j)\eta-\alpha}y\,, \quad m < 0\,, \end{aligned} \tag{9}$$

respectively, where $M = M(t_m)$, $\tilde{M} = \tilde{M}(t_m)$ only depend on the time variable t_m.

Remark: 1. If one chooses $M = 0$, $\tilde{M} = 0$, then $\lambda_{t_m} = 0$. Right now, Eq. (4) becomes an isospectral hierarchy and has the standard Lax pair $Ly = \lambda y$ and $y_{t_m} = W_m y$, $m \in Z$. If one lets $M = I_{N\times N}$, $\tilde{M} = I_{N\times N}$, and $\aleph_J(M) \neq \emptyset$, $\aleph_K(\tilde{M}) \neq \emptyset$, then $\lambda_{t_m} = \lambda^{(m+1)\eta}$, $m \geqslant 0$; $\lambda_{t_m} = \lambda^{m\eta}$, $m < 0$, and (4) expresses a non-isospectral hierarchy and has the standard Lax pair $Ly=\lambda y$ ($\lambda_{t_m}=\lambda^{(m+1)\eta}$, $m\geqslant 0$; $\lambda_{t_m}=\lambda^{m\eta}$, $m<0$) and $y_{t_m}=W_m y$. By the arbitariness of M and $\tilde{M}$, all possible hierarchies of NLEEs associated with the spectral problem (1) and their GLR are unified in Eqs. (4) and (6), respectively. Because of this, Eq. (4) is called 'the hierarchies of GNLEEs'.

2. In the GLR (6), let $m=0$, then as $\eta=1$, and Eq. (6) becomes $L_{t_0}=[W_0,L]+ML$ which is exactly the L–A–B triad representation of integrable system presented by Manakov [26]. Thus, by using our effective procedure and GLR as described above, through choosing many different operators M, we may seek for the L–A–B representations of many integrable NLEEs. The algebraic structure of L–A–B representation was discussed in Ref. [27]. Naturally, we may study the algebraic structure corresponding to (6), which will be reported in another paper [28].

3. For a concrete spectral operator L, injection of L_* and the conditions $\aleph_J(M)\neq\emptyset$ and $\aleph_K(\tilde{M})\neq\emptyset$ can be easily tested. Hence, in order to obtain the GLR of Eq. (4), its key lies in looking for the operator solution $V=V(G)$ of operator equation (5). As $\beta=0$, $\alpha=1$, and $\beta=\alpha-1$, the solution structures of Eq. (5) were studied in Refs. [29–31] and Ref. [32], respectively. But for general $\alpha,\beta\in \mathrm{R}$, this problem is still open.

In the following, we first derive the hierarchies of GNLEEs associated with the Harry–Dym spectral problem, give the corresponding GLR through solving a key operator equation, and then present some new integrable evolution equations and discuss their GLR.

Consider the spectral problem (a special case of the WKI spectral problem [33])

$$y_x=\begin{pmatrix} -i\lambda & (u-1)\lambda \\ -\lambda & i\lambda \end{pmatrix} y\,, \tag{10}$$

with $y=(y_1,y_2)^T$, $i^2=-1$.

Evidently, Eq. (10) is equivalent to the following spectral problem

$$Ly=\lambda y,\ L=L(u)=\frac{1}{u}\begin{pmatrix} i & 1-u \\ 1 & -i \end{pmatrix}\partial,\quad \partial=\frac{\partial}{\partial x}\,, \tag{11}$$

where the potential u decays at infinity or satisfies the periodic condition: $u(x+T)=u(x)$, $T=\mathrm{const}$.

The spectral gradient $\nabla_u\lambda=\delta\lambda/\delta u$ is

$$\nabla_u\lambda=\lambda y_2^2\,. \tag{12}$$

According to the relation $\partial^{-1}u\partial y_2^2=2iy_1y_2+y_2^2-y_1^2$ and Eq. (10), only choosing the following operators K,J

$$K=\partial^3,\quad J=-2(\partial u+u\partial) \tag{13}$$

as the pair of Lenard's operators of Eq. (10), we have

$$K\nabla_u\lambda=\lambda^2\cdot J\nabla_u\lambda\,. \tag{14}$$

The Gateaux derivative operator $L_*(\xi)$ of Eq. (11) in the direction ξ is

$$L_*(\xi)=\frac{\xi}{u^2}\begin{pmatrix}-i & -1\\ -1 & i\end{pmatrix}\partial=\frac{\xi}{u}\begin{pmatrix}0 & -i\\ 0 & -1\end{pmatrix}L\,. \tag{15}$$

Obviously, $L_*(\xi)=0 \Leftrightarrow \xi=0$.

Let $A=A(t_m)$, $B=B(t_m)$ be two arbitary given smooth functions of t_m, and they are independent of x. If and only if we choose

$$M=A\begin{pmatrix}0 & i\\ 0 & 1\end{pmatrix}L,\qquad \tilde{M}=B\begin{pmatrix}0 & -i\\ 0 & -1\end{pmatrix}L\,, \tag{16}$$

the operator equations $L_*(JG_0)=M$, $L_*(KG_{-1})=\tilde{M}$ have the solutions

$$G_0=\tfrac{1}{4}u^{-1/2}\partial^{-1}u^{-1/2}A,\qquad G_{-1}=\partial^{-3}uB,\qquad \partial\partial^{-1}=\partial^{-1}\partial=1\,, \tag{17}$$

respectively. Thus, the Lenard's recursive sequence $\{G_j\}_{j=-\infty}^{\infty}$ is determined by

$$\begin{aligned} G_{j+1}&=\mathscr{L}G_j=\mathscr{L}^{j+1}(\tfrac{1}{4}u^{-1/2}\partial^{-1}u^{-1/2}A),\quad j=0,1,2,\dots\,,\\ G_{j-1}&=\mathscr{L}G_j=\mathscr{L}^{j}(\partial^{-3}uB),\quad j=-1,-2,-3,\dots\,, \end{aligned} \tag{18}$$

where the recursion operator $\mathscr{L}=J^{-1}K=-\frac{1}{4}u^{-1/2}\partial^{-1}u^{-1/2}\partial^3$. The vector fields

$$X_m(u)=\begin{cases} KG_m=K\mathscr{L}^m(\frac{1}{4}u^{-1/2}\partial^{-1}u^{-1/2}A), & m\geqslant 0\,,\\ JG_m=J\mathscr{L}^{m+1}(\partial^{-3}uB), & m<0\,, \end{cases} \tag{19}$$

yield the hierarchies of GNLEEs of Eq. (10)

$$u_{t_m}=X_m(u)=\begin{cases} KG_m=K\mathscr{L}^m(\frac{1}{4}u^{-1/2}\partial^{-1}u^{-1/2}A), & m\geqslant 0\,,\\ JG_m=J\mathscr{L}^{m+1}(\partial^{-3}uB), & m<0\,. \end{cases} \tag{20}$$

Let $G=G(x,t,u)$ be an arbitrary given smooth function. For the spectral problem (10), we consider the following operator equation:

$$[V,L]=L_*(KG)L^{-1}-L_*(JG)L \tag{21}$$

which corresponds to choosing $\beta=-1$, $\alpha=1$ in Eq. (5). In Eq. (21),

$$L^{-1}=\begin{pmatrix}-i\partial^{-1} & \partial^{-1}(u-1)\\ -\partial^{-1} & i\partial^{-1}\end{pmatrix} \tag{22}$$

is the inverse operator of L. Then through calculating the commutator $[V,L]=VL-LV=-W^{-1}V_{0x}+(V_0-W^{-1}V_0W-W^{-1}V_{1x})L+(V_1-W^{-1}V_1W-W^{-1}V_{2x})L^2+(V_2-W^{-1}V_2W)L^3$, we obtain the following result:

Let L, K, J and L_* be defined by Eqs. (11), (13) and (15), respectively. Then Eq. (21) possesses the operator solution as below

$$V=V(G)=G_{xx}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}+G_x\begin{pmatrix}1 & -2i\\ 0 & -1\end{pmatrix}L+2G\begin{pmatrix}-i & u-1\\ -1 & i\end{pmatrix}L^2\,. \tag{23}$$

So, for the spectral problem (10) the hierarchies of GNLEEs (20) possess the following GLR:

$$L_{t_m} = [W_m, L] + \begin{pmatrix} 0 & iA \\ 0 & A \end{pmatrix} L^{2m+3}, \quad m \geqslant 0\,,$$

$$L_{t_m} = [W_m, L] + \begin{pmatrix} 0 & -iB \\ 0 & -B \end{pmatrix} L^{2m+1}, \quad m < 0\,, \tag{24}$$

$$W_m = \begin{cases} \sum_{j=0}^{m} V(G_j) L^{2(m-j)+1}, & m \geqslant 0\,, \\ -\sum_{j=m}^{-1} V(G_j) L^{2(m-j)-1}, & m < 0\,, \end{cases} \tag{25}$$

where $V(G_j)$ is the formula (23) with $G = G_j$ (G_j is the Lenard's recursive sequence (18) or (18)').

Several special cases of Eqs. (20) and (24) (or (24)') are displayed as follows:

(i) Set $A = 0$, $\partial^{-1} 0 = 4$, then $M = 0$ and $G_0 = u^{-1/2} \in \aleph_J(0)$. Thus for $m \geqslant 0$, (20) expresses the positive order NLEEs (i.e. the higher-order isospectral ($\lambda_t = 0$) hierarchy of Eq. (10))

$$u_{t_m} = K \mathscr{L}^m u^{-1/2}, \quad m \geqslant 0\,, \tag{26}$$

which possess the Lax representations: $L_{t_m} = [W_m, L], W_m = \sum_{j=0}^{m} V(G_j) L^{2(m-j)+1}$.

As $m = 0$, Eq. (26) becomes the well-known Harry–Dym equation:

$$u_{t_0} = (u^{-1/2})_{xxx}\,, \tag{27}$$

which has the Lax representation

$$L_{t_0} = [W_0, L]\,, \tag{28}$$

with

$$W_0 = (u^{-1/2})_{xx} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} L + (u^{-1/2})_x \begin{pmatrix} 1 & -2i \\ 0 & -1 \end{pmatrix} L^2 + 2u^{-1/2} \begin{pmatrix} -i & u-1 \\ -1 & i \end{pmatrix} L^3\,,$$

$$L = \frac{1}{u} \begin{pmatrix} i & 1-u \\ 1 & -i \end{pmatrix} \partial\,.$$

Hence, Eq. (26) gives the Harry–Dym hierarchy of NLEEs.

(ii) Set $B = 0$, then $\tilde{M} = 0$ and $G_{-1} = ax^2 + bx + c \in \aleph_K(0)$, $\forall a, b, c \in \mathrm{C}$. Thus, for $m < 0$, Eq. (20) reads the negative order NLEEs (i.e. the lower-order isospectral ($\lambda_t = 0$) hierarchy of Eq. (10))

$$u_{t_m} = J \mathscr{L}^{m+1} (ax^2 + bx + c), \quad m < 0\,, \tag{29}$$

which have the Lax representations $L_{t_m} = [W_m, L]$ with $W_m = -\sum_{j=m}^{-1} V(G_j) L^{2(m-j)-1}$, $m < 0$.

As $m=-1$, Eq. (29) becomes a variant-coefficient linear ordinary differential equation:

$$u_{t_{-1}}=-2(ax^2+bx+c)u_x-4(2ax+b)u \tag{30}$$

which is very easily solved and has the Lax representation

$$L_{t_{-1}}=[W_{-1},L]\,, \tag{31}$$

$$W_{-1}=-2a\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}L^{-1}-2(ax+b)\begin{pmatrix}1 & -2i\\ 0 & -1\end{pmatrix}$$
$$-2(ax^2+bx+c)\begin{pmatrix}i & u-1\\ -1 & i\end{pmatrix}L\,.$$

As $m=-2$, through discussing Eq. (29) in the following cases we can obtain some new integrable NLEEs such as Eqs. (32), (34), (36), which can be possibly applied in theoretical and experimental physics.

(a) Choose $a=b=0$, then

$$G_{-1}=c,\quad G_{-2}=\mathscr{L}^{-1}G_{-1}=-2c\partial^{-2}u,\quad J\mathscr{L}^{-1}G_{-1}=4c(u_x\partial^{-2}u+2u\partial^{-1}u)\,.$$

Order $u=v_{xx}$, then Eq. (29) is changed as

$$v_{xxt_{-2}}=4c(v_{xxx}v+2v_{xx}v_x)\,, \tag{32}$$

which possesses the Lax representation

$$L_{t_{-2}}=[W_{-2},L]\,, \tag{33}$$
$$W_{-2}=-2c\begin{pmatrix}i & 1\\ 1 & -i\end{pmatrix}L^{-1}+2cv_x\begin{pmatrix}1 & -2i\\ 0 & -1\end{pmatrix}+4cv\begin{pmatrix}-i & v_{xx}-1\\ -1 & i\end{pmatrix}L\,,$$
$$L=\frac{1}{v_{xx}}\begin{pmatrix}i & 1-v_{xx}\\ 1 & i\end{pmatrix}\partial\,.$$

(b) Choose $a=c=0$, then

$$G_{-1}=bx,\quad G_{-2}=\mathscr{L}^{-1}G_{-1}=-2b(\partial^{-2}xu+\partial^{-3}u)\,,$$

$$J\mathscr{L}^{-1}G_{-1}=4b(u_x\partial^{-2}(xu)+2u\partial^{-1}(xu)+u_x\partial^{-3}u+2u\partial^{-2}u)\,.$$

Order $u=v_{xxx}$, then Eq. (29) becomes

$$v_{xxxt_{-2}}=4b(v_{xxxx}(xv_x-v)+2xv_{xx}v_{xxx})\,, \tag{34}$$

which has the Lax representation

$$L_{t_{-2}} = [W_{-2}, L]\,, \tag{35}$$

$$W_{-2} = -\,b\begin{pmatrix} 1 & -2i \\ 0 & -1 \end{pmatrix} L^{-2} + 2b\begin{pmatrix} ix & v_{xx}+x \\ x & -ix \end{pmatrix} L^{-1}$$

$$+\,2bv_{xx}\begin{pmatrix} 1 & -2i \\ 0 & -1 \end{pmatrix} + 4b(xv_x - v)\begin{pmatrix} -i & v_{xxx}-1 \\ -1 & i \end{pmatrix} L\,,$$

$$L = \frac{1}{v_{xxx}}\begin{pmatrix} i & 1-v_{xxx} \\ 1 & -i \end{pmatrix}\partial\,.$$

(c) Choose $b=c=0$, then

$$G_{-1} = ax^2, \quad G_{-2} = \mathscr{L}^{-1}G_{-1} = -\,2a(\partial^{-2}(x^2u) + 2\partial^{-3}(xu))\,,$$

$$J\mathscr{L}^{-1}G_{-1} = 4a(u_x\partial^{-2}(x^2u) + 2u\partial^{-1}(x^2u) + 2u_x\partial^{-3}(xu) + 4u\partial^{-2}(xu))\,.$$

Order $u = v_{xxx}/x$, then Eq. (29) reads

$$v_{xxxt_{-2}} = 4a(v_{xxx}v_x + xv_{xxxx}v_x + 2xv_{xxx}v_{xx}) \tag{36}$$

which possesses the Lax representation

$$L_{t_{-2}} = [W_{-2}, L]\,, \tag{37}$$

$$W_{-2} = -\,2a\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} L^{-3} - 2ax\begin{pmatrix} 1 & -2i \\ 0 & -1 \end{pmatrix} L^{-2} + 2a\begin{pmatrix} ix^2 & 2v_{xx}+x^2 \\ x^2 & -ix^2 \end{pmatrix} L^{-1}$$

$$+\,2a(v_x + xv_{xx})\begin{pmatrix} 1 & -2i \\ 0 & -1 \end{pmatrix} + 4av_x\begin{pmatrix} -ix & v_{xxx}-x \\ x & ix \end{pmatrix} L\,,$$

$$L = \frac{x}{v_{xxx}}\begin{pmatrix} i & 1-\frac{v_{xxx}}{x} \\ 1 & -i \end{pmatrix}\partial\,.$$

(iii) Set $A = A(t_m) \neq 0$, $A(t_m)$ is an arbitary differential function of t_m. Then

$$M = A(t_m)\begin{pmatrix} 0 & i \\ 0 & 1 \end{pmatrix} L\,,$$

$$G_0 = \tfrac{1}{4}A(t_m)u^{-1/2}\partial^{-1}u^{-1/2}, \quad G_j = \tfrac{1}{4}A(t_m)\mathscr{L}^j u^{-1/2}\partial^{-1}u^{-1/2}, \qquad j = 0,1,2,\dots\,.$$

For $m \geqslant 0$, Eq. (20) reads the positive order NLEEs

$$u_{t_m} = \tfrac{1}{4}A(t_m)K\mathscr{L}^m u^{-1/2}\partial^{-1}u^{-1/2}, \quad m \geqslant 0\,, \tag{38}$$

which correspond to the non-isospectral case $\lambda_{t_m} = A(t_m)\lambda^{2m+3}$ of Eq. (10) and possess the following GLR

$$L_{t_m} = [W_m, L] + A(t_m)\begin{pmatrix} 0 & i \\ 0 & 1 \end{pmatrix} L^{2m+3}, \quad m \geqslant 0\,, \tag{39}$$

$$W_m = \sum_{j=0}^{m} \left\{ G_{j,xx} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + G_{j,x} \begin{pmatrix} 1 & -2i \\ 0 & -1 \end{pmatrix} L \right.$$

$$\left. +2G_j \begin{pmatrix} -i & u-1 \\ -1 & i \end{pmatrix} L^2 \right\} L^{2(m-j)+1} .$$

A representative equation of Eq. (38) is

$$u_{t_0} = \tfrac{1}{4} A(t_0)[(u^{-1/2})_{xxx} \partial^{-1} u^{-1/2} + 4(u^{-1/2})_{xx} u^{-1/2} + 3(u^{-1/2})_x^2] , \tag{40}$$

which can be reduced to

$$v_{xt_0} = -\tfrac{1}{8} A(t_0)(v_{xxxx} v_x^2 v + 4 v_{xxx} v_x^4 + 3 v_{xx}^2 v_x^3) \tag{41}$$

via the transformation $\partial^{-1} u^{-1/2} = v$. Eq. (41) is a new integrable nonlinear evolution equation and has the GLR

$$L_{t_0} = [W_0, L] + A(t_0) \begin{pmatrix} 0 & i \\ 0 & 1 \end{pmatrix} L^3 , \tag{42}$$

$$W_0 = \frac{1}{4} A(t_0)(v_{xxx} v + 3 v_{xx} v_x) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} L + \frac{1}{4} A(t_0)(v_{xx} + v_x^2) \begin{pmatrix} 1 & -2i \\ 0 & -1 \end{pmatrix} L^2$$

$$+ \frac{1}{2} A(t_0) v_x v \begin{pmatrix} -i & u-1 \\ -1 & i \end{pmatrix} L^3 ,$$

$$L = v_x^2 \begin{pmatrix} i & 1 - v_x^{-2} \\ 1 & -i \end{pmatrix} \partial .$$

(iv) Set $B = B(t_m) \neq 0$, $B(t_m)$ is an arbitrary differential function of t_m. Then

$$\tilde{M} = B(t_m) \begin{pmatrix} 0 & -i \\ 0 & -1 \end{pmatrix} L ,$$

$$G_{-1} = B(t_m) \partial^{-3} u, \quad G_{j-1} = B(t_m) \mathscr{L}^j \partial^{-3} u, \quad j < 0 .$$

For $m < 0$, Eq. (20) reads the negative order NLEEs

$$u_{t_m} = B(t_m) J \mathscr{L}^{m+1} \partial^{-3} u, \quad m < 0 , \tag{43}$$

which correspond to the non-isospectral case $\lambda_{t_m} = B(t_m) \lambda^{2m+1}$ $(m < 0)$ of (10). As $m = -1$, Eq. (43) reads

$$u_{t_{-1}} = -2B(t_{-1})(2u \partial^{-2} u + u_x \partial^{-3} u) \tag{44}$$

which can be changed to

$$v_{xxxt_{-1}} = -2B(t_{-1})(2 v_{xxx} v_x + v_{xxxx} v) \tag{45}$$

via the transformation $u=v_{xxx}$. Eq. (45) is also a new integrable NLEE, and possesses the GLR

$$L_{t_{-1}}=[W_{-1},L]+B(t_{-1})\begin{pmatrix}0 & -i\\ 0 & -1\end{pmatrix}L^{-1}\,, \tag{46}$$

$$\begin{aligned}W_{-1} &= B(t_{-1})v_{xx}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}L^{-1}+B(t_{-1})v_x\begin{pmatrix}1 & -2i\\ 0 & -1\end{pmatrix}\\ &\quad +2B(t_{-1})v\begin{pmatrix}-i & u-1\\ -1 & i\end{pmatrix}L\,,\end{aligned}$$

$$L=\frac{1}{v_{xxx}}\begin{pmatrix}i & 1-v_{xxx}\\ 1 & -i\end{pmatrix}\partial\,.$$

Finally, through some investigations it has been found that this kind of GNLEEs and GLR can connect with the well-known r-matrix in $1+1$ or $1+d$ $(d\geqslant 1)$ dimensional spaces, which is written in Ref. [34]. Additionally, by making use of GNLEEs and GLR we may also generate the Lie algebraic structure of operator related to stationary systems, which has been established in Ref. [16].

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